Lesson 24: Solving Exponential Equations

Classwork

Opening Exercise

In Lesson 7, we modeled a population of bacteria that doubled every day by the function $P(t) = 2^t$, where t was the time in days. We wanted to know the value of t when there were 10 bacteria. Since we didn't yet know about logarithms, we approximated the value of t numerically and we found that $P(t) = 10$ at approximately $t \approx 3.32$ days.

Use your knowledge of logarithms to find an exact value for t when $P(t) = 10$, and then use your calculator to approximate that value to four decimal places.

Exercises

1. Fiona modeled her data from the bean-flipping experiment in Lesson 23 by the function $f(t) = 1.263(1.357)^t$, and Gregor modeled his data with the function $g(t) = 0.972(1.629)^t$.

   a. Without doing any calculating, determine which student, Fiona or Gregor, accumulated 100 beans first. Explain how you know.

   b. Using Fiona's model …

      i. How many trials would be needed for her to accumulate 100 beans?

ii. How many trials would be needed for her to accumulate $1{,}000$ beans?

c. Using Gregor's model …

i. How many trials would be needed for him to accumulate 100 beans?

ii. How many trials would be needed for him to accumulate $1{,}000$ beans?

d. Was your prediction in part (a) correct? If not, what was the error in your reasoning?

2. Fiona wants to know when her model $f(t) = 1.263(1.357)^t$ predicts accumulations of 500, $5{,}000$, and $50{,}000$ beans, but she wants to find a way to figure it out without doing the same calculation three times.

   a. Let the positive number c represent the number of beans that Fiona wants to have. Then solve the equation $1.263(1.357)^t = c$ for t.

   b. Your answer to part (a) can be written as a function M of the number of beans c, where $c > 0$. Explain what this function represents.

   c. When does Fiona's model predict that she will accumulate …

      i. 500 beans?

      ii. 5000 beans?

      iii. $50{,}000$ beans?

3. Gregor states that the function g that he found to model his bean-flipping data can be written in the form $g(t) = 0.972\left(10^{\log(1.629)t}\right)$. Since $\log(1.629) \approx 0.2119$, he is using $g(t) = 0.972(10^{0.2119t})$ as his new model.

   a. Is Gregor correct? Is $g(t) = 0.972\left(10^{\log(1.629)t}\right)$ an equivalent form of his original function? Use properties of exponents and logarithms to explain how you know.

   b. Gregor also wants to find a function that will help him to calculate the number of trials his function g predicts it will take to accumulate 500, $5{,}000$, and $50{,}000$ beans. Let the positive number c represent the number of beans that Gregor wants to have. Solve the equation $0.972(10^{0.2119t}) = c$ for t.

   c. Your answer to part (b) can be written as a function N of the number of beans c, where $c > 0$. Explain what this function represents.

   d. When does Gregor's model predict that he will accumulate …

      i. 500 beans?

ii. 5,000 beans?

iii. 50,000 beans?

4. Helena and Karl each change the rules for the bean experiment. Helena started with four beans in her cup and added one bean for each that landed marked-side up for each trial. Karl started with one bean in his cup but added two beans for each that landed marked-side up for each trial.

 a. Helena modeled her data by the function $h(t) = 4.127(1.468^t)$. Explain why her values of $a = 4.127$ and $b = 1.468$ are reasonable.

 b. Karl modeled his data by the function $k(t) = 0.897(1.992^t)$. Explain why his values of $a = 0.897$ and $b = 1.992$ are reasonable.

c. At what value of t do Karl and Helena have the same number of beans?

d. Use a graphing utility to graph $y = h(t)$ and $y = k(t)$ for $0 < t < 10$.

e. Explain the meaning of the intersection point of the two curves $y = h(t)$ and $y = k(t)$ in the context of this problem.

f. Which student reaches 20 beans first? Does the reasoning you used with whether Gregor or Fiona would get to 100 beans first hold true here? Why or why not?

For the following functions f and g, solve the equation $f(x) = g(x)$. Express your solutions in terms of logarithms.

5. $f(x) = 10(3.7)^{x+1}$, $g(x) = 5(7.4)^{x}$

6. $f(x) = 135(5)^{3x+1}$, $g(x) = 75(3)^{4-3x}$

7. $f(x) = 100^{x^3+x^2-4x}$, $g(x) = 10^{2x^2-6x}$

8. $f(x) = 48\left(4^{x^2+3x}\right)$, $g(x) = 3\left(8^{x^2+4x+4}\right)$

9. $f(x) = e^{\sin^2(x)}$, $g(x) = e^{\cos^2(x)}$

10. $f(x) = (0.49)^{\cos(x)+\sin(x)}$, $g(x) = (0.7)^{2\sin(x)}$

Problem Set

1. Solve the following equations.
   a. $2 \cdot 5^{x+3} = 6250$
   b. $3 \cdot 6^{2x} = 648$
   c. $5 \cdot 2^{3x+5} = 10240$
   d. $4^{3x-1} = 32$
   e. $3 \cdot 2^{5x} = 216$
   f. $5 \cdot 11^{3x} = 120$
   g. $7 \cdot 9^{x} = 5405$
   h. $\sqrt{3} \cdot 3^{3x} = 9$
   i. $\log(400) \cdot 8^{5x} = \log(160000)$

2. Lucy came up with the model $f(t) = 0.701(1.382)^t$ for the first bean activity. When does her model predict that she would have $1{,}000$ beans?

3. Jack came up with the model $g(t) = 1.033(1.707)^t$ for the first bean activity. When does his model predict that he would have $50{,}000$ beans?

4. If instead of beans in the first bean activity you were using fair coins, when would you expect to have $\$1{,}000{,}000$?

5. Let $f(x) = 2 \cdot 3^x$ and $g(x) = 3 \cdot 2^x$.
   a. Which function is growing faster as x increases? Why?
   b. When will $f(x) = g(x)$?

6. A population of *E. coli* bacteria can be modeled by the function $E(t) = 500(11.547)^t$, and a population of *Salmonella* bacteria can be modeled by the function $S(t) = 4000(3.668)^t$, where t measures time in hours.
   a. Graph these two functions on the same set of axes. At which value of t does it appear that the graphs intersect?
   b. Use properties of logarithms to find the time t when these two populations are the same size. Give your answer to two decimal places.

7. Chain emails contain a message suggesting you will have bad luck if you do not forward the email to others. Suppose a student started a chain email by sending the message to 10 friends and asking those friends to each send the same email to 3 more friends exactly one day after receiving the message. Assuming that everyone that gets the email participates in the chain, we can model the number of people who will receive the email on the n^{th} day by the formula $E(n) = 10(3^n)$, where $n = 0$ indicates the day the original email was sent.
   a. If we assume the population of the United States is 318 million people and everyone who receives the email sends it to 3 people who have not received it previously, how many days until there are as many emails being sent out as there are people in the United States?
   b. The population of Earth is approximately 7.1 billion people. On what day will 7.1 billion emails be sent out?

8. Solve the following exponential equations.
   a. $10^{(3x-5)} = 7^x$
   b. $3^{\frac{x}{5}} = 2^{4x-2}$
   c. $10^{x^2+5} = 100^{2x^2+x+2}$
   d. $4^{x^2-3x+4} = 2^{5x-4}$

9. Solve the following exponential equations.
   a. $(2^x)^x = 8^x$
   b. $(3^x)^x = 12$

10. Solve the following exponential equations.
    a. $10^{x+1} - 10^{x-1} = 1287$
    b. $2(4^x) + 4^{x+1} = 342$

11. Solve the following exponential equations.
    a. $(10^x)^2 - 3(10^x) + 2 = 0$ Hint: Let $u = 10^x$ and solve for u before solving for x.
    b. $(2^x)^2 - 3(2^x) - 4 = 0$
    c. $3(e^x)^2 - 8(e^x) - 3 = 0$
    d. $4^x + 7(2^x) + 12 = 0$
    e. $(10^x)^2 - 2(10^x) - 1 = 0$

12. Solve the following systems of equations.
    a. $2^{x+2y} = 8$
       $4^{2x+y} = 1$
    b. $2^{2x+y-1} = 32$
       $4^{x-2y} = 2$
    c. $2^{3x} = 8^{2y+1}$
       $9^{2y} = 3^{3x-9}$

13. Because $f(x) = \log_b(x)$ is an increasing function, we know that if $p < q$, then $\log_b(p) < \log_b(q)$. Thus, if we take logarithms of both sides of an inequality, then the inequality is preserved. Use this property to solve the following inequalities.
    a. $4^x > \frac{5}{3}$
    b. $\left(\frac{2}{7}\right)^x > 9$
    c. $4^x > 8^{x-1}$
    d. $3^{x+2} > 5^{3-2x}$
    e. $\left(\frac{3}{4}\right)^x > \left(\frac{4}{3}\right)^{x+1}$