# Lesson 20: Transformations of the Graphs of Logarithmic and Exponential Functions

# **Student Outcomes**

- Students study transformations of the graphs of logarithmic functions.
- Students use the properties of logarithms and exponents to produce equivalent forms of exponential and logarithmic expressions. In particular, they notice that different types of transformations can produce the same graph due to these properties.

#### **Lesson Notes**

Students revisit the use of transformations to produce graphs of exponential and logarithmic functions (**F-BF.B.3**, **F-IF.B.4**, **F-IF.C.7e**). They make and verify conjectures about why certain transformations of the graphs of functions produce the same graph by applying the properties to produce equivalent expressions (MP.3). This work leads to a general form of both logarithmic and exponential functions where the given parameters can be quickly analyzed to determine key features and to sketch graphs of logarithmic and exponential functions (MP.7, MP.8). This lesson reinforces sketching graphs of functions by applying knowledge of transformations and the properties of logarithms and exponents.

#### Classwork

#### **Opening Exercise (8 minutes)**

Since much of the work on this lesson will involve the connections between scaling and translating graphs of functions, this Opening Exercise presents students with an opportunity to reflect on what they already know about transformations of graphs of functions using a simple polynomial function and the sine function. Observe students carefully as they work on these exercises to gauge how much re-teaching or additional support may be needed in the Exploratory Challenge that follows. If students struggle to recall their knowledge of transformations, you may need to provide additional guidance and practice throughout the lesson. A grid is provided for students to use when sketching the graphs in Opening Exercise, part (a), but students could also complete this exercise using graphing technology.

#### Opening Exercise

а		Sketch the graphs of the three functions $f(x) = x^2$ , $g(x) = (2x)^2 + 1$ , and $h(x) = 4x^2 + 1$ .			
	i	i.	Describe the transformations that will take the graph of $f(x) = x^2$ to the graph of $g(x) = (2x)^2 + 1$ .		
			The graph of g is a horizontal scaling by a factor of $\frac{1}{2}$ and a vertical translation up 1 unit of the graph of f.		
	i	ii.	Describe the transformations that will take the graph of $f(x) = x^2$ to the graph of $h(x) = 4x^2 + 1$ . The graph of $h$ is a vertical scaling by a factor of 4 and a vertical translation up 1 unit of the graph of $f$ .		









Have students share responses and revise their work in small groups. Be sure to emphasize that students need to label graphs since many of them appear on the same set of axes. Lead a brief whole-group discussion that focuses on the responses to Opening Exercise part (a-iii) and part (d). Perhaps have one or two students share their written response with the whole class, using either the board or the document camera. Check to make sure students are actually writing responses to these questions.

Conclude this discussion by helping students to understand the following ideas that, as will be demonstrated later, are also true with graphs of logarithmic and exponential functions.

- Certain transformations of the graph of a function can be identical to other transformations depending on the properties of the given function.
- For example, the function g(x) = x + 1 is either a horizontal translation of 1 unit to the left or a vertical translation of 1 unit up of the graph of f(x) = x. Similarly, g(x) = |-2x + 2| has the same graph as h(x) = 2|x 1|, but could be described using different transformations of the graph of f(x) = |x|.



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Announce to students that in this lesson, they will explore the properties of logarithms and exponents to understand graphing transformations of those types of functions, and they will explore when two different functions have the same graph in order to reinforce those properties.

### **Exploratory Challenge (15 minutes)**

Students should work in small groups to complete this sequence of questions. Provide support to individual groups or students as you move around the classroom. As you circulate, keep questioning students as to the meaning of a logarithm with questions like those below.

- What does log<sub>2</sub>(4) mean?
  - The exponent when the number 4 is written as a power of 2.
- Why is  $\log_2\left(\frac{1}{4}\right)$  negative?
  - It is negative because  $\frac{1}{4} = 2^{-2}$ , and the logarithm is the exponent when the number  $\frac{1}{4}$  is written as a power of 2.
- What will be the domain and range of the function f? Why does this make sense given the definition of a logarithm?
  - <sup>a</sup> The domain is all real numbers greater than 0. The range is all real numbers. This makes sense because the range of the exponential function  $f(x) = 2^x$  is all real numbers greater than 0, the domain is all real numbers, and the logarithmic function base 2 is the inverse of the exponential function base 2.

Scaffolding:

Use technology to support learners who are still struggling with arithmetic and need visual reinforcement. Students can investigate using graphing calculators or online graphing programs. Newer calculators and graphing programs have a  $\log_b(x)$ function built in. On older models, you may need to coach students to use the change of base property to enter these functions (i.e., to graph  $f(x) = \log_2(x)$ , you will need to enter the expression  $\frac{\log(x)}{\log(2)}$  into the graphing calculator).

You can extend this lesson by using graphing software such as GeoGebra to create parameterized graphs with sliders (variables *a* and *b* that can be dynamically changed while viewing graphs). By manipulating the values of *a* and *b* in the functions  $f(x) = \log_2(ax)$  and  $g(x) = \log_2(x) + b$ , you can verify that the graphs of  $f(x) = \log_2(2x)$  and  $g(x) = \log_2(x) + 1$  are the same. This result reinforces properties of logarithms since  $\log_2(2x) = \log_2(x) + \log_2(2)$ . Students and teachers can similarly confirm the other examples in this lesson as well.

Exploratory Challenge a. Sketch the graph of  $f(x) = \log_2(x)$  by identifying and plotting at least five key points. Use the table below to help you get started. The graph of f is blue, and the graph of g and h is red on the solution graph shown below. y









Call the entire class together at this point to debrief their work so far. Make sure students understand that by applying the product or quotient property of logarithms, they can rewrite a single logarithmic expression as a sum or difference. In this way, a horizontal scaling of the graph of a logarithmic function will produce the same graph as a vertical translation. The next three parts can be used to informally assess student understanding of the idea that two different transformations can produce the same graph because of the properties of logarithms.

g.	Describe the graph of $g(x) = \log_2(\frac{x}{4})$ as a vertical translation of the graph of $f(x) = \log_2(x)$ . Justify your response.
	The graph of g is a vertical translation down 2 units of the graph of f because $\log_2(\frac{x}{4}) = \log_2(x) - 2$ .
h.	Describe the graph of $h(x) = \log_2(x) + 3$ as a horizontal scaling of the graph of $f(x) = \log_2(x)$ . Justify your response.
	The graph of h is a horizontal scaling by a factor of $\frac{1}{8}$ of the graph of f because $\log_2(x) + 3 = \log_2(x) + \log_2(8) = \log_2(8x)$ .
	g. h.



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MP.3

**MP.7** 



ALGEBRA II

i. Do the functions  $f(x) = \log_2(x) + \log_2(4)$  and  $g(x) = \log_2(x+4)$  have the same graphs? Justify your reasoning.

No, they do not. By substituting 1 for x in both f and g, you can see that the graphs of the two functions will not have the same y-coordinate at this point. Therefore, the graphs cannot be the same if at least one point is different.

Carefully review the answers to the preceding questions to check for student understanding. Students may struggle with expressing 3 as  $\log_2(8)$  in part (h). In the last portion of the Exploratory Challenge, students turn their attention to exponential functions and apply the properties of exponents to explain why graphs of certain exponential functions are identical. Circulate around the room while students are working and encourage them to create the graph of the "parent" function by plotting key points on the graph of the function and then transforming those key points according to the transformation they described.





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ALGEBRA II

Have groups volunteer to present their findings on the last three parts of the Exploratory Challenge. When debriefing, model both transformations for students by marking the sketch as shown in the solutions above.

Discuss these transformations.

- In part (k), how do you see the transformations that produce the graphs of f and g from the graph of  $y = 2^x$ ?
  - <sup>a</sup> I see the horizontal translation of 2 units to the left, but others might see the vertical scaling that takes each *y*-value and multiplies it by 4.
- In part (I), how do the transformations validate the definition of a negative exponent?

\_h

Since the graphs of g and h were identical, we have visual confirmation that  $\left(\frac{1}{2}\right)^x = 2^{-x}$ , which can only be true if  $\frac{1}{2} = 2^{-1}$ .

Then, have students respond to the reflection question below in writing or with a partner.

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- How do the properties of logarithms and exponents justify the fact that different transformations of the graph of a function can sometimes produce the same graph?
  - We can use the properties to rewrite logarithmic and exponential expressions in equivalent forms that then represent different transformations of the same original function.

If time permits, you can also tie these transformations to a simple real-world context. For example, if we rewrite the function  $f(x) = 2^{x+3}$  as  $f(x) = 8 \cdot 2^x$ , students can see that adding 3 to x would be like going three years forward in time. This means the population doubled three times, which is why we are multiplying by 8.



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#### Example 1 (4 minutes): Graphing Transformations of the Logarithm Functions



Introduce the general form of a logarithm function, noting that we do not need a horizontal scaling parameter since a horizontal scaling can always be rewritten as a vertical translation. Continue to reinforce learning from the previous lessons by asking students why the restrictions on b and x - h are necessary. Students should be able to work through part (a) without your assistance, but monitor their work to make sure that all students have the correct answer to refer to when they work the Problem Set. Model your expectations for sketching the graphs of logarithm functions in part (b) so students are able to produce accurate and precise graphs. Demonstrate how to plot the key points, and then transform the individual points to produce the graph of g.



The common logarithm function is shown in blue, and the graph of f is shown in red. Notice the key points that students should include on their hand-drawn sketches.



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#### Example 2 (4 minutes): Graphing Transformations of Exponential Functions

Introduce the general form of the exponential function, noting that we do not need a horizontal scaling or a horizontal translation since these can always be rewritten using the properties of exponents. Demonstrate, or let students work with a partner on part (a), and make sure students have correct work to refer to when they work on the Problem Set. Since earlier lessons applied transformations to graphing exponential functions, parts (b), (c), and (d) should move along rather quickly. Continue to reinforce your expectations for sketching graphs of functions using transformations.







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#### Exercises 1-4 (4 minutes)

Students can work on these exercises independently or with a partner. Monitor their work by circulating around the classroom and checking for accuracy. Encourage students to describe the graph of g as a transformation of the graph of f in more than one way and to justify their answer analytically. In particular, emphasize how rewriting the expression using the properties of logarithms can make sketching the graphs easier because a horizontal scaling is revealed to have the same effect as a vertical translation when graphing logarithm functions.





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After a few minutes, have different groups share how they saw the transformations and discuss when it is advantageous to rewrite an expression before graphing and when it is not. For example, it might be easier in Exercise 10 to simply translate the graph 1 unit to the right rather than scale it by a factor of  $\frac{2}{3}$ . Also, make sure students are including a sketch of the end behavior of the functions.

# **Closing (5 minutes)**

Provide students with an opportunity to summarize their learning with a partner by responding to the questions below. Their summaries will provide you with additional evidence of their understanding of this lesson.

- How do you apply properties of logarithms or exponents to rewrite  $f(x) = \log_2(5x)$  and  $g(x) = 3^{x+2} + 2$  in general form?
  - Using the product properties:  $\log_2(5x) = \log_2(5) + \log_2(x)$ , so  $f(x) = \log_2(5) + \log_2(x)$  in general form where  $k = \log_2(5)$ , a = 1, and h = 0 in the general form.
  - Using the product properties:  $3^{x+2} = 3^x \cdot 3^2$ , so  $g(x) = 9 \cdot 3^x + 2$  in general form where a = 9 and k = 2.
- How do transformations help you to sketch quick and accurate graphs of functions?
  - If you can make a basic logarithm or exponential function graph for a given base, then transformations can be used to quickly sketch a new function that is based on the original function.

A summary of the key points of this lesson is provided. Review them with your class before they begin the Exit Ticket.

#### Lesson Summary

GENERAL FORM OF A LOGARITHMIC FUNCTION:  $f(x) = k + a \log_b(x - h)$  such that a, h, and k are real numbers, b is any positive number not equal to 1, and x - h > 0.

GENERAL FORM OF AN EXPONENTIAL FUNCTION:  $f(x) = a \cdot b^x + k$  such that a and k are real numbers, and b is any positive number not equal to 1.

The properties of logarithms and exponents can be used to rewrite expressions for functions in equivalent forms that can then be graphed by applying transformations.

#### **Exit Ticket (5 minutes)**









Name

Date

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# **Exit Ticket**

- 1. Express  $g(x) = -\log_4(2x)$  in the general form of a logarithmic function,  $f(x) = k + a \log_b(x h)$ . Identify a, b, h, and k.
- 2. Use the structure of g when written in general from to describe the graph of g as a transformation of the graph of  $h(x) = \log_4(x)$ .
- 3. Graph g anad h on the same coordinate axes.





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### **Exit Ticket Sample Solutions**





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#### **Problem Set Sample Solutions**





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5. Using the function  $f(x) = 2^x$ , create a new function g whose graph is a series of transformations of the graph of f with the following characteristics: The graph of g is increasing for all real numbers. The equation for the horizontal asymptote is y = 5. The y-intercept is 4. One possible solution is  $g(x) = -(2^{-x}) + 5$ . Given the function  $g(x) = \left(\frac{1}{4}\right)^{x-3}$ : 6. Write the function g as an exponential function with base 4. Describe the transformations that would take the graph of  $f(x) = 4^x$  to the graph of g.  $\left(\frac{1}{4}\right)^{x-3} = (4^{-1})^{x-3} = 4^{-x+3} = 4^3 \cdot 4^{-x}$ Thus,  $g(x) = 64 \cdot 4^{-x}$ . The graph of g is the graph of f reflected about the vertical axis and scaled vertically by a factor of 64. b. Write the function g as an exponential function with base 2. Describe two different series of transformations that would take the graph of  $f(x) = 2^x$  to the graph of g.  $\left(\frac{1}{4}\right)^{x-3} = (2^{-2})^{x-3} = 2^{-2(x-3)} = 2^{-2x+6} = 64 \cdot 2^{-2x}$ Thus,  $g(x) = 64 \cdot 2^{-2x}$ , or  $g(x) = 2^{-2(x-3)}$ . To obtain the graph of g from the graph of f, you can scale the graph horizontally by a factor of  $\frac{1}{2}$ , reflect the graph about the vertical axis, and scale it vertically by a factor of 64. Or, you can scale the graph horizontally by a factor of  $\frac{1}{2}$ , reflect the graph about the vertical axis, and translate the resulting graph horizontally 3 units to the right. Explore the graphs of functions in the form  $f(x) = \log(x^n)$  for n > 1. Explain how the graphs of these functions 7. change as the values of n increase. Use a property of logarithms to support your reasoning. The graphs appear to be a vertical scaling of the common logarithm function by a factor of n. This is true because of the property of logarithms that states  $log(x^n) = n log(x)$ . 8. Use a graphical approach to solve each equation. If the equation has no solution, explain why.  $\log(x) = \log(x - 2)$ a. This equation has no solution because the graphs of  $y = \log(x)$  and  $y = \log(x-2)$  are horizontal translations of each other. Thus, their graphs do not intersect, and the corresponding equation has no solution. b.  $\log(x) = \log(2x)$ This equation has no solution because log(2x) = log(2) + log(x), which means that the graphs of  $y = \log(x)$  and  $y = \log(2x)$  are a vertical translation of each other. Thus, their graphs do not intersect, and the corresponding equation has no solution.



MP 3

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