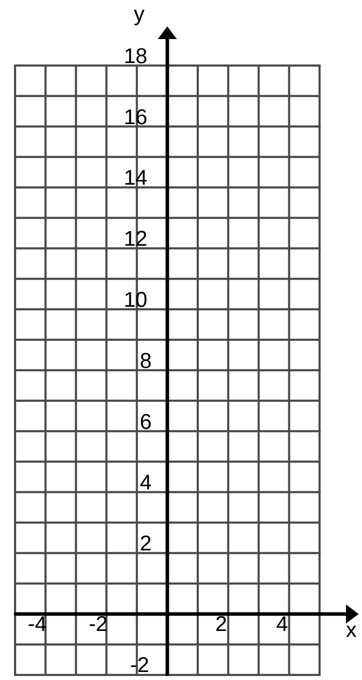
Lesson 20: Transformations of the Graphs of Logarithmic and Exponential Functions

Classwork

Opening Exercise

a. Sketch the graphs of the three functions $f(x) = x^2$, $g(x) = (2x)^2 + 1$, and $h(x) = 4x^2 + 1$.



i. Describe the transformations that will take the graph of $f(x) = x^2$ to the graph of $g(x) = (2x)^2 + 1$.

ii. Describe the transformations that will take the graph of $f(x) = x^2$ to the graph of $h(x) = 4x^2 + 1$.

iii. Explain why g and h from parts (i) and (ii) are equivalent functions.

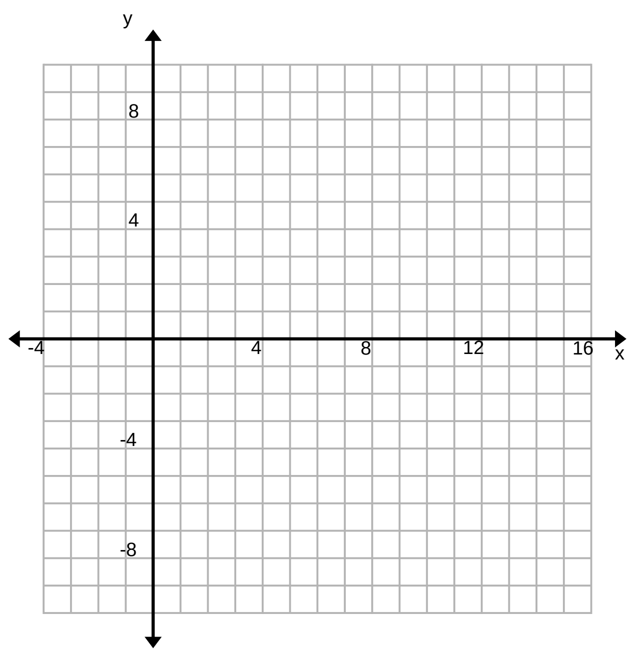
b. Describe the transformations that will take the graph of $f(x) = \sin(x)$ to the graph of $g(x) = \sin(2x) - 3$.

c. Describe the transformations that will take the graph of $f(x) = \sin(x)$ to the graph of $h(x) = 4\sin(x) - 3$.

d. Explain why g and h from parts (b)–(c) are *not* equivalent functions.

Exploratory Challenge

a. Sketch the graph of $f(x) = \log_2(x)$ by identifying and plotting at least five key points. Use the table below to help you get started.



b. Describe the transformations that will take the graph of f to the graph of $g(x) = \log_2(4x)$.

c. Describe the transformations that will take the graph of f to the graph of $h(x) = 2 + \log_2(x)$.

d. Complete the table below for f, g, and h and describe any patterns that you notice.

$\boldsymbol{x}$	$\boldsymbol{f(x)}$	$\boldsymbol{g(x)}$	$\boldsymbol{h(x)}$
$\frac{1}{4}$			
$\frac{1}{2}$			
1			
2			
4			
8			

e. Graph the three functions on the same coordinate axes and describe any patterns that you notice. Use a property of logarithms to show that g and h are equivalent.

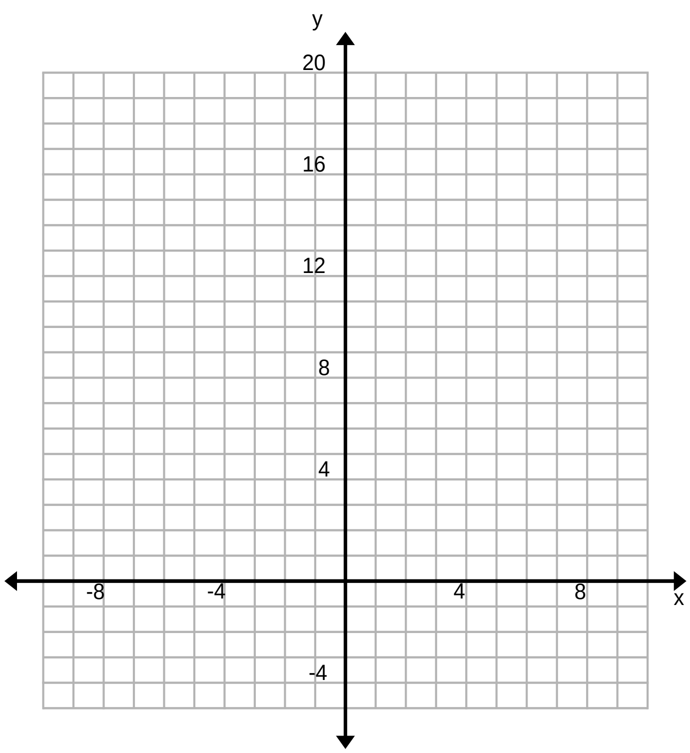
f. Describe the graph of $g(x) = \log_2\left(\frac{x}{4}\right)$ as a vertical translation of the graph of $f(x) = \log_2(x)$. Justify your response.

g. Describe the graph of $h(x) = \log_2(x) + 3$ as a horizontal scaling of the graph of $f(x) = \log_2(x)$. Justify your response.

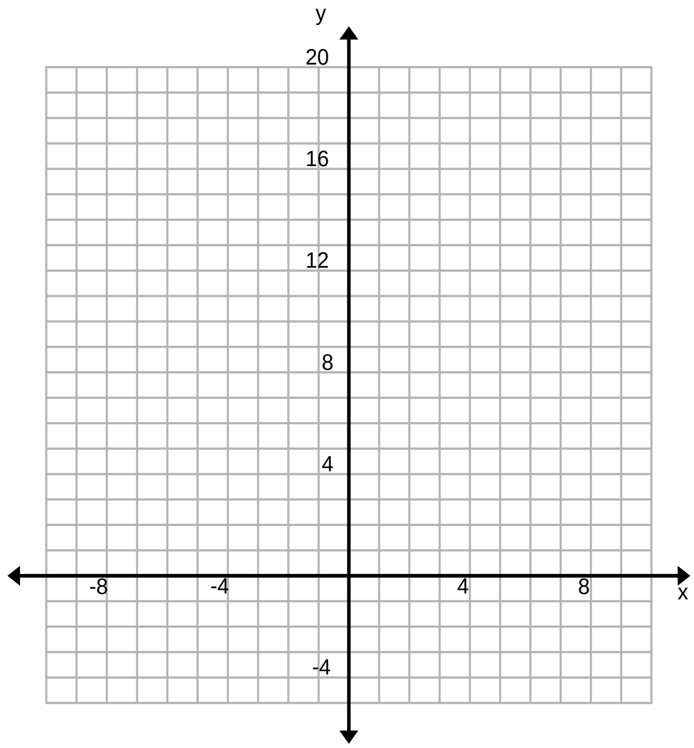
h. Do the functions $f(x) = \log_2(x) + \log_2(4)$ and $g(x) = \log_2(x + 4)$ have the same graphs? Justify your reasoning.

i. Use the properties of exponents to explain why the graphs of $f(x) = 4^x$ and $g(x) = 2^{2x}$ are identical.

j. Use the properties of exponents to predict what the graphs of $f(x) = 4 \cdot 2^x$ and $g(x) = 2^{x+2}$ will look like compared to one another. Describe the graphs of f and g as transformations of the graph of $f = 2^x$. Confirm your prediction by graphing f and g on the same coordinate axes.



k. Graph $f(x) = 2^x$, $g(x) = 2^{-x}$, and $h(x) = \left(\frac{1}{2}\right)^x$ on the same coordinate axes. Describe the graphs of g and h as transformations of the graph of f. Use the properties of exponents to explain why g and h are equivalent.

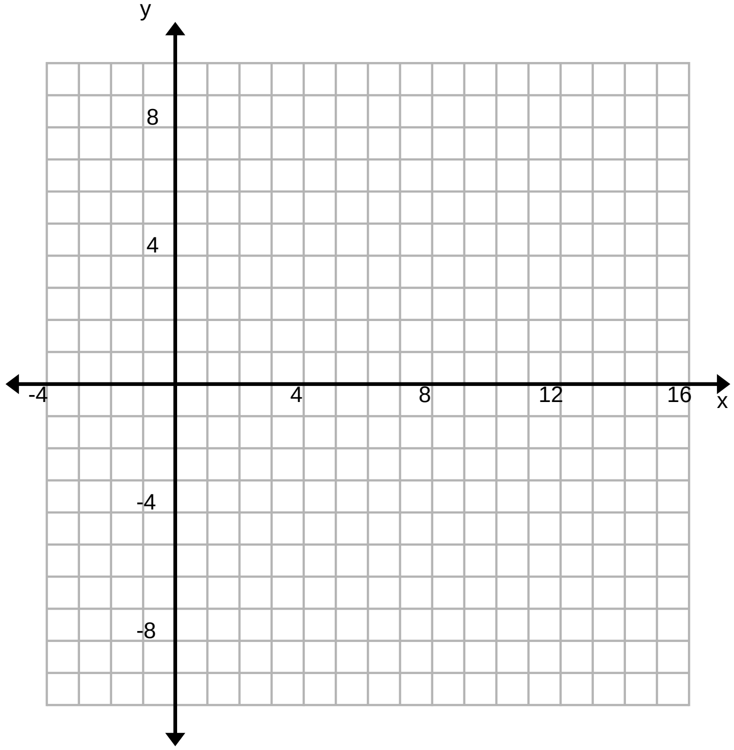


Example 1: Graphing Transformations of the Logarithm Functions

The general form of a logarithm function is given by $f(x) = k + a\log_b(x - h)$, where a, b, k, and h are real numbers such that b is a positive number not equal to 1, and $x - h > 0$.

a. Given $g(x) = 3 + 2\log(x - 2)$, describe the graph of g as a transformation of the common logarithm function.

b. Graph the common logarithm function and g on the same coordinate axes.



Example 2: Graphing Transformations of Exponential Functions

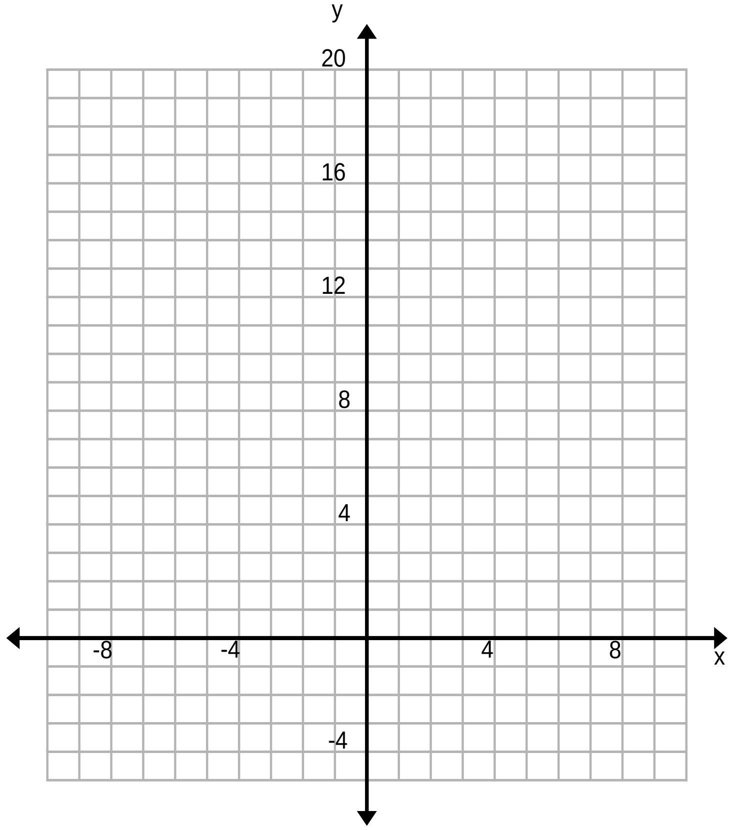
The general form of the exponential function is given by $f(x) = a \cdot b^x + k$, where a, b, and k are real numbers such that b is a positive number not equal to 1.

a. Use the properties of exponents to transform the function $g(x) = 3^{2x+1} - 2$ to the general form, and then graph it. What are the values of a, b, and k?

b. Describe the graph of g as a transformation of the graph of $h(x) = 9^x$.

c. Describe the graph of g as a transformation of the graph of $h(x) = 3^x$.

d. Graph g using transformations.



Exercises 1–4

Graph each pair of functions by first graphing f and then graphing g by applying transformations of the graph of f. Describe the graph of g as a transformation of the graph of f.

1. $f(x) = \log_3(x)$ and $g(x) = 2\log_3(x-1)$

2. $f(x) = \log(x)$ and $g(x) = \log(100x)$

3. $f(x) = \log_5 x$ and $g(x) = -\log_5\big(5(x+2)\big)$

4. $f(x) = 3^x$ and $g(x) = -2 \cdot 3^{x-1}$

Lesson Summary

GENERAL FORM OF A LOGARITHMIC FUNCTION: $f(x) = k + a\log_b(x - h)$ such that a, h, and k are real numbers, b is any positive number not equal to 1, and $x - h > 0$.

GENERAL FORM OF AN EXPONENTIAL FUNCTION: $f(x) = a \cdot b^x + k$ such that a and k are real numbers, and b is any positive number not equal to 1.

The properties of logarithms and exponents can be used to rewrite expressions for functions in equivalent forms that can then be graphed by applying transformations.

Problem Set

1. Describe each function as a transformation of the graph of a function in the form $f(x) = \log_b(x)$. Sketch the graph of f and the graph of g by hand. Label key features such as intercepts, increasing or decreasing intervals, and the equation of the vertical asymptote.

   a. $g(x) = \log_2(x - 3)$
   b. $g(x) = \log_2(16x)$
   c. $g(x) = \log_2\left(\frac{8}{x}\right)$
   d. $g(x) = \log_2((x - 3)^2)$

2. Each function graphed below can be expressed as a transformation of the graph of $f(x) = \log(x)$. Write an algebraic function for g and h and state the domain and range.

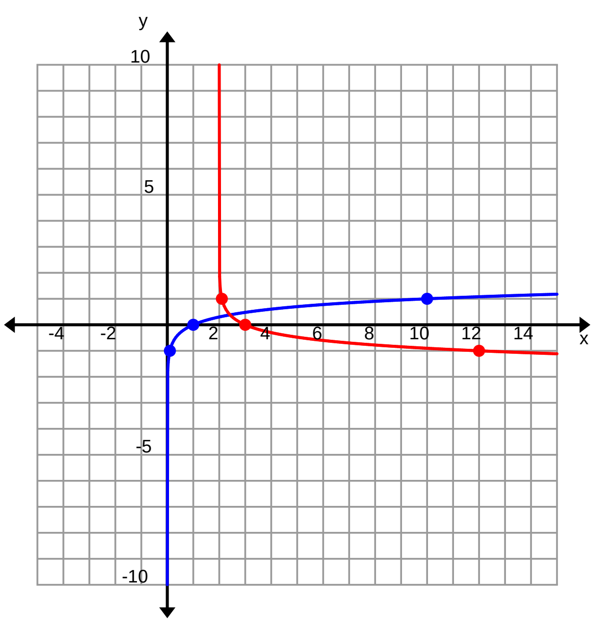


Figure 1: Graphs of $f(x) = \log(x)$ and the function g

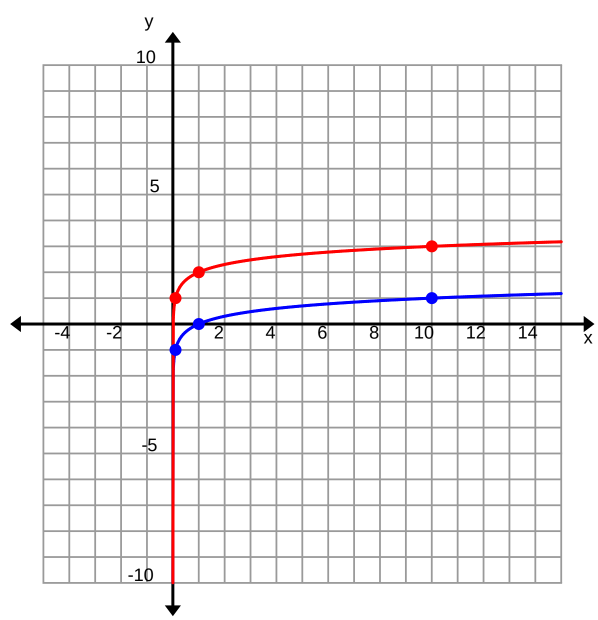


Figure 2: Graphs of $f(x) = \log(x)$ and the function h

3. Describe each function as a transformation of the graph of a function in the form $f(x) = b^x$. Sketch the graph of f and the graph of g by hand. Label key features such as intercepts, increasing or decreasing intervals, and the horizontal asymptote. (Estimate when needed from the graph.)

   a. $g(x) = 2 \cdot 3^x - 1$

   b. $g(x) = 2^{2x} + 3$

   c. $g(x) = 3^{x-2}$

   d. $g(x) = -9^{\frac{x}{2}} + 1$

4. Using the function $f(x) = 2^x$, create a new function g whose graph is a series of transformations of the graph of f with the following characteristics:

   - The graph of g is decreasing for all real numbers.
   - The equation for the horizontal asymptote is $y = 5$.
   - The y-intercept is 7.

5. Using the function $f(x) = 2^x$, create a new function g whose graph is a series of transformations of the graph of f with the following characteristics:

   - The graph of g is increasing for all real numbers.
   - The equation for the horizontal asymptote is $y = 5$.
   - The y-intercept is 4.

6. Given the function $g(x) = \left(\frac{1}{4}\right)^{x-3}$:

   a. Write the function g as an exponential function with base 4. Describe the transformations that would take the graph of $f(x) = 4^x$ to the graph of g.

   b. Write the function g as an exponential function with base 2. Describe two different series of transformations that would take the graph of $f(x) = 2^x$ to the graph of g.

7. Explore the graphs of functions in the form $f(x) = \log(x^n)$ for $n > 1$. Explain how the graphs of these functions change as the values of n increase. Use a property of logarithms to support your reasoning.

8. Use a graphical approach to solve each equation. If the equation has no solution, explain why.

   a. $\log(x) = \log(x - 2)$

   b. $\log(x) = \log(2x)$

   c. $\log(x) = \log\left(\frac{2}{x}\right)$

   d. Show algebraically that the exact solution to the equation in part (c) is $\sqrt{2}$.

9. Make a table of values for $f(x) = x^{\frac{1}{\log(x)}}$ for $x > 1$. Graph this function for $x > 1$. Use properties of logarithms to explain what you see in the graph and the table of values.