Lesson 15: Why Were Logarithms Developed?

Student Outcomes

- Students use logarithm tables to calculate products and quotients of multi-digit numbers without technology.
- Students understand that logarithms were developed to speed up arithmetic calculations by reducing multiplication and division to the simpler operations of addition and subtraction.
- Students solve logarithmic equations of the form $\log(X) = \log(Y)$ by equating $X = Y$.

Lesson Notes

This final lesson in Topic B includes two procedures that seem to be different but are closely related mathematically. First, students work with logarithm tables to see how applying logarithms simplified calculations in the days before computing machines and electronic technology. They also learn a bit of the history of how and why logarithms first appeared—a history often obscured when logarithmic functions are introduced as inverses of exponential functions. The last two pages of this document contain a base 10 table of logarithms that can be copied and distributed; such tables are also available on the Internet.

Then, students learn to solve the final type of logarithmic equation, $\log(X) = \log(Y)$, where X and Y are either real numbers or expressions that take on positive real values (**A.SSE.A.2**, **F-LE.A.4**). Using either technique requires that we know that the logarithm is a one-to-one function; that is, if $\log(X) = \log(Y)$, then $X = Y$. Students do not yet have the vocabulary to be told this directly, but we do state it as fact in this lesson, and they will further explore the idea of one-to-one functions in Precalculus. As with Lessons 10 and 12, this lesson involves only base 10 logarithms, but the problem set does require that students do some work with logarithms base e and base 2. Remind students to check for extraneous solutions when solving logarithmic equations.

Classwork

Discussion (4 minutes): How to Read a Table of Logarithms

- For this lesson, we will pretend that we live in the time when logarithms were discovered, before there were calculators or computing machines. In this time, scientists, merchants, and sailors needed to make calculations for both astronomical observation and navigation. Logarithms made these calculations much easier, faster, and more accurate than calculation by hand. In fact, noted mathematician Pierre-Simon LaPlace (France, circa 1800) said that "[logarithms are an] admirable artifice which, by reducing to a few days the labour of many months, doubles the life of the astronomer, and spares him the errors and disgust inseparable from long calculations."
- A typical table of common logarithms, like the table at the end of this document, has many rows of numbers arranged in ten columns. The numbers in the table are decimals. In our table, they are given to four decimal places, and there are 90 rows of them (some tables of logarithms have 900 rows). Down the left-hand side of the table are the numbers from 1.0 to 9.9. Across the top of the table are the numbers from 0 to 9. To read the table, you locate the number whose logarithm you want using the numbers down the left of the table followed by the numbers across the top.

MP.5 & MP.6

- What does the number in the third row and second column represent (the entry for 1.21)?
  - *The logarithm of* 1.21*, which is* 0.0828.
- The logarithm of numbers larger than 9.9 and smaller than 1.0 can also be found using this table. Suppose you want to find $\log(365)$. Is there any way we can rewrite this number to show a number between 1.0 and 9.9?
  - *Rewrite* 365 *in scientific notation:* 3.65×10^2.
- Can we simplify $\log(3.65 \times 10^2)$?
  - *We can apply the formula for the logarithm of a product. Then, we have* $\log(10^2) + \log(3.65) = 2 + \log(3.65)$.
- Now, all that is left is to find the value of $\log(3.65)$ using the table. What is the value of $\log(365)$?
  - *The table entry is* 0.5623. *That means* $\log(365) \approx 2 + 0.5623$, *so* $\log(365) \approx 2.5623$.
- How would you find $\log(0.365)$?
  - *In scientific notation,* $0.365 = 3.65 \times 10^{-1}$. *So, once again you would find the row for* 3.6 *and the column for* 5, *and you would again find the number* 0.5623. *But this time, you would have* $\log(0.365) \approx -1 + 0.5623$, *so* $\log(0.365) \approx 0.4377$.

Example 1 (7 minutes)

Students will multiply multi-digit numbers without technology, and then use a table of logarithms to find the same product using logarithms.

- Find the product 3.42×2.47 without using a calculator.
  - *Using paper and pencil, and without any rounding, students should get* 8.4474. *The point is to show how much time the multiplication of multi-digit numbers can take.*
- How could we use logarithms to find this product?
  - *If we take the logarithm of the product, we can rewrite the product as a sum of logarithms.*
- Rewrite the logarithm of the product as the sum of logarithms.
  - $\log(3.42 \times 2.47) = \log(3.42) + \log(2.47)$
- Use the table of logarithms to look up the values of $\log(3.42)$ and $\log(2.47)$.
  - *According to the table,* $\log(3.42) \approx 0.5340$, and $\log(2.47) \approx 0.3927$.
- Approximate the logarithm $\log(3.42 \times 2.47)$.
  - *The approximate sum is*

$$\begin{aligned}\log(3.42 \times 2.47) &\approx 0.5340 + 0.3927\\ &\approx 0.9267.\end{aligned}$$

- What if there is more than one number that has a logarithm of 0.9267? Suppose that there are two numbers X and Y that satisfy $\log(X) = 0.9267$ and $\log(Y) = 0.9267$. Then, $10^{0.9267} = X$, and $10^{0.9267} = Y$, so that $X = Y$. This means that there is only one number that has the logarithm 0.9267. So, what is that number?

> *Scaffolding:*
> - Students may need to be reminded that if the logarithm is greater than 1, a power of 10 greater than 1 is involved, and only the decimal part of the number will be found in the table.
> - Struggling students should attempt a simpler product such as 1.20×6.00 to illustrate the process.
> - Advanced students may use larger or more precise numbers as a challenge. To multiply a product such as 34.293×107.9821, students will have to employ scientific notation and the property for the logarithm of a product.

- Can we find the exact number that has logarithm 0.9267 using the table?
  - *The table says that* $\log(8.44) \approx 0.9263$, *and* $\log(8.45) \approx 0.9269$.
- Which is closer?
  - $\log(8.45) \approx 0.9267$
- Since $\log(3.42 \times 2.47) \approx \log(8.45)$, what can we conclude is an approximate value for 3.42×2.47?
  - *Since* $\log(3.42 \times 2.47) \approx \log(8.45)$, *we know that* $3.42 \times 2.47 \approx 8.45$.
- Does this agree with the product you found when you did the calculation by hand?
  - *Yes, by hand we found that the product is* 8.4474, *which is approximately* 8.45.

Discussion (2 minutes)

In the above example, we showed that there was only one number that had logarithm 0.9267. This result generalizes to any number and any base of the logarithm: If $\log_b(X) = \log_b(Y)$, then $X = Y$. We need to know this property both to use a logarithm table to look up values that produce a certain logarithmic value and to solve logarithmic equations later in the lesson.

> If X and Y are positive real numbers, or expressions that take on the value of positive real numbers, and $\log_b(X) = \log_b Y$, then $X = Y$.

Example 2 (4 minutes)

This example is a continuation of the first example, with the addition of scientific notation to further explain the power of logarithms. Because much of the reasoning was explained in Example 1, this should take much less time to work through.

- Now, what if we needed to calculate $(3.42 \times 10^{14}) \times (5.76 \times 10^{12})$?
- Take the logarithm of this product, and find its approximate value using the logarithm table.
  - $$\begin{aligned} \log\left((3.42 \times 10^{14}) \times (5.76 \times 10^{12})\right) &= \log(3.42) + \log(10^{14}) + \log(5.76) + \log(10^{12}) \\ &= \log(3.42) + \log(5.76) + 14 + 12 \\ &\approx 0.5340 + 0.7604 + 26 \\ &\approx 27.2944 \end{aligned}$$
- Look up 0.2944 in the logarithm table.
  - *Since* $\log(1.97) \approx 0.2945$, *we can say that* $0.2944 \approx \log(1.97)$.
- How does that tell us which number has a logarithm approximately equal to 27.2944?
  - $\log(1.97 \times 10^{27}) = 27 + \log(1.97)$, *so* $\log(1.97 \times 10^{27}) \approx 27.2944$.
- Finally, what is an approximate value of the product $(3.42 \times 10^{14}) \times (5.76 \times 10^{12})$?
  - $(3.42 \times 10^{14}) \times (5.76 \times 10^{12}) \approx 1.97 \times 10^{27}$.

Example 3 (6 minutes)

- According to one estimate, the mass of the earth is roughly 5.28×10^{24} kg, and the mass of the moon is about 7.35×10^{22} kg. Without using a calculator but using the table of logarithms, find how many times greater the mass of earth is than the mass of the moon.
  - *Let R be the ratio of the two masses. Then $R = \frac{5.28 \times 10^{24}}{7.35 \times 10^{22}} = \frac{5.28}{7.35} \cdot 10^2$*

    Taking the logarithm of each side,

$$\begin{aligned}\log(R) &= \log\left(\frac{5.28}{7.35} \cdot 10^2\right)\\ &= 2 + \log\left(\frac{5.28}{7.35}\right)\\ &= 2 + \log(5.28) - \log(7.35)\\ &\approx 2 + 0.7226 - 0.8663\\ &\approx 1.8563.\end{aligned}$$

- Find 0.8563 in the table entries to estimate R.
  - *In the table, 0.8563 is closest to $\log(7.18)$.*
  - *So, $\log(71.8) \approx 1.8563$, and therefore, the mass of earth is approximately 71.8 times that of the moon.*
- Logarithms turn out to be very useful in dealing with especially large or especially small numbers. Scientific notation was probably developed as an attempt to do arithmetic using logarithms. How does it help to have those numbers expressed in scientific notation if we are going to use a logarithm table to perform multiplication or division?
  - *Again, answers will differ, but students should at least recognize that scientific notation is helpful in working with very large or very small numbers. Using scientific notation, we can express each number as the product of a number between 1 and 10, and a power of 10. Taking the logarithm of the number allows us to use properties of logarithms base 10 to handle more easily any number n where $0 < n < 1$ or $n \geq 10$. The logarithm of the number between 1 and 10 can be read from the table, and the exponent of the power of 10 can then be added to it.*
- Whenever we have a number of the form $k \times 10^n$ where n is an integer and k is a number between 1.0 and 9.9, the logarithm of this number will always be $n + \log(k)$ and can be evaluated using a table of logarithms like the one included in this lesson.

Discussion (4 minutes)

Logarithms were devised by the Scottish mathematician John Napier (1550–1617) with the help of the English mathematician Henry Briggs (1561–1630) to simplify arithmetic computations with multi-digit numbers by turning multiplication and division into addition and subtraction. The basic idea is that while a sequence of powers like $2^0, 2^1, 2^2, 2^3, 2^4, 2^5, \ldots$ is increasing multiplicatively, the sequence of its exponents is increasing additively. If numbers can be represented as the powers of a base, they can be multiplied by adding their exponents and divided by subtracting their exponents. Napier and Briggs published the first tables of what came to be called base 10 or common logarithms.

- It was Briggs's idea to base the logarithms on the number 10. Why do you think he made that choice?
  - *The number 10 is the base of our number system. So, taking 10 as the base of common logarithms makes hand calculations with logarithms easier. It is really the same argument that makes scientific notation helpful: Powers of 10 are easy to use in calculations.*

Exercises 1–2 (12 minutes)

Now that we know that if two logarithmic expressions with the same base are equal, then the arguments inside of the logarithms are equal, and we can solve a wider variety of logarithmic equations without invoking the definition each time. Due to the many logarithmic properties that the students now know, there are multiple approaches to solving these equations. Discuss different approaches with the students and their responses to Exercise 2.

Exercises 1–2

1. Solve the following equations. Remember to check for extraneous solutions because logarithms are only defined for positive real numbers.

a. $\log(x^2) = \log(49)$

$$x^2 = 49$$
$$x = 7 \text{ or } x = -7$$

Check: Both solutions are valid since 7^2 and $(-7)^2$ are both positive numbers.

The two solutions are 7 and -7.

b. $\log(x+1) + \log(x+2) = \log(7x-17)$

$$\log\big((x+1)(x-2)\big) = \log(7x-17)$$
$$(x+1)(x-2) = 7x-17$$
$$x^2 - x - 2 = 7x - 17$$
$$x^2 - 8x + 15 = 0$$
$$(x-5)(x-3) = 0$$
$$x = 3 \text{ or } x = 5$$

Check: Since $x+1$, $x-2$, and $7x-17$ are all positive for either $x=3$ or $x=5$, both solutions are valid.

Thus, the solutions to this equation are 3 and 5.

c. $\log(x^2+1) = \log\big(x(x-2)\big)$

$$x^2 + 1 = x(x-2)$$
$$= x^2 - 2x$$
$$1 = -2x$$
$$x = -\frac{1}{2}$$

Check: Both $\left(-\frac{1}{2}\right)^2 + 1 > 0$ and $-\frac{1}{2}\left(-\frac{1}{2} - 2\right) > 0$, so the solution $-\frac{1}{2}$ is valid.

Thus, $-\frac{1}{2}$ is the only valid solution to this equation.

Scaffolding:

If the class seems to be struggling with the process to solve logarithmic equations, then either encourage them to create a graphic organizer that summarizes the types of problems and approaches that they should use in each case, or hang one on the board for reference. A sample graphic organizer is included.

Rewrite problem in the form…	
$\log_b(Y) = L$	$\log_b(Y) = \log_b(Z)$
Then…	
$b^L = Y$	$Y = Z$

d. $\log(x+4) + \log(x-1) = \log(3x)$

$$\log\big((x+4)(x-1)\big) = \log(3x)$$
$$(x+4)(x-1) = 3x$$
$$x^2 + 3x - 4 = 3x$$
$$x^2 - 4 = 0$$
$$x = 2 \text{ or } x = -2$$

Check: Since $\log(3x)$ is undefined when $x = -2$, there is an extraneous solution of $x = -2$.

The only valid solution to this equation is 2.

e. $\log(x^2 - x) - \log(x-2) = \log(x-3)$

$$\log(x-2) + \log(x-3) = \log(x^2 - x)$$
$$\log\big((x-2)(x-3)\big) = \log(x^2 - x)$$
$$(x-2)(x-3) = x^2 - x$$
$$x^2 - 5x + 6 = x^2 - x$$
$$4x = 6$$
$$x = \frac{3}{2}$$

Check: When $x = \frac{3}{2}$, we have $x - 2 < 0$, so $\log(x-2)$, $\log(x-3)$, and $\log(x^2 - x)$ are all undefined. So, the solution $x = \frac{3}{2}$ is extraneous.

There are no valid solutions to this equation.

f. $\log(x) + \log(x-1) + \log(x+1) = 3\log(x)$

$$\log\big(x(x-1)(x+1)\big) = \log(x^3)$$
$$\log(x^3 - x) = \log(x^3)$$
$$x^3 - x = x^3$$
$$x = 0$$

Since $\log(0)$ is undefined, $x = 0$ is an extraneous solution.

There are no valid solutions to this equation.

g. $\log(x-4) = -\log(x-2)$

Two possible approaches to solving this equation are shown.

$$\log(x-4) = \log\left(\frac{1}{x-2}\right)$$
$$x - 4 = \frac{1}{x-2}$$
$$(x-4)(x-2) = 1$$
$$x^2 - 6x + 8 = 1$$
$$x^2 - 6x + 7 = 0$$
$$x = 3 \pm \sqrt{2}$$

$$\log(x-4) + \log(x-2) = 0$$
$$\log\big((x-4)(x-2)\big) = \log(1)$$
$$(x-4)(x-2) = 1$$
$$x^2 - 6x + 8 = 1$$
$$x^2 - 6x + 7 = 0$$
$$x = 3 \pm \sqrt{2}$$

Check: If $x = 3 - \sqrt{2}$, then $x < 2$, so $\log(x-2)$ is undefined. Thus, $3 - \sqrt{2}$ is an extraneous solution.

The only valid solution to this equation is $3 + \sqrt{2}$.

2. **How do you know if you need to use the definition of logarithm to solve an equation involving logarithms as we did in Lesson 15 or if you can use the methods of this lesson?**

   If the equation involves only logarithmic expressions, then it can be reorganized to be of the form $\log(X) = \log(Y)$ and then solved by equating $X = Y$. If there are constants involved, then the equation can be solved by the definition.

Closing (2 minutes)

Ask students the following questions and after coming to a consensus, have students record the answers in their notebooks.

- How do we use a table of logarithms to compute a product of two numbers x and y?
  - *We look up approximations to $\log(x)$ and $\log(y)$ in the table, add those logarithms, and then look up the sum in the table to extract the approximate product.*
- Does this process provide an exact answer? Explain how you know.
  - *It is only an approximation because the table only allows us to look up x to two decimal places and $\log(x)$ to four decimal places.*
- How do we solve an equation in which every term contains a logarithm?
  - *We rearrange the terms to get an equation of the form $\log(X) = \log(Y)$, then equate $X = Y$, and solve from there.*
- How does that differ from solving an equation that contains constant terms?
  - *If an equation has constant terms, then we rearrange the equation to the form $\log(X) = c$, apply the definition of the logarithm, and solve from there.*

Lesson Summary

A table of base 10 logarithms can be used to simplify multiplication of multi-digit numbers:

1. **To compute $A \times B$ for positive real numbers A and B, look up the values $\log(A)$ and $\log(B)$ in the logarithm table.**
2. **Add $\log(A)$ and $\log(B)$. The sum can be written as $k + d$, where k is an integer and $0 \le d < 1$ is the decimal part.**
3. **Look back at the table and find the entry closest to the decimal part, d.**
4. **The product of that entry and 10^k is an approximation to $A \times B$.**

A similar process simplifies division of multi-digit numbers:

1. **To compute $A \div B$ for positive real numbers A and B, look up the values $\log(A)$ and $\log(B)$ in the logarithm table.**
2. **Calculate $\log(A) - \log(B)$. The difference can be written as $k + d$, where k is an integer and $0 \le d < 1$ is the decimal part.**
3. **Look back at the table to find the entry closest to the decimal part, d.**
4. **The product of that entry and 10^k is an approximation to $A \div B$.**

For any positive values X and Y, if $\log_b(X) = \log_b(Y)$, we can conclude that $X = Y$. This property is the essence of how a logarithm table works, and it allows us to solve equations with logarithmic expressions on both sides of the equation.

Exit Ticket (4 minutes)

Name ____ Date ____________________

Lesson 15: Why Were Logarithms Developed?

Exit Ticket

The surface area of Jupiter is $6.14 \times 10^{10}\ \mathrm{km}^2$, and the surface area of Earth is $5.10 \times 10^{8}\ \mathrm{km}^2$. Without using a calculator but using the table of logarithms, find how many times greater the surface area of Jupiter is than the surface area of Earth.

Exit Ticket Sample Solutions

The surface area of Jupiter is 6.14×10^{10} km^2, and the surface area of Earth is 5.10×10^{8} km^2. Without using a calculator but using the table of logarithms, find how many times greater the surface area of Jupiter is than the surface area of Earth.

Let R be the ratio of the two surface areas. Then, $R = \frac{6.14 \times 10^{10}}{5.10 \times 10^{8}} = \frac{6.14}{5.10} \cdot 10^2$.

Taking the logarithm of each side,

$$\begin{aligned}\log(R) &= \log\left(\frac{6.14}{5.10} \cdot 10^2\right)\\ &= 2 + \log\left(\frac{6.14}{5.10}\right)\\ &= 2 + \log(6.14) - \log(5.10)\\ &\approx 2 + 0.7882 - 0.7076\\ &\approx 2.0806.\end{aligned}$$

Find 0.0806 in the table entries to estimate R.

Look up 0.0806, which is closest to $\log(1.20)$. Note that $2 + 0.0806 \approx \log(100) + \log(1.20)$, so $\log(120) \approx 2.0806$. Therefore, the surface area of Jupiter is approximately 120 times that of Earth.

Problem Set Sample Solutions

These problems give students additional practice using base 10 logarithms to perform arithmetic calculations and solve equations.

1. **Use the table of logarithms to approximate solutions to the following logarithmic equations.**

   a. $\log(x) = 0.5044$

   In the table, 0.5044 is closest to $\log(3.19)$, so $\log(x) \approx \log(3.19)$.
   Therefore, $x \approx 3.19$.

   b. $\log(x) = -0.5044$ **(Hint: Begin by writing -0.5044 as $[(-0.5044) + 1] - 1$.)**

   $$\begin{aligned}\log(x) &= [(-0.5044) + 1] - 1\\ &= 0.4956 - 1\end{aligned}$$

   In the table, 0.4956 is closest to $\log(3.13)$, so

   $$\begin{aligned}\log(x) &\approx \log(3.13) - 1\\ &\approx \log(3.13) - \log(10)\\ &\approx \log\left(\frac{3.13}{10}\right)\\ &\approx \log(0.313).\end{aligned}$$

   Therefore, $x \approx 0.313$.

   Alternatively, -0.5044 is the opposite of 0.5044, so x is the reciprocal of the answer in part (a). Thus, $x \approx 3.19^{-1} \approx 0.313$.

c. $\log(x) = 35.5044$

$$\begin{aligned}\log(x) &= 35 + 0.5044\\ &= \log(10^{35}) + 0.5044\\ &\approx \log(10^{35}) + \log(3.19)\\ &\approx \log(3.19 \times 10^{35})\end{aligned}$$

Therefore, $x \approx 3.19 \times 10^{35}$.

d. $\log(x) = 4.9201$

$$\begin{aligned}\log(x) &= 4 + 0.9201\\ &= \log(10^{4}) + 0.9201\\ &\approx \log(10^{4}) + \log(8.32)\\ &\approx \log(8.32 \times 10^{4})\end{aligned}$$

Therefore, $x \approx 83,200$.

2. Use logarithms and the logarithm table to evaluate each expression.

a. $\sqrt{2.33}$

$$\log\left((2.33)^{\frac{1}{2}}\right) = \frac{1}{2}\log(2.33) = \frac{1}{2}(0.3674) = 0.1837$$

Thus, $\log(\sqrt{2.33}) = 0.1837$*, and locating* 0.1837 *in the logarithm table gives a value approximately* 1.53. *Therefore,* $\sqrt{2.33} \approx 1.53$.

b. $13,500 \cdot 3,600$

$$\begin{aligned}\log(1.35 \cdot 10^4 \cdot 3.6 \cdot 10^3) &= \log(1.35) + \log(3.6) + 4 + 3\\ &\approx 0.1303 + 0.5563 + 7\\ &\approx 7.6866\end{aligned}$$

Thus, $\log(13,500 \cdot 3,600) \approx 7.6866$. *Locating* 0.6866 *in the logarithm table gives a value of* 4.86. *Therefore, the product is approximately* 4.86×10^7.

c. $\dfrac{7.2\times 10^9}{1.3\times 10^5}$

$$\log(7.2) + \log(10^9) - \log(1.3) - \log(10^5) \approx 0.8573 + 9 - 0.1139 - 5 \approx 4.7434$$

Locating 0.7434 *in the logarithm table gives* 5.54. *So, the quotient is approximately* 5.54×10^4.

3. Solve for x: $\log(3) + 2\log(x) = \log(27)$.

$$\begin{aligned}\log(3) + 2\log(x) &= \log(27)\\ \log(3) + \log(x^2) &= \log(27)\\ \log(3x^2) &= \log(27)\\ 3x^2 &= 27\\ x^2 &= 9\\ x &= 3\end{aligned}$$

4. Solve for x: $\log(3x) + \log(x+4) = \log(15)$.

$$\begin{aligned} \log(3x^2 + 12x) &= \log(15) \\ 3x^2 + 12x &= 15 \\ 3x^2 + 12x - 15 &= 0 \\ 3x^2 + 15x - 3x - 15 &= 0 \\ 3x(x+5) - 3(x+5) &= 0 \\ (3x-3)(x+5) &= 0 \end{aligned}$$

Thus, 1 and -5 solve the quadratic equation, but -5 is an extraneous solution. Hence, 1 is the only solution.

5. Solve for x.

a. $\log(x) = \log(y+z) + \log(y-z)$

$$\begin{aligned} \log(x) &= \log(y+z) + \log(y-z) \\ \log(x) &= \log\big((y+z)(y-z)\big) \\ \log(x) &= \log(y^2 - z^2) \\ x &= y^2 - z^2 \end{aligned}$$

b. $\log(x) = \big(\log(y) + \log(z)\big) + \big(\log(y) - \log(z)\big)$

$$\begin{aligned} \log(x) &= \big(\log(y) + \log(z)\big) + \big(\log(y) - \log(z)\big) \\ \log(x) &= 2\log(y) \\ \log(x) &= \log(y^2) \\ x &= y^2 \end{aligned}$$

6. If x and y are positive real numbers, and $\log(y) = 1 + \log(x)$, express y in terms of x.

Since $\log(10x) = 1 + \log(x)$, we see that $\log(y) = \log(10x)$. Then $y = 10x$.

7. If x, y, and z are positive real numbers, and $\log(x) - \log(y) = \log(y) - \log(z)$, express y in terms of x and z.

$$\begin{aligned} \log(x) - \log(y) &= \log(y) - \log(z) \\ \log(x) + \log(z) &= 2\log(y) \\ \log(xz) &= \log(y^2) \\ xz &= y^2 \\ y &= \sqrt{xz} \end{aligned}$$

8. If x and y are positive real numbers, and $\log(x) = y\big(\log(y+1) - \log(y)\big)$, express x in terms of y.

$$\begin{aligned} \log(x) &= y\big(\log(y+1) - \log(y)\big) \\ \log(x) &= y\left(\log\left(\frac{y+1}{y}\right)\right) \\ \log(x) &= \log\left(\left(\frac{y+1}{y}\right)^y\right) \\ x &= \left(\frac{y+1}{y}\right)^y \end{aligned}$$

9. If x and y are positive real numbers, and $\log(y) = 3 + 2\log(x)$, express y in terms of x.

Since $\log(1000x^2) = 3 + \log(x^2) = 3 + 2\log(x)$, we see that $\log(y) = \log(1000x^2)$. Thus, $y = 1000x^2$.

10. If x, y, and z are positive real numbers, and $\log(z) = \log(y) + 2\log(x) - 1$, express z in terms of x and y.

Since $\log\left(\frac{x^2y}{10}\right) = \log(y) + 2\log(x) - 1$, we see that $\log(z) = \log\left(\frac{x^2y}{10}\right)$. Thus, $z = \frac{x^2y}{10}$.

11. Solve the following equations.

a. $\ln(10) - \ln(7 - x) = \ln(x)$

$$\begin{aligned}
\ln\left(\frac{10}{7-x}\right) &= \ln(x)\\
\frac{10}{7-x} &= x\\
10 &= x(7-x)\\
x^2 - 7x + 10 &= 0\\
(x-5)(x-2) &= 0\\
x = 2 &\text{ or } x = 5
\end{aligned}$$

Check: If $x = 2$ or $x = 5$, then the expressions x and $7 - x$ are positive.

Thus, both 2 and 5 are valid solutions to this equation.

b. $\ln(x + 2) + \ln(x - 2) = \ln(9x - 24)$

$$\begin{aligned}
\ln\big((x+2)(x-2)\big) &= \ln(9x - 24)\\
x^2 - 4 &= 9x - 24\\
x^2 - 9x + 20 &= 0\\
(x-4)(x-5) &= 0\\
x = 4 &\text{ or } x = 5
\end{aligned}$$

Check: If $x = 4$ or $x = 5$, then the expressions $x + 2$, $x - 2$, and $9x - 24$ are all positive.

Thus, both 4 and 5 are valid solutions to this equation.

c. $\ln(x + 2) + \ln(x - 2) = \ln(-2x - 1)$

$$\begin{aligned}
\ln\big((x+2)(x-2)\big) &= \ln(-2x - 1)\\
\ln(x^2 - 4) &= \ln(-2x - 1)\\
x^2 - 4 &= -2x - 1\\
x^2 + 2x - 3 &= 0\\
(x+3)(x-1) &= 0\\
x = -3 &\text{ or } x = 1
\end{aligned}$$

So, $x = -3$ or $x = 1$, but $x = -3$ makes the input to both logarithms on the left-hand side negative, and $x = 1$ makes the input to the second and third logarithms negative. Thus, there are no solutions to the original equation.

12. Suppose the formula $P = P_0(1+r)^t$ gives the population of a city P growing at an annual percent rate r, where P_0 is the population t years ago.

a. Find the time t it takes this population to double.

Let $P = 2P_0$; then,

$$\begin{aligned} 2P_0 &= P_0(1+r)^t \\ 2 &= (1+r)^t \\ \log(2) &= \log(1+r)^t \\ \log(2) &= t\log(1+r) \\ t &= \frac{\log(2)}{\log(1+r)}. \end{aligned}$$

b. Use the structure of the expression to explain why populations with lower growth rates take a longer time to double.

If r is a decimal between 0 and 1, then the denominator will be a number between 0 and $\log(2)$. Thus, the value of t will be large for small values of r and getting closer to 1 as r increases.

c. Use the structure of the expression to explain why the only way to double the population in one year is if there is a 100 percent growth rate.

For the population to double, we need to have $t = 1$. This happens if $\log(2) = \log(1+r)$, and then we have $2 = 1 + r$ and $r = 1$.

13. If $x > 0$, $a + b > 0$, $a > b$, and $\log(x) = \log(a+b) + \log(a-b)$, find x in terms of a and b.

Applying properties of logarithms, we have

$$\begin{aligned} \log(x) &= \log(a+b) + \log(a-b) \\ &= \log\big((a+b)(a-b)\big) \\ &= \log(a^2 - b^2). \end{aligned}$$

So, $x = a^2 - b^2$.

MP.3

14. Jenn claims that because $\log(1) + \log(2) + \log(3) = \log(6)$, then $\log(2) + \log(3) + \log(4) = \log(9)$.

a. Is she correct? Explain how you know.

Jenn is not correct. Even though $\log(1) + \log(2) + \log(3) = \log(1 \cdot 2 \cdot 3) = \log(6)$, the logarithm properties give $\log(2) + \log(3) + \log(4) = \log(2 \cdot 3 \cdot 4) = \log(24)$. Since $9 < 10$, we know that $\log(9) < 1$, and since $24 > 10$, we know that $\log(24) > 1$, so clearly $\log(9) \neq \log(24)$.

MP.3

b. If $\log(a) + \log(b) + \log(c) = \log(a + b + c)$, express c in terms of a and b. Explain how this result relates to your answer to part (a).

$$\begin{aligned} \log(a) + \log(b) + \log(c) &= \log(a + b + c) \\ \log(abc) &= \log(a + b + c) \\ abc &= a + b + c \\ abc - c &= a + b \\ c(ab - 1) &= a + b \\ c &= \frac{a + b}{ab - 1} \end{aligned}$$

If $\log(2) + \log(3) + \log(4)$ were equal to $\log(9)$, then we would have $4 = \frac{2+3}{2\cdot 3 - 1}$. However, $\frac{2+3}{2\cdot 3-1} = \frac{5}{5} = 1 \neq 4$, so we know that $\log(2) + \log(3) + \log(4) \neq \log(9)$.

c. Find other values of a, b, and c so that $\log(a) + \log(b) + \log(c) = \log(a + b + c)$.

Many answers are possible; in fact, any positive values of a and b where $ab \neq 1$ will produce c so that $\log(a) + \log(b) + \log(c) = \log(a + b + c)$. One such answer is $a = 3$, $b = 7$, and $c = \frac{1}{2}$.

15. In Problem 7 of the Lesson 12 Problem Set, you showed that for $x \geq 1$, $\log\left(x + \sqrt{x^2 - 1}\right) + \log\left(x - \sqrt{x^2 - 1}\right) = 0$. It follows that $\log\left(x + \sqrt{x^2 - 1}\right) = -\log\left(x - \sqrt{x^2 - 1}\right)$. What does this tell us about the relationship between the expressions $x + \sqrt{x^2 - 1}$ and $x - \sqrt{x^2 - 1}$?

Since we know $\log\left(x + \sqrt{x^2 - 1}\right) = -\log\left(x - \sqrt{x^2 - 1}\right)$, and $-\log\left(x - \sqrt{x^2 - 1}\right) = \log\left(\frac{1}{x - \sqrt{x^2 - 1}}\right)$, we know that $\log\left(x + \sqrt{x^2 - 1}\right) = \log\left(\frac{1}{x - \sqrt{x^2 - 1}}\right)$. Then, $x + \sqrt{x^2 - 1} = \frac{1}{x - \sqrt{x^2 - 1}}$. We can verify that these expressions are reciprocals by multiplying them together:

$$\begin{aligned} \left(x + \sqrt{x^2 - 1}\right)\left(x - \sqrt{x^2 - 1}\right) &= x^2 + x\sqrt{x^2 - 1} - x\sqrt{x^2 - 1} - \left(\sqrt{x^2 - 1}\right)^2 \\ &= x^2 - (x^2 - 1) \\ &= 1. \end{aligned}$$

16. Use the change of base formula to solve the following equations.

a. $\log(x) = \log_{100}(x^2 - 2x + 6)$

$$\begin{aligned} \log(x) &= \frac{\log(x^2 - 2x + 6)}{\log(100)} \\ \log(x) &= \frac{1}{2}\log(x^2 - 2x + 6) \\ 2\log(x) &= \log(x^2 - 2x + 6) \\ \log(x^2) &= \log(x^2 - 2x + 6) \\ x^2 &= x^2 - 2x + 6 \\ 2x &= 6 \\ x &= 3 \end{aligned}$$

Since both sides of the equation are defined for $x = 3$, the only solution to this equation is 3.

b. $\log(x-2) = \log_{100}(14-x)$

$$\log(x-2) = \frac{\log(14-x)}{\log(100)}$$
$$\log(x-2) = \frac{1}{2}\log(14-x)$$
$$2\log(x-2) = \log(14-x)$$
$$\log((x-2)^2) = \log(14-x)$$
$$(x-2)^2 = 14-x$$
$$x^2-4x+4 = 14-x$$
$$x^2-3x-10 = 0$$
$$(x-5)(x+2) = 0$$

Thus, either $x=5$ or $x=-2$. Since the left side of the equation is undefined when $x=-2$, but both sides are defined for $x=5$, the only solution to the equation is 5.

c. $\log_2(x+1) = \log_4(x^2+3x+4)$

$$\log_2(x+1) = \log_4(x^2+3x+4)$$
$$\log_2(x+1) = \frac{\log_2(x^2+3x+4)}{\log_2(4)}$$
$$2\log_2(x+1) = \log_2(x^2+3x+4)$$
$$\log_2((x+1)^2) = \log_2(x^2+3x+4)$$
$$(x+1)^2 = x^2+3x+4$$
$$x^2+2x+1 = x^2+3x+4$$
$$2x+1 = 3x+4$$
$$x = -3$$

Since the left side of the equation is undefined for $x=-3$, there is no solution to this equation.

d. $\log_2(x-1) = \log_8(x^3-2x^2-2x+5)$

$$\log_2(x-1) = \frac{\log_2(x^3-2x^2-2x+5)}{\log_2(8)}$$
$$3\log_2(x-1) = \log_2(x^3-2x^2-2x+5)$$
$$\log_2((x-1)^3) = \log_2(x^3-2x^2-2x+5)$$
$$(x-1)^3 = x^3-2x^2-2x+5$$
$$x^3-3x^2+3x-1 = x^3-2x^2-2x+5$$
$$x^2-5x+6 = 0$$
$$(x-3)(x-2) = 0$$

Since both sides of the equation are defined for $x=3$ and $x=2$, 2 and 3 are both valid solutions to this equation.

17. Solve the following equation: $\log(9x) = \frac{2\ln(3)+\ln(x)}{\ln(10)}$.

Rewrite the left-hand side using the change-of-base formula:

$$\begin{aligned}\log(9x) &= \frac{\ln(9x)}{\ln(10)}\\ &= \frac{\ln(9)+\ln(x)}{\ln(10)}\\ &= \frac{\ln(3^2)+\ln(x)}{\ln(10)}.\end{aligned}$$

Thus, the equation is true for all $x > 0$.

Common Logarithm Table

N	0	1	2	3	4	5	6	7	8	9
1.0	0.0000	0.0043	0.0086	0.0128	0.0170	0.0212	0.0253	0.0294	0.0334	0.0374
1.1	0.0414	0.0453	0.0492	0.0531	0.0569	0.0607	0.0645	0.0682	0.0719	0.0755
1.2	0.0792	0.0828	0.0864	0.0899	0.0934	0.0969	0.1004	0.1038	0.1072	0.1106
1.3	0.1139	0.1173	0.1206	0.1239	0.1271	0.1303	0.1335	0.1367	0.1399	0.1430
1.4	0.1461	0.1492	0.1523	0.1553	0.1584	0.1614	0.1644	0.1673	0.1703	0.1732
1.5	0.1761	0.1790	0.1818	0.1847	0.1875	0.1903	0.1931	0.1959	0.1987	0.2014
1.6	0.2041	0.2068	0.2095	0.2122	0.2148	0.2175	0.2201	0.2227	0.2253	0.2279
1.7	0.2304	0.2330	0.2355	0.2380	0.2405	0.2430	0.2455	0.2480	0.2504	0.2529
1.8	0.2553	0.2577	0.2601	0.2625	0.2648	0.2672	0.2695	0.2718	0.2742	0.2765
1.9	0.2788	0.2810	0.2833	0.2856	0.2878	0.2900	0.2923	0.2945	0.2967	0.2989
2.0	0.3010	0.3032	0.3054	0.3075	0.3096	0.3118	0.3139	0.3160	0.3181	0.3201
2.1	0.3222	0.3243	0.3263	0.3284	0.3304	0.3324	0.3345	0.3365	0.3385	0.3404
2.2	0.3424	0.3444	0.3464	0.3483	0.3502	0.3522	0.3541	0.3560	0.3579	0.3598
2.3	0.3617	0.3636	0.3655	0.3674	0.3692	0.3711	0.3729	0.3747	0.3766	0.3784
2.4	0.3802	0.3820	0.3838	0.3856	0.3874	0.3892	0.3909	0.3927	0.3945	0.3962
2.5	0.3979	0.3997	0.4014	0.4031	0.4048	0.4065	0.4082	0.4099	0.4116	0.4133
2.6	0.4150	0.4166	0.4183	0.4200	0.4216	0.4232	0.4249	0.4265	0.4281	0.4298
2.7	0.4314	0.4330	0.4346	0.4362	0.4378	0.4393	0.4409	0.4425	0.4440	0.4456
2.8	0.4472	0.4487	0.4502	0.4518	0.4533	0.4548	0.4564	0.4579	0.4594	0.4609
2.9	0.4624	0.4639	0.4654	0.4669	0.4683	0.4698	0.4713	0.4728	0.4742	0.4757
3.0	0.4771	0.4786	0.4800	0.4814	0.4829	0.4843	0.4857	0.4871	0.4886	0.4900
3.1	0.4914	0.4928	0.4942	0.4955	0.4969	0.4983	0.4997	0.5011	0.5024	0.5038
3.2	0.5051	0.5065	0.5079	0.5092	0.5105	0.5119	0.5132	0.5145	0.5159	0.5172
3.3	0.5185	0.5198	0.5211	0.5224	0.5237	0.5250	0.5263	0.5276	0.5289	0.5302
3.4	0.5315	0.5328	0.5340	0.5353	0.5366	0.5378	0.5391	0.5403	0.5416	0.5428
3.5	0.5441	0.5453	0.5465	0.5478	0.5490	0.5502	0.5514	0.5527	0.5539	0.5551
3.6	0.5563	0.5575	0.5587	0.5599	0.5611	0.5623	0.5635	0.5647	0.5658	0.5670
3.7	0.5682	0.5694	0.5705	0.5717	0.5729	0.5740	0.5752	0.5763	0.5775	0.5786
3.8	0.5798	0.5809	0.5821	0.5832	0.5843	0.5855	0.5866	0.5877	0.5888	0.5899
3.9	0.5911	0.5922	0.5933	0.5944	0.5955	0.5966	0.5977	0.5988	0.5999	0.6010
4.0	0.6021	0.6031	0.6042	0.6053	0.6064	0.6075	0.6085	0.6096	0.6107	0.6117
4.1	0.6128	0.6138	0.6149	0.6160	0.6170	0.6180	0.6191	0.6201	0.6212	0.6222
4.2	0.6232	0.6243	0.6253	0.6263	0.6274	0.6284	0.6294	0.6304	0.6314	0.6325
4.3	0.6335	0.6345	0.6355	0.6365	0.6375	0.6385	0.6395	0.6405	0.6415	0.6425
4.4	0.6435	0.6444	0.6454	0.6464	0.6474	0.6484	0.6493	0.6503	0.6513	0.6522
4.5	0.6532	0.6542	0.6551	0.6561	0.6571	0.6580	0.6590	0.6599	0.6609	0.6618
4.6	0.6628	0.6637	0.6646	0.6656	0.6665	0.6675	0.6684	0.6693	0.6702	0.6712
4.7	0.6721	0.6730	0.6739	0.6749	0.6758	0.6767	0.6776	0.6785	0.6794	0.6803
4.8	0.6812	0.6821	0.6830	0.6839	0.6848	0.6857	0.6866	0.6875	0.6884	0.6893
4.9	0.6902	0.6911	0.6920	0.6928	0.6937	0.6946	0.6955	0.6964	0.6972	0.6981
5.0	0.6990	0.6998	0.7007	0.7016	0.7024	0.7033	0.7042	0.7050	0.7059	0.7067
5.1	0.7076	0.7084	0.7093	0.7101	0.7110	0.7118	0.7126	0.7135	0.7143	0.7152
5.2	0.7160	0.7168	0.7177	0.7185	0.7193	0.7202	0.7210	0.7218	0.7226	0.7235
5.3	0.7243	0.7251	0.7259	0.7267	0.7275	0.7284	0.7292	0.7300	0.7308	0.7316
5.4	0.7324	0.7332	0.7340	0.7348	0.7356	0.7364	0.7372	0.7380	0.7388	0.7396

N	0	1	2	3	4	5	6	7	8	9
5.5	0.7404	0.7412	0.7419	0.7427	0.7435	0.7443	0.7451	0.7459	0.7466	0.7474
5.6	0.7482	0.7490	0.7497	0.7505	0.7513	0.7520	0.7528	0.7536	0.7543	0.7551
5.7	0.7559	0.7566	0.7574	0.7582	0.7589	0.7597	0.7604	0.7612	0.7619	0.7627
5.8	0.7634	0.7642	0.7649	0.7657	0.7664	0.7672	0.7679	0.7686	0.7694	0.7701
5.9	0.7709	0.7716	0.7723	0.7731	0.7738	0.7745	0.7752	0.7760	0.7767	0.7774
6.0	0.7782	0.7789	0.7796	0.7803	0.7810	0.7818	0.7825	0.7832	0.7839	0.7846
6.1	0.7853	0.7860	0.7868	0.7875	0.7882	0.7889	0.7896	0.7903	0.7910	0.7917
6.2	0.7924	0.7931	0.7938	0.7945	0.7952	0.7959	0.7966	0.7973	0.7980	0.7987
6.3	0.7993	0.8000	0.8007	0.8014	0.8021	0.8028	0.8035	0.8041	0.8048	0.8055
6.4	0.8062	0.8069	0.8075	0.8082	0.8089	0.8096	0.8102	0.8109	0.8116	0.8122
6.5	0.8129	0.8136	0.8142	0.8149	0.8156	0.8162	0.8169	0.8176	0.8182	0.8189
6.6	0.8195	0.8202	0.8209	0.8215	0.8222	0.8228	0.8235	0.8241	0.8248	0.8254
6.7	0.8261	0.8267	0.8274	0.8280	0.8287	0.8293	0.8299	0.8306	0.8312	0.8319
6.8	0.8325	0.8331	0.8338	0.8344	0.8351	0.8357	0.8363	0.8370	0.8376	0.8382
6.9	0.8388	0.8395	0.8401	0.8407	0.8414	0.8420	0.8426	0.8432	0.8439	0.8445
7.0	0.8451	0.8457	0.8463	0.8470	0.8476	0.8482	0.8488	0.8494	0.8500	0.8506
7.1	0.8513	0.8519	0.8525	0.8531	0.8537	0.8543	0.8549	0.8555	0.8561	0.8567
7.2	0.8573	0.8579	0.8585	0.8591	0.8597	0.8603	0.8609	0.8615	0.8621	0.8627
7.3	0.8633	0.8639	0.8645	0.8651	0.8657	0.8663	0.8669	0.8675	0.8681	0.8686
7.4	0.8692	0.8698	0.8704	0.8710	0.8716	0.8722	0.8727	0.8733	0.8739	0.8745
7.5	0.8751	0.8756	0.8762	0.8768	0.8774	0.8779	0.8785	0.8791	0.8797	0.8802
7.6	0.8808	0.8814	0.8820	0.8825	0.8831	0.8837	0.8842	0.8848	0.8854	0.8859
7.7	0.8865	0.8871	0.8876	0.8882	0.8887	0.8893	0.8899	0.8904	0.8910	0.8915
7.8	0.8921	0.8927	0.8932	0.8938	0.8943	0.8949	0.8954	0.8960	0.8965	0.8971
7.9	0.8976	0.8982	0.8987	0.8993	0.8998	0.9004	0.9009	0.9015	0.9020	0.9025
8.0	0.9031	0.9036	0.9042	0.9047	0.9053	0.9058	0.9063	0.9069	0.9074	0.9079
8.1	0.9085	0.9090	0.9096	0.9101	0.9106	0.9112	0.9117	0.9122	0.9128	0.9133
8.2	0.9138	0.9143	0.9149	0.9154	0.9159	0.9165	0.9170	0.9175	0.9180	0.9186
8.3	0.9191	0.9196	0.9201	0.9206	0.9212	0.9217	0.9222	0.9227	0.9232	0.9238
8.4	0.9243	0.9248	0.9253	0.9258	0.9263	0.9269	0.9274	0.9279	0.9284	0.9289
8.5	0.9294	0.9299	0.9304	0.9309	0.9315	0.9320	0.9325	0.9330	0.9335	0.9340
8.6	0.9345	0.9350	0.9355	0.9360	0.9365	0.9370	0.9375	0.9380	0.9385	0.9390
8.7	0.9395	0.9400	0.9405	0.9410	0.9415	0.9420	0.9425	0.9430	0.9435	0.9440
8.8	0.9445	0.9450	0.9455	0.9460	0.9465	0.9469	0.9474	0.9479	0.9484	0.9489
8.9	0.9494	0.9499	0.9504	0.9509	0.9513	0.9518	0.9523	0.9528	0.9533	0.9538
9.0	0.9542	0.9547	0.9552	0.9557	0.9562	0.9566	0.9571	0.9576	0.9581	0.9586
9.1	0.9590	0.9595	0.9600	0.9605	0.9609	0.9614	0.9619	0.9624	0.9628	0.9633
9.2	0.9638	0.9643	0.9647	0.9652	0.9657	0.9661	0.9666	0.9671	0.9675	0.9680
9.3	0.9685	0.9689	0.9694	0.9699	0.9703	0.9708	0.9713	0.9717	0.9722	0.9727
9.4	0.9731	0.9736	0.9741	0.9745	0.9750	0.9754	0.9759	0.9763	0.9768	0.9773
9.5	0.9777	0.9782	0.9786	0.9791	0.9795	0.9800	0.9805	0.9809	0.9814	0.9818
9.6	0.9823	0.9827	0.9832	0.9836	0.9841	0.9845	0.9850	0.9854	0.9859	0.9863
9.7	0.9868	0.9872	0.9877	0.9881	0.9886	0.9890	0.9894	0.9899	0.9903	0.9908
9.8	0.9912	0.9917	0.9921	0.9926	0.9930	0.9934	0.9939	0.9943	0.9948	0.9952
9.9	0.9956	0.9961	0.9965	0.9969	0.9974	0.9978	0.9983	0.9987	0.9991	0.9996