Lesson 13: Changing the Base

Student Outcomes

- Students understand how to change logarithms from one base to another.
- Students calculate logarithms with any base using a calculator that computes only logarithms base 10 and base e.
- Students justify properties of logarithms with any base.

Lesson Notes

The lesson begins by showing how to change logarithms from one base to another and develops properties of logarithms for the general base b. We introduce the use of a calculator instead of a table to approximate logarithms, and then we define $\ln(x) = \log_e(x)$. One goal of the lesson is to explain why the calculator only has a LOG and an LN key. Students solve exponential equations by applying the appropriate logarithm (**F-LE.A.4**).

Materials

Students will need access either to graphing calculators or computer software facilities such as the Wolfram|Alpha engine for finding logarithms with base 10 and base e.

Classwork

Example 1 (5 minutes)

The purpose of this example is to show how to find $\log_2(x)$ using $\log(x)$.

- We have been working primarily with base 10 logarithms, but in Lesson 7 we defined logarithms for any base b. For example, the number 2 might be the base. When logarithms have bases other than 10, it often helps to be able to rewrite the logarithm in terms of base-10 logarithms. Let $L = \log_2(x)$, and show that $L = \frac{\log(x)}{\log(2)}$.
  - *Let* $L = \log_2(x)$.
    Then $2^L = x$.
    Taking the logarithm of each side, we get
    $$\log(2^L) = \log(x)$$
    $$L \cdot \log(2) = \log(x)$$
    $$L = \frac{\log(x)}{\log(2)}.$$
    Therefore, $\log_2(x) = \frac{\log(x)}{\log(2)}$.

> *Scaffolding:*
> - Students who struggle with the first step of this example might need to be reminded of the definition of *logarithm* from Lesson 7: $L = \log_b(x)$ means $b^L = x$. Therefore, $L = \log_2(x)$ means $2^L = x$.
> - Advanced learners may want to immediately start with the second part of the example, converting $\log_b(x)$ into $\frac{\log(x)}{\log(b)}$. Alternatively, students may explore the scale change by $\log(b)$ by finding different values of $\log(b)$ and discussing the effects on the graph of $y = \log(x)$.

Remember that $\log(2)$ is a number, so this shows that $\log_2(x)$ is a rescaling of $\log(x)$.

- The example shows how we can convert $\log_2(x)$ to an expression involving $\log(x)$. More generally, suppose we are given a logarithm with base b. What is $\log_b(x)$ in terms of $\log(x)$?
  - *Let* $L = \log_b(x)$.
    Then $b^L = x$.
    Taking the logarithm of each side, we get
    $$\begin{aligned} \log(b^L) &= \log(x) \\ L \cdot \log(b) &= \log(x) \\ L &= \frac{\log(x)}{\log(b)}. \end{aligned}$$
    Therefore, $\log_b(x) = \frac{\log(x)}{\log(b)}$.
- This equation not only allows us to change from $\log_b(x)$ to $\log(x)$, but to change the base in the other direction as well: $\log(x) = \log_b(x) \cdot \log(b)$.

Exercise 1 (3 minutes)

The first exercise deals with the general formula for changing the base of a logarithm. It follows the same pattern as Example 1. Take time for students to share their results from Exercise 1 in a class discussion before moving on to Exercise 2 so that all students understand how the base of a logarithm is changed. Ask students to work in pairs on this exercise.

> *Scaffolding:*
> If students have difficulty with Exercise 1, they should review the argument in Example 1, noting that it deals with base 10, whereas this exercise generalizes that base to a.

Exercises

1. **Assume that x, a, and b are all positive real numbers, so that $a \neq 1$ and $b \neq 1$. What is $\log_b(x)$ in terms of $\log_a(x)$? The resulting equation allows us to change the base of a logarithm from a to b.**

   Let* $L = \log_b(x)$. *Then* $b^L = x$. *Taking the logarithm base* a *of each side, we get
   $$\begin{aligned} \log_a(b^L) &= \log_a(x) \\ L \cdot \log_a(b) &= \log_a(x) \\ L &= \frac{\log_a(x)}{\log_a(b)}. \end{aligned}$$
   ***Therefore,* $\log_b(x) = \frac{\log_a(x)}{\log_a(b)}$.**

Discussion (2 minutes)

Ask a student to present the solution to Exercise 1 to the class to ensure that all students understand how to change the base of a logarithm and how the formula comes from the definition of the logarithm as an exponential equation. Be sure that students record the formula in their notebooks.

> Change of Base Formula for Logarithms
>
> If x, a, and b are all positive real numbers with $a \neq 1$ and $b \neq 1$, then
> $$\log_b(x) = \frac{\log_a(x)}{\log_a(b)}.$$

Exercise 2 (2 minutes)

In the second exercise, students practice changing bases. They will need a calculator with the ability to calculate logarithms base 10. Later in the lesson, students will need to calculate natural logarithms as well. Students should work in pairs on this exercise, with one student using the calculator and the other keeping track of the computation. Students should share their results for Exercise 2 in a class discussion before moving on to Exercise 3.

2. Approximate each of the following logarithms to four decimal places. Use the $\boxed{\text{LOG}}$ key on your calculator rather than logarithm tables, first changing the base of the logarithm to 10 if necessary.

a. $\log(3^2)$

$\log(3^2) = \log(9) \approx 0.9542$

Therefore, $\log(3^2) \approx 0.9542$.

OR

$\log(3^2) = 2\log(3) \approx 2 \cdot 0.4771 \approx 0.9542$

Therefore, $\log(3^2) \approx 0.9542$.

b. $\log_3(3^2)$

$$\log_3(3^2) = \frac{2\log(3)}{\log(3)} = 2$$

Therefore, $\log_3(3^2) = 2.0000$.

c. $\log_2(3^2)$

$$\log_2(3^2) = \log_2(9) = \frac{\log(9)}{\log(2)} \approx 3.1699$$

Therefore, $\log_2(3^2) \approx 3.1699$.

Scaffolding:

Students who are not familiar with the $\boxed{\text{LOG}}$ key on the calculator can check how it works by finding the following:

$\log(1) = 0$;
$\log(10) = 1$;
$\log(10^3) = 3$.

Exercise 3 (8 minutes)

3. In Lesson 12, we justified a number of properties of base 10 logarithms. Working in pairs, justify the following properties of base-b logarithms.

a. $\log_b(1) = 0$

Because $L = \log_b(x)$ *means* $b^L = x$*, then when* $x = 1$, $L = 0$.

b. $\log_b(b) = 1$

Because $L = \log_b(x)$ *means* $b^L = x$*, then when* $x = b$, $L = 1$.

c. $\log_b(b^r) = r$

Because $L = \log_b(x)$ *means* $b^L = x$*, then when* $x = b^r$, $L = r$.

d. $b^{\log_b(x)} = x$

Because $L = \log_b(x)$ *means* $b^L = x$*, then* $x = b^{\log_b(x)}$.

Scaffolding:

By working in pairs, students should be able to reconstruct the arguments they used in Lesson 10. If they have trouble, they should be encouraged to use the definition and properties already justified.

e. $\log_b(x \cdot y) = \log_b(x) + \log_b(y)$

By the rule $a^q \cdot a^r = a^{q+r}$, $b^{\log_b(x)} \cdot b^{\log_b(y)} = b^{\log_b(x)+\log_b(y)}$.

By property 4, $b^{\log_b(x)} \cdot b^{\log_b(y)} = x \cdot y$.

Therefore, $x \cdot y = b^{\log_b(x)+\log_b(y)}$. *By property 4 again,* $x \cdot y = b^{\log_b(x \cdot y)}$.

So, the exponents must be equal, and $\log_b(x \cdot y) = \log_b(x) + \log_b(y)$.

f. $\log_b(x^r) = r \cdot \log_b(x)$

By the rule $(a^q)^r = a^{qr}$, $b^{r\log_b(x)} = \left(b^{\log_b(x)}\right)^r$.

By property 4, $\left(b^{\log_b(x)}\right)^r = x^r$.

Therefore, $x^r = b^{kr\log_b(x)}$. *By property 4 again,* $x^r = b^{\log_b(x^r)}$.

So, the exponents must be equal, and $\log_b(x^r) = r \cdot \log_b(x)$.

g. $\log_b\left(\frac{1}{x}\right) = -\log_b(x)$

By property 6, $\log_b(x^k) = k \cdot \log_b(x)$.

Let $k = -1$, *then for* $x \neq 0$, $\log_b(x^{-1}) = (-1) \cdot \log_b(x)$.

Thus, $\log_b\left(\frac{1}{x}\right) = -\log_b(x)$.

h. $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$

By property 5, $\log_b(x \cdot y) = \log_b(x) + \log_b(y)$.

By property 7, for $y \neq 0$, $\log_b(y^{-1}) = (-1) \cdot \log_b(y)$.

Therefore, $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$.

Discussion (2 minutes)

The topic is the definition of the natural logarithm. Students often misinterpret the symbol "ln" as the word "in". Emphasize that the notation is an *L* followed by an *N*, which comes from the French for natural logarithm: *le logarithme naturel*.

- Recall Euler's number e from Lesson 5, which is an irrational number approximated by $e \approx 2.71828$ .... This number plays an important role in many parts of mathematics, and it is frequently used as the base of logarithms. When e is taken as the base, the logarithm of a number x is abbreviated as $\ln(x)$. Because e is so often used to model growth and change in the natural world, it is not surprising that $\ln(x)$ is called the natural logarithm of x.
- Specifically, we say $\ln(x) = \log_e(x)$. What is the value of $\ln(1)$?
  - $\ln(1) = 0$
- What is the value of $\ln(e)$? The value of $\ln(e^2)$? Of $\ln(e^3)$?
  - $\ln(e) = 1$, $\ln(e^2) = 2$, *and* $\ln(e^3) = 3$.

- Because scientists primarily use logarithms base 10 and base e, calculators only have two logarithm buttons; LOG for calculating $\log(x)$ and LN for calculating $\ln(x)$. With the change of base formula, you can use either the common logarithm (base 10) or the natural logarithm (base e) to calculate the value of a logarithm with any allowable base b, so technically we only need one of those two buttons. However, each base has important uses, so most calculators are able to calculate logarithms in either base.

Exercise 4 (3 minutes)

Exercise 4 introduces the natural logarithm. Students will need a calculator with an LN key. They should work in pairs on these exercises, with one student using the calculator and the other keeping track of the computation. They should share their results for Exercise 4 in a class discussion before moving on.

Scaffolding:

Students who are not familiar with the LN key on the calculator can check how it works by finding the following:

$\ln(1) = 0;$

$\ln(e) = 1;$

$\ln(e^3) = 3.$

4. Find each of the following to four decimal places. Use the LN key on your calculator rather than a table.

a. $\ln(3^2)$

$\ln(3^2) = \ln(9) \approx 2.1972$

b. $\ln(2^4)$

$\ln(2^4) = \ln(16) \approx 2.7726$

Example 2 (4 minutes)

This example introduces more complicated expressions involving logarithms and showcases the power of logarithms in rearranging logarithmic expressions. Students have done exercises like this in their homework in prior lessons for base 10 logarithms, so this example and the following exercises demonstrate how the same procedures apply to logarithms with any base.

- Write as an expression containing only one logarithm: $\ln(k^2) + \ln\left(\frac{1}{k^2}\right) - \ln\left(\sqrt{k}\right)$.
  - $\ln(k^2) + \ln\left(\frac{1}{k^2}\right) - \ln\left(\sqrt{k}\right) = 2\ln(k) - 2\ln(k) - \frac{1}{2}\cdot\ln(k) = -\frac{1}{2}\ln(k)$
- Therefore, $\ln(k^2) + \ln\left(\frac{1}{k^2}\right) - \ln\left(\sqrt{k}\right) = -\frac{1}{2}\ln(k)$.

Exercises 5–6 (7 minutes)

Exercise 5 follows Example 3 by introducing somewhat more complicated expressions to be simplified that involve natural logarithms. In Exercise 5, students condense a sum of logarithmic expressions to an expression containing only one logarithm, while in Exercise 6, students take a single complicated logarithm and break it up into simpler parts. Students should work in pairs on these exercises, sharing their results in a class discussion before the closing.

5. **Write as a single logarithm.**

   a. $\ln(4) - 3\ln\left(\frac{1}{3}\right) + \ln(2)$

$$\begin{aligned}\ln(4) - 3\ln\left(\frac{1}{3}\right) + \ln(2) &= \ln(4) + \ln(3^3) + \ln(2)\\ &= \ln(4 \cdot 3^3 \cdot 2)\\ &= \ln(216)\\ &= \ln(6^3)\\ &= 3\ln(6)\end{aligned}$$

   Any of the last three expressions is an acceptable final answer.

   b. $\ln(5) + \frac{3}{5}\ln(32) - \ln(4)$

$$\begin{aligned}\ln(5) + \frac{3}{5}\ln(32) - \ln(4) &= \ln(5) + \ln(8) - \ln(4)\\ &= \ln(5 \cdot 8) - \ln(4)\\ &= \ln\left(\frac{40}{4}\right)\\ &= \ln(10)\end{aligned}$$

   Therefore, $\ln(5) + \frac{3}{5}\ln(32) - \ln(4) = \ln(10)$.

6. **Write each expression as a sum or difference of constants and logarithms of simpler terms.**

   a. $\ln\left(\frac{\sqrt{5x^3}}{e^2}\right)$

$$\begin{aligned}\ln\left(\frac{\sqrt{5x^3}}{e^2}\right) &= \ln(\sqrt{5}) + \ln\left(\sqrt{x^3}\right) - \ln(e^2)\\ &= \frac{1}{2}\ln(5) + \frac{3}{2}\ln(x) - 2\end{aligned}$$

   b. $\ln\left(\frac{(x+y)^2}{x^2+y^2}\right)$

$$\begin{aligned}\ln\left(\frac{(x+y)^2}{x^2+y^2}\right) &= \ln(x+y)^2 - \ln(x^2+y^2)\\ &= 2\ln(x+y) - \ln(x^2+y^2)\end{aligned}$$

   The point of this simplification is that neither of these terms can be simplified further.

Closing (4 minutes)

Have students summarize the lesson by discussing the following questions and coming to a consensus before students record the answers in their notebook.

- What is the definition of the logarithm base b?
  - *If there exist numbers b, L, and x so that $b^L = x$, then $L = \log_b(x)$.*
- What does $\ln(x)$ represent?
  - *The notation $\ln(x)$ represents the logarithm of x base e; that is, $\ln(x) = \log_e(x)$.*

- How can we use a calculator to approximate a logarithm to a base other than 10 or e?
  - *Use the change of base formula to convert a logarithm with base b to one with base 10 or base e; then, use the appropriate calculator button.*

Lesson Summary

We have established a formula for changing the base of logarithms from b to a:

$$\log_b(x) = \frac{\log_a(x)}{\log_a(b)}.$$

In particular, the formula allows us to change logarithms base b to common or natural logarithms, which are the only two kinds of logarithms that calculators compute:

$$\log_b(x) = \frac{\log(x)}{\log(b)} = \frac{\ln(x)}{\ln(b)}.$$

We have also established the following properties for base b logarithms. If x, y, a and b are all positive real numbers with $a \neq 1$ and $b \neq 1$ and r is any real number, then:

$$\log_b(1) = 0$$
$$\log_b(b) = 1$$
$$\log_b(b^r) = r$$
$$b^{\log_b(x)} = x$$
$$\log_b(x \cdot y) = \log_b(x) + \log_b(y)$$
$$\log_b(x^r) = r \cdot \log_b(x)$$
$$\log_b\left(\frac{1}{x}\right) = -\log_b(x)$$
$$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$$

Exit Ticket (5 minutes)

Name ______________________________ Date ____________________

Lesson 13: Changing the Base

Exit Ticket

1. Are there any properties that hold for base 10 logarithms that would not be valid for the logarithm base e? Why? Are there any properties that hold for base 10 logarithms that would not be valid for some positive base b, such that $b \neq 1$?

2. Write each logarithm as an equivalent expression involving only logarithms base 10.
   a. $\log_3(25)$

   b. $\log_{100}(x^2)$

3. Rewrite each expression as an equivalent expression containing only one logarithm.
   a. $3\ln(p+q) - 2\ln(q) - 7\ln(p)$

   b. $\ln(xy) - \ln\left(\frac{x}{y}\right)$

Exit Ticket Sample Solutions

1. Are there any properties that hold for base 10 logarithms that would not be valid for the logarithm base e? Why? Are there any properties that hold for base 10 logarithms that would not be valid for some positive base b, such that $b \neq 1$?

   No. Any property that is true for a base 10 logarithm will be true for a base e logarithm. The only difference between a common logarithm and a natural logarithm is a scale change, because $\log(x) = \frac{\ln(x)}{\ln(10)}$, and $\ln(x) = \frac{\log(x)}{\log(e)}$.

   Since $\log_b(x) = \frac{\log(x)}{\log(b)}$, we would only encounter a problem if $\log(b) = 0$, but this only happens when $b = 1$, and 1 is not a valid base for logarithms.

2. Write each logarithm as an equivalent expression involving only logarithms base 10.

   a. $\log_3(25)$

   $$\log_3(25) = \frac{\log(25)}{\log(3)}$$

   b. $\log_{100}(x^2)$

   $$\begin{aligned}\log_{100}(x^2) &= \frac{\log(x^2)}{\log(100)}\\ &= \frac{2\log(x)}{2}\\ &= \log(x)\end{aligned}$$

3. Rewrite each expression as an equivalent expression containing only one logarithm.

   a. $3\ln(p+q) - 2\ln(q) - 7\ln(p)$

   $$\begin{aligned}3\ln(p+q) - 2\ln(q) - 7\ln(p) &= \ln((p+q)^3) - \left(\ln(q^2) + \ln(p^7)\right)\\ &= \ln((p+q)^3) - \ln(q^2p^7)\\ &= \ln\left(\frac{(p+q)^3}{q^2p^7}\right)\end{aligned}$$

   b. $\ln(xy) - \ln\left(\frac{x}{y}\right)$

   $$\begin{aligned}\ln(xy) - \ln\left(\frac{x}{y}\right) &= \ln(x) + \ln(y) - \ln(x) + \ln(y)\\ &= 2\ln(y)\\ &= \ln(y^2)\end{aligned}$$

   Therefore, $\ln(xy) - \ln\left(\frac{x}{y}\right)$ is equivalent to both $2\ln(y)$ and $\ln(y^2)$.

Problem Set Sample Solutions

1. Evaluate each of the following logarithmic expressions, approximating to four decimal places if necessary. Use the $\boxed{\text{LN}}$ or $\boxed{\text{LOG}}$ key on your calculator rather than a table.

   a. $\log_8(16)$

   $$\begin{aligned}\log_8(16) &= \frac{\log(16)}{\log(8)}\\ &= \frac{\log(2^4)}{\log(2^3)}\\ &= \frac{4\cdot\log(2)}{3\cdot\log(2)}\\ &= \frac{4}{3}\end{aligned}$$

   Therefore, $\log_8(16) = \frac{4}{3}$.

   b. $\log_7(11)$

   $$\begin{aligned}\log_7(11) &= \frac{\log(11)}{\log(7)}\\ &\approx 1.2323\end{aligned}$$

   Therefore, $\log_7(11) \approx 1.2323$.

   c. $\log_3(2) + \log_2(3)$

   $$\begin{aligned}\log_3(2) + \log_2(3) &= \frac{\log(2)}{\log(3)} + \frac{\log(3)}{\log(2)}\\ &\approx 2.2159\end{aligned}$$

   Therefore, $\log_3(2) + \log_2(3) \approx 2.2159$.

2. Use logarithmic properties and the fact that $\ln(2) \approx 0.69$ and $\ln(3) \approx 1.10$ to approximate the value of each of the following logarithmic expressions. Do not use a calculator.

   a. $\ln(e^4)$

   $$\begin{aligned}\ln(e^4) &= 4\ln(e)\\ &= 4\end{aligned}$$

   Therefore, $\ln(e^4) = 4$.

   b. $\ln(6)$

   $$\begin{aligned}\ln(6) &= \ln(2) + \ln(3)\\ &\approx 0.69 + 1.10\\ &\approx 1.79\end{aligned}$$

   Therefore, $\ln(6) \approx 1.79$.

c. $\ln(108)$

$$\begin{aligned}\ln(108) &= \ln(4\cdot 27)\\ &= \ln(4)+\ln(27)\\ &\approx 2\ln(2)+3\ln(3)\\ &\approx 1.38+3.30\\ &\approx 4.68\end{aligned}$$

Therefore, $\ln(108)\approx 4.68$.

d. $\ln\left(\frac{8}{3}\right)$

$$\begin{aligned}\ln\left(\frac{8}{3}\right) &= \ln(8)-\ln(3)\\ &= \ln(2^3)-\ln(3)\\ &\approx 3(0.69)-1.10\\ &\approx 0.97\end{aligned}$$

Therefore, $\ln\left(\frac{8}{3}\right)\approx 0.97$.

3. Compare the values of $\log_{\frac{1}{9}}(10)$ and $\log_9\left(\frac{1}{10}\right)$ without using a calculator.

Using the change of base formula,

$$\begin{aligned}\log_{\frac{1}{9}}(10) &= \frac{\log_9(10)}{\log_9\left(\frac{1}{9}\right)}\\ &= \frac{\log_9(10)}{-1}\\ &= -\log_9(10)\\ &= \log_9\left(\frac{1}{10}\right).\end{aligned}$$

Thus, $\log_{\frac{1}{9}}(10)=\log_9\left(\frac{1}{10}\right)$.

4. Show that for any positive numbers a and b with $a\neq 1$ and $b\neq 1$, $\log_a(b)\cdot\log_b(a)=1$.

Using the change of base formula,

$$\log_a(\mathrm{b})=\frac{\log_b(\mathrm{b})}{\log_b(\mathrm{a})}=\frac{1}{\log_b(\mathrm{a})}.$$

Thus,

$$\log_a(b)\cdot\log_b(a)=\frac{1}{\log_b(a)}\cdot\log_b(a)=1.$$

5. Express x in terms of a, e, and y if $\ln(x)-\ln(y)=2a$.

$$\begin{aligned}\ln(x)-\ln(y) &= 2a\\ \ln\left(\frac{x}{y}\right) &= 2a\\ \frac{x}{y} &= e^{2a}\\ x &= y\,e^{2a}\end{aligned}$$

6. Rewrite each expression in an equivalent form that only contains one base 10 logarithm.

a. $\log_2(800)$

$$\frac{\log(800)}{\log(2)} = \frac{\log(2^3)+2}{\log(2)} = 3 + \frac{2}{\log(2)}$$

b. $\log_x\left(\frac{1}{10}\right)$, for positive real values of $x \neq 1$

$$\frac{\log\left(\frac{1}{10}\right)}{\log(x)} = -\frac{1}{\log(x)}$$

c. $\log_5(12,500)$

$$\frac{\log(5^3 \cdot 10^2)}{\log(5)} = \frac{3\log(5)+2}{\log(5)} = 3 + \frac{2}{\log(5)}$$

d. $\log_3(0.81)$

$$\frac{\log\left(\frac{81}{100}\right)}{\log(3)} = \frac{4\log(3)-2}{\log(3)} = 4 - \frac{2}{\log(3)}$$

7. Write each number in terms of natural logarithms, and then use the properties of logarithms to show that it is a rational number.

a. $\log_9(\sqrt{27})$

$$\frac{\ln(\sqrt{27})}{\ln(9)} = \frac{\ln\left(3^{\frac{3}{2}}\right)}{\ln(3^2)} = \frac{\frac{3}{2}\ln(3)}{2\ln(3)} = \frac{3}{4}$$

b. $\log_8(32)$

$$\frac{\ln(32)}{\ln(8)} = \frac{\ln(2^5)}{\ln(2^3)} = \frac{5}{3}$$

c. $\log_4\left(\frac{1}{8}\right)$

$$\frac{\ln\left(\frac{1}{8}\right)}{\ln(4)} = \frac{\ln(2^{-3})}{\ln(2^2)} = -\frac{3}{2}$$

8. Write each expression as an equivalent expression with a single logarithm. Assume x, y, and z are positive real numbers.

a. $\ln(x) + 2\ln(y) - 3\ln(z)$

$$\ln\left(\frac{xy^2}{z^3}\right)$$

b. $\frac{1}{2}(\ln(x+y) - \ln(z))$

$\ln\left(\sqrt{\frac{x+y}{z}}\right)$

c. $(x+y) + \ln(z)$

$(x+y)\ln(e) + \ln(z) = \ln(e^{x+y}) + \ln(z) = \ln(e^{x+y} \cdot z)$

9. Rewrite each expression as sums and differences in terms of $\ln(x)$, $\ln(y)$, and $\ln(z)$.

a. $\ln(xyz^3)$

$\ln(x) + \ln(y) + 3\ln(z)$

b. $\ln\left(\frac{e^3}{xyz}\right)$

$3 - \ln(x) - \ln(y) - \ln(z)$

c. $\ln\left(\sqrt{\frac{x}{y}}\right)$

$\frac{1}{2}(\ln(x) - \ln(y))$

10. Solve the following equations in terms of base 5 logarithms. Then, use the change of base properties and a calculator to estimate the solution to the nearest 1000^{th}. If the equation has no solution, explain why.

a. $5^{2x} = 20$

$2x = \log_5(20)$

$x = \frac{1}{2}\log_5(20)$

$x = \frac{\log(20)}{2\log(5)}$

$x \approx 0.931$

b. $75 = 10 \cdot 5^{x-1}$

$7.5 = 5^{x-1}$

$x = \log_5(7.5) + 1$

$x \approx 2.252$

c. $5^{2+x} - 5^x = 10$

$5^x(5^2 - 1) = 10$

$5^x = \frac{10}{24}$

$x = \log_5\left(\frac{10}{24}\right)$

$x \approx -0.544$

d. $5^{x^2} = 0.25$

$x^2 = \log_5(0.25)$

$x^2 = \dfrac{\log(0.25)}{\log(5)}$

This equation has no real solution because $\dfrac{\log(0.25)}{\log(5)}$ *is negative.*

11. In Lesson 6, you discovered that $\log(x \cdot 10^k) = k + \log(x)$ by looking at a table of logarithms. Use the properties of logarithms to justify this property for an arbitrary base $b > 0$ with $b \neq 1$. That is, show that $\log_b(x \cdot b^k) = k + \log_b(x)$.

$$\begin{aligned}\log_b(x \cdot b^k) &= \log_b(x) + \log_b(b^k)\\ &= k + log_b(x)\end{aligned}$$

MP.3

12. Larissa argued that since $\log_2(2) = 1$ and $\log_2(4) = 2$, then it must be true that $\log_2(3) = 1.5$. Is she correct? Explain how you know.

Larissa is not correct. According to the calculator and the change of base formula, $\log_2(3) = \dfrac{\log(3)}{\log(2)} \approx 1.585$. *If* $\log_2(x) = 1.5$ *then* $2^{1.5} = x$*, so* $x = \sqrt{8} = 2\sqrt{2}$. *Since* $3 \neq 2\sqrt{2}$*, Larissa's calculation is not correct. Larissa is assuming that the logarithm function behaves like a linear function, which it does not.*

13. Extension: Suppose that there is some positive number b so that

$$\log_b(2) = 0.36$$
$$\log_b(3) = 0.57$$
$$\log_b(5) = 0.84.$$

a. Use the given values of $\log_b(2)$, $\log_b(3)$, and $\log_b(5)$ to evaluate the following logarithms.

i. $\log_b(6)$

$$\begin{aligned}\log_b(6) &= \log_b(2 \cdot 3)\\ &= \log_b(2) + \log_b(3)\\ &= 0.36 + 0.57\\ &= 0.93\end{aligned}$$

ii. $\log_b(8)$

$$\begin{aligned}\log_b(8) &= \log_b(2^3)\\ &= 3 \cdot \log_b(2)\\ &= 3 \cdot 0.36\\ &= 1.08\end{aligned}$$

iii. $\log_b(10)$

$$\begin{aligned}\log_b(10) &= \log_b(2 \cdot 5)\\ &= \log_b(2) + \log_b(5)\\ &= 0.36 + 0.84\\ &= 1.20\end{aligned}$$

iv. $\log_b(600)$

$$\begin{aligned}\log_b(600) &= \log_b(6\cdot 100)\\ &= \log_b(6)+\log_b(100)\\ &= 0.93+2\log_b(10)\\ &= 0.93+2(1.20)\\ &= 0.93+2.40\\ &= 3.33\end{aligned}$$

b. Use the change of base formula to convert $\log_b(10)$ to base 10, and solve for b. Give your answer to four decimal places.

From part (iii) above, $\log_b(10) = 1.20$. *Then,*

$$\begin{aligned}1.20 &= \log_b(10)\\ 1.20 &= \frac{\log_{10}(10)}{\log_{10}(b)}\\ 1.20 &= \frac{1}{\log_{10}(b)}\\ \frac{1}{1.20} &= \log_{10}(b)\\ b &= 10^{\frac{1}{1.20}}\\ b &\approx 6.8129.\end{aligned}$$

14. Solve the following exponential equations.

a. $2^{3x} = 16$

$$\begin{aligned}\log_2(2^{3x}) &= \log_2(16)\\ 3x &= 4\\ x &= \frac{4}{3}\end{aligned}$$

b. $2^{x+3} = 4^{3x}$

$$\begin{aligned}\log_2(2^{x+3}) &= \log_2(4^{3x})\\ x+3 &= 3x\cdot\log_2(4)\\ x+3 &= 3x\cdot 2\\ 5x &= 3\\ x &= \frac{3}{5}\end{aligned}$$

c. $3^{4x-2} = 27^{x+2}$

$$\begin{aligned}\log_3(3^{4x-2}) &= \log_3(27^{x+2})\\ (4x-2)\log_3(3) &= (x+2)\log_3(27)\\ 4x-2 &= 3(x+2)\\ 4x-2 &= 3x+6\\ x &= 8\end{aligned}$$

d. $4^{2x} = \left(\frac{1}{4}\right)^{3x}$

$$\log_4(4^{2x}) = \log_4\left(\left(\frac{1}{4}\right)^{3x}\right)$$
$$2x\log_4(4) = 3x\log_4\left(\frac{1}{4}\right)$$
$$2x = 3x(-1)$$
$$5x = 0$$
$$x = 0$$

e. $5^{0.2x+3} = 625$

$$\log_5(5^{0.2x+3}) = \log_5(625)$$
$$(0.2x+3)\log_5(5) = \log_5(5^4)$$
$$0.2x + 3 = 4$$
$$0.2x = 1$$
$$x = 5$$

15. Solve each exponential equation.

a. $3^{2x} = 81$

$x = 2$

b. $6^{3x} = 36^{x+1}$

$x = 2$

c. $625 = 5^{3x}$

$x = \frac{4}{3}$

d. $25^{4-x} = 5^{3x}$

$x = \frac{8}{5}$

e. $32^{x-1} = \frac{1}{2}$

$x = \frac{4}{5}$

f. $\frac{4^{2x}}{2^{x-3}} = 1$

$x = -1$

g. $\frac{1}{8^{2x-4}} = 1$

$x = 2$

h. $2^x = 81$

$x = \frac{\ln(81)}{\ln(2)}$

i. $8 = 3^x$

$x = \frac{\ln(8)}{\ln(3)}$

j. $6^{x+2} = 12$

$x = -2 + \frac{\log(12)}{\log(6)}$

k. $10^{x+4} = 27$

$x = -4 + log(27)$

l. $2^{x+1} = 3^{1-x}$

$x = \frac{\log(3) - \log(2)}{\log(2) + \log(3)}$

m. $3^{2x-3} = 2^{x+4}$

$$x = \frac{4\log(2) + 3\log(3)}{3\log(3) - \log(2)}$$

n. $e^{2x} = 5$

$$x = \frac{\ln(5)}{2}$$

o. $e^{x-1} = 6$

$$x = -1 + 3\ln(2)$$

16. In Problem 9(e) of Lesson 12, you solved the equation $3^x = 7^{-3x+2}$ using the logarithm base 10.

a. Solve $3^x = 7^{-3x+2}$ using the logarithm base 3.

$$\begin{aligned}\log_3(3^x) &= \log_3(7^{-3x+2})\\ x &= (-3x+2)\log_3(7)\\ x &= -3x\log_3(7) + 2\log_3(7)\\ x + 3x\log_3(7) &= 2\log_3(7)\\ x\big(1+3\log_3(7)\big) &= 2\log_3(7)\\ x &= \frac{2\log_3(7)}{1+3\log_3(7)}\end{aligned}$$

b. Apply the change of base formula to show that your answer to part (a) agrees with your answer to Problem 9(e) of Lesson 12.

Changing from base 3 to base 10, we see that

$$\log_3(7) = \frac{\log(7)}{\log(3)}.$$

Then,

$$\begin{aligned}\frac{2\log_3(7)}{1+3\log_3(7)} &= \frac{2\left(\frac{\log(7)}{\log(3)}\right)}{1+3\left(\frac{\log(7)}{\log(3)}\right)}\\ &= \frac{2\log(7)}{\log(3) + 3\log(7)},\end{aligned}$$

which was the answer from Problem 9(e) of Lesson 12.

c. Solve $3^x = 7^{-3x+2}$ using the logarithm base 7.

$$\begin{aligned}\log_7(3^x) &= \log_7(7^{-3x+2})\\ x\log_7(3) &= -3x + 2\\ 3x + x\log_7(3) &= 2\\ x\big(3+\log_7(3)\big) &= 2\\ x &= \frac{2}{3+\log_7(3)}\end{aligned}$$

d. Apply the change of base formula to show that your answer to part (c) also agrees with your answer to Problem 9(e) of Lesson 12.

Changing from base 7 to base 10, we see that

$$\log_7(3) = \frac{\log(3)}{\log(7)}.$$

Then,

$$\frac{2}{3+\log_7(3)} = \frac{2}{3+\frac{\log(3)}{\log(7)}}$$
$$= \frac{2\log(7)}{3\log(7)+\log(3)},$$

which was the answer from Problem 9(e) of Lesson 12.

17. Pearl solved the equation $2^x = 10$ as follows:

$$\log(2^x) = \log(10)$$
$$x\log(2) = 1$$
$$x = \frac{1}{\log(2)}.$$

Jess solved the equation $2^x = 10$ as follows:

$$\log_2(2^x) = \log_2(10)$$
$$x\log_2(2) = \log_2(10)$$
$$x = \log_2(10).$$

Is Pearl correct? Is Jess correct? Explain how you know.

MP.3

Both Pearl and Jess are correct. If we take Jess's solution and apply the change of base formula, we have

$$x = \log_2(10)$$
$$= \frac{\log(10)}{\log(2)}$$
$$= \frac{1}{\log(2)}.$$

Thus, the two solutions are equivalent, and both students are correct.