# Lesson 13: Changing the Base

## Classwork

### Exercises

1. Assume that $x$, $a$, and $b$ are all positive real numbers, so that $a \neq 1$ and $b \neq 1$. What is $\log_b(x)$ in terms of $\log_a(x)$? The resulting equation allows us to change the base of a logarithm from $a$ to $b$.

2. Approximate each of the following logarithms to four decimal places. Use the $\boxed{\text{LOG}}$ key on your calculator rather than logarithm tables, first changing the base of the logarithm to 10 if necessary.

   a. $\log(3^2)$

   b. $\log_3(3^2)$

   c. $\log_2(3^2)$

3. In Lesson 12, we justified a number of properties of base 10 logarithms. Working in pairs, justify the following properties of base $b$ logarithms.

   a. $\log_b(1) = 0$

   b. $\log_b(b) = 1$

   c. $\log_b(\mathrm{b}^r) = r$

   d. $b^{\log_b(x)} = x$

   e. $\log_b(x \cdot y) = \log_b(x) + \log_b(y)$

f. $\log_b(x^r) = r \cdot \log_b(x)$

g. $\log_b\left(\frac{1}{x}\right) = -\log_b(x)$

h. $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$

4. Find each of the following to four decimal places. Use the $\boxed{\text{LN}}$ key on your calculator rather than a table.

a. $\ln(3^2)$

b. $\ln(2^4)$

5. Write as a single logarithm:

   a. $\ln(4) - 3\ln\left(\frac{1}{3}\right) + \ln(2)$.

   b. $\ln(5) + \frac{3}{5}\ln(32) - \ln(4)$.

6. Write each expression as a sum or difference of constants and logarithms of simpler terms.

   a. $\ln\left(\frac{\sqrt{5x^3}}{e^2}\right)$

   b. $\ln\left(\frac{(x+y)^2}{x^2+y^2}\right)$

### Lesson Summary

We have established a formula for changing the base of logarithms from $b$ to $a$:

$$\log_b(x) = \frac{\log_a(x)}{\log_a(b)}.$$

In particular, the formula allows us to change logarithms base $b$ to common or natural logarithms, which are the only two kinds of logarithms that calculators compute:

$$\log_{\mathrm{b}}(x) = \frac{\log(x)}{\log(b)} = \frac{\ln(x)}{\ln(b)}.$$

We have also established the following properties for base $b$ logarithms. If $x$, $y$, $a$, and $b$ are all positive real numbers with $a \neq 1$ and $b \neq 1$ and $r$ is any real number, then:

1. $\log_b(1) = 0$
2. $\log_b(b) = 1$
3. $\log_b(b^r) = r$
4. $b^{\log_b(x)} = x$
5. $\log_b(x \cdot y) = \log_b(x) + \log_b(y)$
6. $\log_b(x^r) = r \cdot \log_b(x)$
7. $\log_b\left(\frac{1}{x}\right) = -\log_b(x)$
8. $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$

## Problem Set

1. Evaluate each of the following logarithmic expressions, approximating to four decimal places if necessary. Use the LN or LOG key on your calculator rather than a table.
   a. $\log_8(16)$
   b. $\log_7(11)$
   c. $\log_3(2) + \log_2(3)$

2. Use logarithmic properties and the fact that $\ln(2) \approx 0.69$ and $\ln(3) \approx 1.10$ to approximate the value of each of the following logarithmic expressions. Do not use a calculator.
   a. $\ln(e^4)$
   b. $\ln(6)$
   c. $\ln(108)$
   d. $\ln\left(\frac{8}{3}\right)$

3. Compare the values of $\log_{\frac{1}{9}}(10)$ and $\log_9\left(\frac{1}{10}\right)$ without using a calculator.

4. Show that for any positive numbers $a$ and $b$ with $a \neq 1$ and $b \neq 1$, $\log_a(b) \cdot \log_b(a) = 1$.

5. Express $x$ in terms of $a$, $e$, and $y$ if $\ln(x) - \ln(y) = 2a$.

6. Rewrite each expression in an equivalent form that only contains one base $10$ logarithm.
   a. $\log_2(800)$
   b. $\log_x\left(\frac{1}{10}\right)$, for positive real values of $x \neq 1$
   c. $\log_5(12{,}500)$
   d. $\log_3(0.81)$

7. Write each number in terms of natural logarithms, and then use the properties of logarithms to show that it is a rational number.
   a. $\log_9\left(\sqrt{27}\right)$
   b. $\log_8(32)$
   c. $\log_4\left(\frac{1}{8}\right)$

8. Write each expression as an equivalent expression with a single logarithm. Assume $x$, $y$, and $z$ are positive real numbers.
   a. $\ln(x) + 2\ln(y) - 3\ln(z)$
   b. $\frac{1}{2}(\ln(x+y) - \ln(z))$
   c. $(x+y) + \ln(z)$

9. Rewrite each expression as sums and differences in terms of $\ln(x)$, $\ln(y)$, and $\ln(z)$.
   a. $\ln(xyz^3)$
   b. $\ln\left(\frac{e^3}{xyz}\right)$
   c. $\ln\left(\sqrt{\frac{x}{y}}\right)$

10. Solve the following equations in terms of base 5 logarithms. Then, use the change of base properties and a calculator to estimate the solution to the nearest $1000^{\text{th}}$. If the equation has no solution, explain why.
    a. $5^{2x} = 20$
    b. $75 = 10 \cdot 5^{x-1}$
    c. $5^{2+x} - 5^x = 10$
    d. $5^{x^2} = 0.25$

11. In Lesson 6, you discovered that $\log(x \cdot 10^k) = k + \log(x)$ by looking at a table of logarithms. Use the properties of logarithms to justify this property for an arbitrary base $b > 0$ with $b \neq 1$. That is, show that $\log_b(x \cdot b^k) = k + \log_b(x)$.

12. Larissa argued that since $\log_2(2) = 1$ and $\log_2(4) = 2$, then it must be true that $\log_2(3) = 1.5$. Is she correct? Explain how you know.

13. Extension: Suppose that there is some positive number $b$ so that

$$\log_b(2) = 0.36$$
$$\log_b(3) = 0.57$$
$$\log_b(5) = 0.84.$$

   a. Use the given values of $\log_b(2)$, $\log_b(3)$, and $\log_b(5)$ to evaluate the following logarithms.
      i. $\log_b(6)$
      ii. $\log_b(8)$
      iii. $\log_b(10)$
      iv. $\log_b(600)$
   b. Use the change of base formula to convert $\log_b(10)$ to base $10$, and solve for $b$. Give your answer to four decimal places.

14. Solve the following exponential equations.
   a. $2^{3x} = 16$
   b. $2^{x+3} = 4^{3x}$
   c. $3^{4x-2} = 27^{x+2}$
   d. $4^{2x} = \left(\frac{1}{4}\right)^{3x}$
   e. $5^{0.2x+3} = 625$

15. Solve each exponential equation.
   a. $3^{2x} = 81$
   b. $6^{3x} = 36^{x+1}$
   c. $625 = 5^{3x}$
   d. $25^{4-x} = 5^{3x}$
   e. $32^{x-1} = \frac{1}{2}$
   f. $\frac{4^{2x}}{2^{x-3}} = 1$
   g. $\frac{1}{8^{2x-4}} = 1$
   h. $2^x = 81$
   i. $8 = 3^x$
   j. $6^{x+2} = 12$
   k. $10^{x+4} = 27$
   l. $2^{x+1} = 3^{1-x}$
   m. $3^{2x-3} = 2^{x+4}$
   n. $e^{2x} = 5$
   o. $e^{x-1} = 6$

16. In Problem 9(e) of Lesson 12, you solved the equation $3^x = 7^{-3x+2}$ using the logarithm base $10$.
    a. Solve $3^x = 7^{-3x+2}$ using the logarithm base $3$.
    b. Apply the change of base formula to show that your answer to part (a) agrees with your answer to Problem 9(e) of Lesson 12.
    c. Solve $3^x = 7^{-3x+2}$ using the logarithm base $7$.
    d. Apply the change of base formula to show that your answer to part (c) also agrees with your answer to Problem 9(e) of Lesson 12.

17. Pearl solved the equation $2^x = 10$ as follows:

$$\begin{aligned} \log(2^x) &= \log(10) \\ x\log(2) &= 1 \\ x &= \frac{1}{\log(2)}. \end{aligned}$$

Jess solved the equation $2^x = 10$ as follows:

$$\begin{aligned} \log_2(2^x) &= \log_2(10) \\ x\log_2(2) &= \log_2(10) \\ x &= \log_2(10). \end{aligned}$$

Is Pearl correct? Is Jess correct? Explain how you know.