Lesson 11: The Most Important Property of Logarithms

Student Outcomes

- Students construct a table of logarithms base 10 and observe patterns that indicate properties of logarithms.

Lesson Notes

In the previous lesson, students discovered that for logarithms base 10, $\log(10^k \cdot x) = k + \log(x)$. In this lesson, we extend this result to develop the most important property of logarithms: $\log(xy) = \log(x) + \log(y)$. Additionally, students will discover the reciprocal property of logarithms: $\log\left(\frac{1}{x}\right) = -\log(x)$. Students will continue to hone their skills at observing and generalizing patterns in this lesson as they create tables of logarithms and observe patterns, practicing MP.8. In the next lesson, these logarithmic properties will be formalized and generalized for any base, but for this lesson we will solely focus on logarithms base 10. Understanding deeply the properties of logarithms will help prepare students to rewrite expressions based on their structure (**A-SSE.A.2**), solve exponential equations (**F-LE.A.4**), and interpret transformations of graphs of logarithmic functions (**F-BF.B.3**).

Materials Needed

Students will need access to a calculator or other technological tool able to compute exponents and logarithms base 10.

Classwork

Opening (1 minute)

In the previous lesson, students discovered the logarithmic property $\log(10^k \cdot x) = k + \log(x)$, which is a special case of the additive property $\log(xy) = \log(x) + \log(y)$ that they will discover today. The Opening Exercise reminds students of how we can use this property to compute logarithms of numbers not in our table. By the end of today's lesson, students will be able to calculate any logarithm base 10 using just a table of values of $\log(x)$ for prime integers x. The only times in this lesson that calculators should be used is to create the tables in Exercises 1 and 6.

Opening Exercise (4 minutes)

Students should complete this exercise without the use of a calculator.

Opening Exercise

Use the logarithm table below to calculate the specified logarithms.

x	$\log(x)$
1	0
2	0.3010
3	0.4771
4	0.6021
5	0.6990
6	0.7782
7	0.8451
8	0.9031
9	0.9542

Scaffolding:

- For struggling students, model the decomposition of 80 with the whole class before starting or with a small group.

$$\begin{aligned}\log(80) &= \log(10\cdot 8)\\ &= \log(10)+\log(8)\\ &= 1+\log(8)\end{aligned}$$

- Ask advanced students to write an expression for $\log(10^k \cdot x)$ independently.

a. $\log(80)$

$\log(80) = \log(10^1 \cdot 8) = 1 + \log(8) \approx 1.9031$

b. $\log(7000)$

$\log(7000) = \log(10^3 \cdot 7) = 3 + \log(7) \approx 3.8451$

c. $\log(0.00006)$

$\log(0.00006) = \log(10^{-5} \cdot 6) = -5 + \log(6) \approx -4.2218$

d. $\log(3.0 \times 10^{27})$

$\log(3.0 \times 10^{27}) = \log(10^{27} \cdot 3) = 27 + \log(3) \approx 27.4771$

e. $\log(9.0 \times 10^k)$ for an integer k

$\log(9.0 \times 10^k) = \log(10^k \cdot 9) = k + \log(9) \approx k + 0.9542$

Discussion (3 minutes)

Use this discussion to review the formulas discovered in the previous lesson. Tell the class that they will be using more logarithmic tables to discover some other interesting properties of logarithms.

In the next set of exercises, we want students to discover the additive property of logarithms, which is not as readily apparent as the patterns observed yesterday. Plant the seed of the idea by restating the previous property in the additive format.

- What was the formula we developed in the last class?
  - $\log(10^k \cdot x) = k + \log(x)$

- What is the value of $\log(10^k)$?
  - $\log(10^k) = k$
- So, what is another way we can write the formula $\log(10^k \cdot x) = k + \log(x)$?
  - $\log(10^k \cdot x) = \log(10^k) + \log(x)$
- Keep this statement of the formula in mind as you progress through the next set of exercises.

Exercises 1–5 (6 minutes)

Students may be confused by the fact that the formulas do not appear to be exact–for example, the table shows that $\log(4) = 0.6021$, and $2\log(2) = 0.6020$. If this question arises, remind students that since we have made approximations to irrational numbers, there is some error in rounding off the decimal expansions to four decimal places. Students should question this in part (f) of Exercise 2.

Exercises 1–5

1. **Use your calculator to complete the following table. Round the logarithms to four decimal places.**

x	$\log(x)$
1	0
2	0.3010
3	0.4771
4	0.6021
5	0.6990
6	0.7782
7	0.8451
8	0.9031
9	0.9542

x	$\log(x)$
10	1.0000
12	1.0792
16	1.2041
18	1.2553
20	1.3010
25	1.3979
30	1.4771
36	1.5563
100	2.0000

2. **Calculate the following values. Do they appear anywhere else in the table?**

   a. $\log(2) + \log(4)$

   We see that $\log(2) + \log(4) \approx 0.9031$*, which is approximately* $\log(8)$*.*

   b. $\log(2) + \log(6)$

   We see that $\log(2) + \log(6) \approx 1.0792$*, which is approximately* $\log(12)$*.*

   c. $\log(3) + \log(4)$

   We see that $\log(3) + \log(4) \approx 1.0792$*, which is approximately* $\log(12)$*.*

   d. $\log(6) + \log(6)$

   We see that $\log(6) + \log(6) \approx 1.5663$*, which is approximately* $\log(36)$*.*

   e. $\log(2) + \log(18)$

   We see that $\log(2) + \log(18) \approx 1.5663$*, which is approximately* $\log(36)$*.*

f. $\log(3) + \log(12)$

We see that $\log(3) + \log(12) \approx 1.5664$*, which is approximately* $\log(36)$*.*

3. **What pattern(s) can you see in Exercise 2 and the table from Exercise 1? Write them using logarithmic notation.**

I found the pattern $\log(xy) = \log(x) + \log(y)$*.*

MP.8

4. **What pattern would you expect to find for $\log(x^2)$? Make a conjecture, and test it to see whether or not it appears to be valid.**

I would guess that $\log(x^2) = \log(x) + \log(x) = 2\log(x)$*. This is verified by the fact that* $\log(4) \approx 0.6021 \approx 2\log(2)$*,* $\log(9) \approx 0.9542 \approx 2\log(3)$*,* $\log(16) \approx 1.2041 \approx 2\log(4)$*, and* $\log(25) \approx 1.3980 \approx 2\log(5)$*.*

5. **Make a conjecture for a logarithm of the form $\log(xyz)$, where x, y, and z are positive real numbers. Provide evidence that your conjecture is valid.**

It appears that $\log(xyz) = \log(x) + \log(y) + \log(z)$*. This is due to applying the property from Exercise 3 twice.*

$$\begin{aligned}\log(xyz) &= \log(xy \cdot z)\\ &= \log(xy) + \log(z)\\ &= \log(x) + \log(y) + \log(z)\end{aligned}$$

OR

It appears that $\log(xyz) = \log(x) + \log(y) + \log(z)$*. We can see that*

$\log(18) \approx 1.2553 \approx 0.3010 + 0.3010 + 0.4771 \approx \log(2) + \log(2) + \log(3)$,
$\log(20) \approx 1.3010 \approx 0.3010 + 0.3010 + 0.6990 \approx \log(2) + \log(2) + \log(5)$*, and*
$\log(36) \approx 1.5563 \approx 0.3010 + 0.4771 + 0.7782 \approx \log(2) + \log(3) + \log(6)$.

Discussion (2 minutes)

Ask groups to share the patterns and conjectures they formed in Exercises 3–5 with the class; emphasize that the pattern discovered in Exercise 3 is the *most important property* of logarithms base 10. Ensure that all students have the correct statements recorded in their notebooks or journals before continuing to the next example.

Example 1 (5 minutes)

Lead the class through these four logarithmic calculations, relying only on the values in the table from Exercise 1. Notice that since we do not have a value for $\log(11)$ in the table, we do not have enough information to calculate $\log(121)$. Allow students to figure this out for themselves.

Example 1

Use the logarithm table from Exercise 1 to approximate the following logarithms:

a. $\log(14)$

$\log(14) = \log(2) + \log(7) \approx 0.3010 + 0.8451$*, so* $\log(14) \approx 1.1461$*.*

b. $\log(35)$

$\log(35) = \log(5) + \log(7) \approx 0.6990 + 0.8451$*, so* $\log(35) \approx 1.5441$*.*

c. $\log(72)$

$\log(72) = \log(8) + \log(9) \approx 0.9031 + 0.9542$, *so* $\log(72) \approx 1.8573$.

d. $\log(121)$

$\log(121) = \log(11) + \log(11)$, *but we do not have a value for* $\log(11)$ *in the table, so we cannot evaluate* $\log(121)$.

Discussion (3 minutes)

- Suppose we are building a logarithm table, and we have already approximated the values of $\log(2)$ and $\log(3)$. What other values of $\log(x)$ for $4 \leq x \leq 20$ can we approximate by applying the additive property developed in Exercise 5?
  - *Since the table contains* $\log(2)$ *and* $\log(3)$, *we can figure out approximations of* $\log(4)$, $\log(6)$, $\log(8)$, $\log(9)$, $\log(12)$, *and* $\log(18)$ *since the only factors of* 4, 6, 8, 9, 12, 16, *and* 18 *are* 2 *and* 3.
- In order to develop the entire logarithm table for all integers between 1 and 20, what is the smallest set of logarithmic values that we need to know?
  - *We need to know the values of the logarithms for the prime numbers:* 2, 3, 5, 7, 11, 13, 17, *and* 19.
- Why does the additive property make sense based on what we know about exponents?
  - *We know that when you multiply two powers of the same base together, the exponents are added. For example,* $10^4 \cdot 10^5 = 10^{4+5} = 10^9$, *so* $\log(10^4 \cdot 10^5) = \log(10^9) = 9$.

Exercises 6–8 (7 minutes)

Have students again work individually to complete the table in Exercise 6 and to check their tables against each other before they proceed to discuss and answer Exercise 7 in groups. Ensure that there is enough time for a volunteer to present justification for the conjecture in Exercise 8.

Scaffolding:

Ask students who struggle with decimal to fraction conversion to convert the decimal values in the second table to fractions before looking for the pattern, or present the table with values as fractions.

Exercises 6–8

6. Use your calculator to complete the following table. Round the logarithms to four decimal places.

x	$\log(x)$
2	0.3013
4	0.6021
5	0.6990
8	0.9031
10	1.0000
16	1.2041
20	1.3010
50	1.6990
100	2.0000

x	$\log(x)$
0.5	-0.3013
0.25	-0.6021
0.2	-0.6990
0.125	-0.9031
0.1	-1.0000
0.0625	-1.2041
0.05	-1.3010
0.02	-1.6990
0.01	-2.0000

MP.8

7. **What pattern(s) can you see in the table from Exercise 6? Write a conjecture using logarithmic notation.**

For any real number $x > 0$, $\log\left(\frac{1}{x}\right) = -\log(x)$.

8. **Use the definition of logarithm to justify the conjecture you found in Exercise 7.**

If $\log\left(\frac{1}{x}\right) = a$ for some number a, then $10^a = \frac{1}{x}$. So, $10^{-a} = x$, and thus $\log(x) = -a$. We then have $\log\left(\frac{1}{x}\right) = -\log(x)$.

Discussion (3 minutes)

Ask groups to share the conjecture they formed in Exercise 7 with the class. Ensure that all students have the correct conjecture recorded in their notebooks or journals before continuing to the next example.

Example 2 (5 minutes)

Lead the class through these calculations. You may decide to let them work either alone or in groups on parts (b)-(d) after you have led them through part (a).

Example 2

Use the logarithm tables and the rules we discovered to estimate the following logarithms to four decimal places.

a. $\log(2100)$

$$\begin{aligned}\log(2100) &= \log(10^2 \cdot 21)\\ &= 2 + \log(21)\\ &= 2 + \log(3) + \log(7)\\ &\approx 2 + 0.4771 + 0.8451\\ &\approx 3.3222\end{aligned}$$

b. $\log(0.00049)$

$$\begin{aligned}\log(0.00049) &= \log(10^{-5} \cdot 49)\\ &= -5 + \log(49)\\ &= -5 + \log(7) + \log(7)\\ &\approx -5 + 0.8451 + 0.8451\\ &\approx -3.3098\end{aligned}$$

c. $\log(42{,}000{,}000)$

$$\begin{aligned}\log(42{,}000{,}000) &= \log(10^6 \cdot 42)\\ &= 6 + \log(42)\\ &= 6 + \log(6) + \log(7)\\ &\approx 6 + 0.7782 + 0.8451\\ &\approx 7.6233\end{aligned}$$

d. $\log\left(\frac{1}{640}\right)$

$$\begin{aligned}\log\left(\frac{1}{640}\right) &= -\log(640)\\ &= -\big(\log(10\cdot 64)\big)\\ &= -\big(1+\log(64)\big)\\ &= -\big(1+\log(8)+\log(8)\big)\\ &\approx -(1+0.9031+0.9031)\\ &\approx -2.8062\end{aligned}$$

Closing (2 minutes)

Ask students to summarize the important parts of the lesson, either in writing, to a partner, or as a class. Use this as an opportunity to informally assess understanding of the lesson. The following are some important summary elements:

Lesson Summary

- **The notation $\log(x)$ is used to represent $\log_{10}(x)$.**
  - **The most important property of logarithms base 10 is that for positive real numbers x and y,** $$\log(xy) = \log(x) + \log(y).$$
- **For positive real numbers x,** $$\log\left(\frac{1}{x}\right) = -\log(x).$$

Exit Ticket (4 minutes)

Name ______________________________ Date ____________________

Lesson 11: The Most Important Property of Logarithms

Exit Ticket

1. Use the table below to approximate the following logarithms to four decimal places. Do not use a calculator.

x	$\log(x)$
2	0.3010
3	0.4771
5	0.6990
7	0.8451

a. $\log(9)$

b. $\log\left(\frac{1}{15}\right)$

c. $\log(45{,}000)$

2. Suppose that k is an integer, a is a positive real number, and you know the value of $\log(a)$. Explain how to find the value of $\log(10^k \cdot a^2)$.

Exit Ticket Sample Solutions

1. Use the table below to approximate the following logarithms to four decimal places. Do not use a calculator.

x	$\log(x)$
2	0.3010
3	0.4771
5	0.6990
7	0.8451

a. $\log(9)$

$$\begin{aligned}\log(9) &= \log(3)+\log(3)\\ &\approx 0.4771+0.4771\\ &\approx 0.9542\end{aligned}$$

b. $\log\left(\frac{1}{15}\right)$

$$\begin{aligned}\log\left(\frac{1}{15}\right) &= -\log(15)\\ &= -(\log(3)+\log(5))\\ &\approx -(0.4771+0.6990)\\ &\approx -1,1761\end{aligned}$$

c. $\log(45,000)$

$$\begin{aligned}\log(45,000) &= \log(10^3\cdot 45)\\ &= 3+\log(45)\\ &= 3+\log(5)+\log(9)\\ &\approx 3+0.6990+0.9542\\ &\approx 4.6532\end{aligned}$$

2. Suppose that k is an integer, a is a positive real number, and you know the value of $\log(a)$. Explain how to find the value of $\log(10^k\cdot a^2)$.

Applying the rule for the logarithm of a number multiplied by a power of 10, and then the rule for the logarithm of a product, we have

$$\begin{aligned}\log(10^k\cdot a^2) &= k+\log(a^2)\\ &= k+\log(a)+\log(a)\\ &= k+2\log(a).\end{aligned}$$

Problem Set Sample Solutions

All of the exercises in this problem set should be completed without the use of a calculator.

1. Use the table of logarithms at right to estimate the value of the logarithms in parts (a)–(h).

x	$\log(x)$
2	0.30
3	0.48
5	0.70
7	0.85
11	1.04
13	1.11

a. $\log(25)$

1.40

b. $\log(27)$

1.44

c. $\log(33)$

1.52

d. $\log(55)$

1.74

e. $\log(63)$

1.81

f. $\log(75)$

1.88

g. $\log(81)$

1.92

h. $\log(99)$

2.00

2. **Use the table of logarithms at right to estimate the value of the logarithms in parts (a)–(f).**

a. $\log(350)$

2.55

b. $\log(0.0014)$

-2.85

c. $\log(0.077)$

-1.11

d. $\log(49,000)$

4.70

e. $\log(1.69)$

0.22

f. $\log(6.5)$

0.81

3. **Use the table of logarithms at right to estimate the value of the logarithms in parts (a)–(f).**

a. $\log\left(\frac{1}{30}\right)$

-1.48

b. $\log\left(\frac{1}{35}\right)$

-1.55

c. $\log\left(\frac{1}{40}\right)$

-1.60

d. $\log\left(\frac{1}{42}\right)$

-1.63

e. $\log\left(\frac{1}{50}\right)$

-1.70

f. $\log\left(\frac{1}{64}\right)$

-1.80

4. **Reduce each expression to a single logarithm of the form $\log(x)$.**

a. $\log(5) + \log(7)$

$log(35)$

b. $\log(3) + \log(9)$

$log(27)$

c. $\log(15) - \log(5)$

$log(3)$

d. $\log(8) + \log\left(\frac{1}{4}\right)$

$log(2)$

5. **Use properties of logarithms to write the following expressions involving logarithms of only prime numbers.**

a. $\log(2500)$

$2 + 2\,log(5)$

b. $\log(0.00063)$

$-5 + 2\,log(3) + log(7)$

c. $\log(1250)$

$1 + 3\,log(5)$

d. $\log(26,000,000)$

$6 + log(2) + log(13)$

6. Use properties of logarithms to show that $\log(26) = \log(2) - \log\left(\frac{1}{13}\right)$.

$$\begin{aligned}\log(2) - \log\left(\frac{1}{13}\right) &= \log(2) - \log(13^{-1})\\ &= \log(2) + \log(13)\\ &= \log(26)\end{aligned}$$

7. Use properties of logarithms to show that $\log(3) + \log(4) + \log(5) - \log(6) = 1$.

There are multiple ways to solve this problem.

$$\begin{aligned}\log(3) + \log(4) + \log(5) - \log(6) &= \log(3) + \log(4) + \log(5) + \log\left(\frac{1}{6}\right)\\ &= \log\left(3 \cdot 4 \cdot 5 \cdot \frac{1}{6}\right)\\ &= \log(10)\\ &= 1\end{aligned}$$

OR

$$\begin{aligned}\log(3) + \log(4) + \log(5) &= \log(60)\\ &= \log(10 \cdot 6)\\ &= \log(10) + \log(6)\\ &= 1 + \log(6)\\ \log(3) + \log(4) + \log(5) - \log(6) &= 1\end{aligned}$$

8. Use properties of logarithms to show that $-\log(3) = \log\left(\frac{1}{2} - \frac{1}{3}\right) + \log(2)$.

$$\begin{aligned}\log\left(\frac{1}{2} - \frac{1}{3}\right) + \log(2) &= \log\left(\frac{1}{6}\right) + \log(2)\\ &= -\log(6) + \log(2)\\ &= -\big(\log(2) + \log(3)\big) + \log(2)\\ &= -\log(3)\end{aligned}$$

9. Use properties of logarithms to show that $\log\left(\frac{1}{3} - \frac{1}{4}\right) + \left(\log\left(\frac{1}{3}\right) - \log\left(\frac{1}{4}\right)\right) = -2\log(3)$.

$$\begin{aligned}\log\left(\frac{1}{3} - \frac{1}{4}\right) + \left(\log\left(\frac{1}{3}\right) - \log\left(\frac{1}{4}\right)\right) &= \log\left(\frac{1}{12}\right) + \log\left(\frac{1}{3}\right) - \log\left(\frac{1}{4}\right)\\ &= -\log(12) - \log(3) + \log(4)\\ &= -\big(\log(3) + \log(4)\big) - \log(3) + \log(4)\\ &= -2\log(3)\end{aligned}$$