Lesson 10: Building Logarithmic Tables

Student Outcomes

- Students construct a table of logarithms base 10 and observe patterns that indicate properties of logarithms.

Lesson Notes

In the previous lesson, students were introduced to the concept of the logarithm by finding the power to which we need to raise a base b in order to produce a given number, which we originally called the $\mathrm{WhatPower}_b$ function. In this lesson and the next, students will build their own base 10 logarithm tables using their calculators. By taking the time to construct the table themselves (as opposed to being handed a pre-prepared table), students will have a better opportunity to observe patterns in the table and practice MP.7. These observed patterns will lead to formal statements of the properties of logarithms in upcoming lessons. Using logarithmic properties to rewrite logarithmic expressions satisfies the foundational standard **A-SSE.A.2**.

Materials Needed

Students will need access to a calculator or other technological tool able to compute exponents and logarithms base 10.

Classwork

Opening Exercise (3 minutes)

In this quick Opening Exercise, we ask students to recall the $\mathrm{WhatPower}_b$ function from the previous lesson and reinforce that the logarithm base b is the formal name of the $\mathrm{WhatPower}_b$ function. We will consider only base 10 logarithms in this lesson as we construct our table, so this Opening Exercise is constrained to base 10 calculations.

At the end of this exercise, announce to the students that the notation $\log(x)$ without the little b in the subscript means $\log_{10}(x)$. This is called the *common logarithm.*

> *Scaffolding:*
>
> Prompt struggling students to restate the logarithmic equation $\log_{10}(10^3) = x$ as the exponential equation $10^x = 10^3$.

Opening Exercise

Find the value of the following expressions without using a calculator.

$\text{WhatPower}_{10}(1000)$	$= 3$	$\log_{10}(1000)$	$= 3$
$\text{WhatPower}_{10}(100)$	$= 2$	$\log_{10}(100)$	$= 2$
$\text{WhatPower}_{10}(10)$	$= 1$	$\log_{10}(10)$	$= 1$
$\text{WhatPower}_{10}(1)$	$= 0$	$\log_{10}(1)$	$= 0$
$\text{WhatPower}_{10}\left(\frac{1}{10}\right)$	$= -1$	$\log_{10}\left(\frac{1}{10}\right)$	$= -1$
$\text{WhatPower}_{10}\left(\frac{1}{100}\right)$	$= -2$	$\log_{10}\left(\frac{1}{100}\right)$	$= -2$

Formulate a rule based on your results above: If k is an integer, then $\log_{10}(10^k) =$ _______.

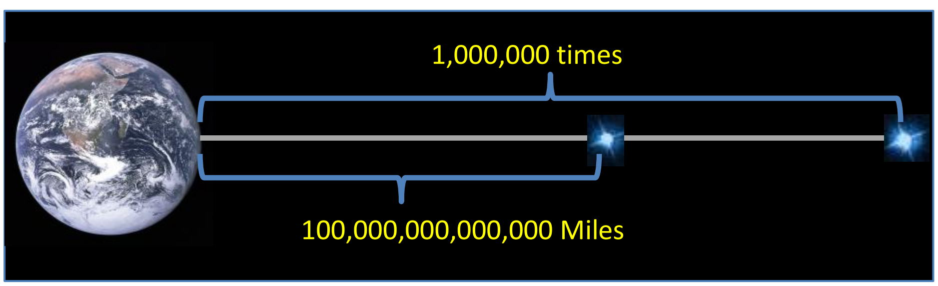
$\log_{10}(10^k) = k$

Example 1 (6 minutes)

In this example, we get our first glimpse of the property $\log_b(xy) = \log_b(x) + \log_b(y)$. Be careful not to give this formula away; by the end of the next lesson, students should have discovered it for themselves.

- Suppose that you are an astronomer, and you measure the distance to a star as $100{,}000{,}000{,}000{,}000$ miles. A second star is collinear with the first star and the Earth and is $1{,}000{,}000$ times farther away from Earth than the first star is. How many miles is the second star from Earth? Note: The figure is not to scale.

Example 1



  - $(100{,}000{,}000{,}000{,}000)(1{,}000{,}000) = 100{,}000{,}000{,}000{,}000{,}000{,}000$, *so the second star is* 100 *quintillion miles away from Earth.*
- How did you arrive at that figure?
  - *I counted the zeros; there are* 14 *zeros in* $100{,}000{,}000{,}000{,}000$ *and* 6 *zeros in* $1{,}000{,}000$, *so there must be* 20 *zeros in the product.*
- Can we restate that in terms of exponents?
  - $(10^{14})(10^6) = 10^{20}$
- How are the exponents related?
  - $14 + 6 = 20$

- What are $\log(10^{14})$, $\log(10^{6})$, and $\log(10^{20})$?
  - 14, 6, *and* 20
- In this case, can we state an equivalent expression for $\log(10^{14} \cdot 10^{6})$?
  - $\log(10^{14} \cdot 10^{6}) = \log(10^{14}) + \log(10^{6})$
- Why is this equation true?
  - $\log(10^{14} \cdot 10^{6}) = \log(10^{20}) = 20 = 14 + 6 = \log(10^{14}) + \log(10^{6})$
- Generalize to find an equivalent expression for $\log(10^{m} \cdot 10^{n})$ for integers m and n. Why is this equation true?
  - $\log(10^{m} \cdot 10^{n}) = \log(10^{m+n}) = m + n = \log(10^{m}) + \log(10^{n})$
  - *This equation is true because when we multiply powers of* 10 *together, the resulting product is a power of* 10 *whose exponent is the sum of the exponents of the factors.*
- Keep this result in mind as we progress through the lesson.

MP.8

Exercises 1–6 (8 minutes)

Historically, logarithms were calculated using tables because there were no calculators or computers to do the work. Every scientist and mathematician kept a book of logarithmic tables on hand to use for calculation. It is very easy to find the value of a base 10 logarithm for a number that is a power of 10, but what about for the other numbers? In this exercise, students will find an approximate value of $\log(30)$ using exponentiation, the same way we approximated $\log_2(10)$ in Lesson 6. After this exercise, we will rely on the logarithm button on the calculator to compute logarithms base 10 for the remainder of this lesson. Emphasize to the students that logarithms are generally irrational numbers, so that the results produced by the calculator are only decimal approximations. As such, we should be careful to use the approximation symbol, $\approx$, when writing out a decimal expansion of a logarithm.

Exercises

1. **Find two consecutive powers of 10 so that 30 is between them. That is, find an integer exponent k so that $10^{k} < 30 < 10^{k+1}$.**

   Since $10 < 30 < 100$, *we have* $k = 1$.

2. **From your result in Exercise 1, $\log(30)$ is between which two integers?**

   Since 30 *is some power of* 10 *between* 1 *and* 2, $1 < \log(30) < 2$.

3. **Find a number k to one decimal place so that $10^{k} < 30 < 10^{k+0.1}$, and use that to find under and over estimates for $\log(30)$.**

   Since $10^{1.4} \approx 25.1188$ *and* $10^{1.5} \approx 31.6228$, *we have* $10^{1.4} < 30 < 10^{1.5}$. *Then* $1.4 < \log(30) < 1.5$, *and* $k \approx 1.4$.

4. **Find a number k to two decimal places so that $10^{k} < 30 < 10^{k+0.01}$, and use that to find under and over estimates for $\log(30)$.**

   Since $10^{1.47} \approx 29.5121$, *and* $10^{1.48} \approx 30.1995$, *we have* $10^{1.47} < 30 < 10^{1.48}$ *so that* $1.47 < \log(30) < 1.48$. *So,* $k \approx 1.47$.

5. **Repeat this process to approximate the value of $\log(30)$ to 4 decimal places.**

   Since $10^{1.477} \approx 29.9916$, *and* $10^{1.478} \approx 30.0608$, *we have* $10^{1.477} < 30 < 10^{1.478}$ *so that* $1.477 < \log(30) < 1.478$.

   Since $10^{1.4771} \approx 29.9985$, *and* $10^{1.4772} \approx 30.0054$, *we have* $10^{1.4771} < 30 < 10^{1.4772}$ *so that* $1.4771 < \log(30) < 1.4772$.

   Since $10^{1.47712} \approx 29.99991$, *and* $10^{1.47713} \approx 30.0006$, *we have* $10^{1.47712} < 30 < 10^{1.47713}$ *so that* $1.47712 < \log(30) < 1.47713$.

   Thus, to four decimal places, $\log(30) \approx 1.4771$.

6. **Verify your result on your calculator, using the LOG button.**

   The calculator gives $\log(30) \approx 1.477121255$.

Discussion (1 minute)

In the next exercises, students will use their calculators to create a table of logarithms that they will analyze to look for patterns that will lead to the discovery of the logarithmic properties. The process of identifying and generalizing the observed patterns provides students with an opportunity to practice MP.7.

- Historically, since there were no calculators or computers, logarithms were calculated using a complicated algorithm involving multiple square roots. Thankfully, we have calculators and computers to do this work for us now.
- We will use our calculators to create a table of values of base 10 logarithms. Once the table is made, see what patterns you can observe.

Exercises 7–10 (6 minutes)

Put students in pairs or small groups, but have students work individually to complete the table in Exercise 7. Before working on Exercises 8–10 in groups, have students check their tables against each other. You may need to remind students that $\log(x)$ means $\log_{10}(x)$.

7. **Use your calculator to complete the following table. Round the logarithms to 4 decimal places.**

x	$\log(x)$
1	0
2	0.3010
3	0.4771
4	0.6021
5	0.6990
6	0.7782
7	0.8451
8	0.9031
9	0.9542

x	$\log(x)$
10	1
20	1.3010
30	1.4771
40	1.6021
50	1.6990
60	1.7782
70	1.8451
80	1.9031
90	1.9542

x	$\log(x)$
100	2
200	2.3010
300	2.4771
400	2.6021
500	2.6990
600	2.7782
700	2.8451
800	2.9031
900	2.9542

MP.7

8. What pattern(s) can you see in the table from Exercise 7 as x is multiplied by 10? Write the pattern(s) using logarithmic notation.

   I found the patterns $\log(10x) = 1 + \log(x)$ *and* $\log(100x) = 2 + \log(x)$. *I also noticed that* $\log(100x) = 1 + \log(10x)$.

9. What pattern would you expect to find for $\log(1000x)$? Make a conjecture and test it to see whether or not it appears to be valid.

   I would guess that the values of $\log(1000x)$ *will all start with* 3. *That is,* $\log(1000x) = 3 + \log(x)$. *This appears to be the case since* $\log(2000) \approx 3.3010$, $\log(500) \approx 3.6990$, *and* $\log(800) \approx 3.9031$.

10. Use your results from Exercises 8 and 9 to make a conjecture about the value of $\log(10^k \cdot x)$ for any positive integer k.

    It appears that $\log(10^k \cdot x) = k + \log(x)$, *for any positive integer* k.

Discussion (3 minutes)

Ask groups to share the patterns they observed in Exercise 8 and the conjectures they made in Exercises 9 and 10. Ensure that all students have the correct conjectures recorded in their notebooks or journals before continuing to the next set of exercises, which extend the result from Exercise 10 to all integers k (and not just positive values of k).

Exercises 11–14 (8 minutes)

In this set of exercises, students will discover a rule for calculating logarithms of the form $\log(10^k \cdot x)$, where k is any integer. Have students again work individually to complete the table in Exercise 11 and to check their tables against each other before they proceed to discuss and answer Exercises 12–14 in groups.

Scaffolding:

If students are having difficulty seeing the pattern in the table for Exercise 12, nudge them to add together $\log(x)$ and $\log\left(\frac{x}{10}\right)$ for some values of x in the table.

11. Use your calculator to complete the following table. Round the logarithms to 4 decimal places.

x	$\log(x)$	x	$\log(x)$	x	$\log(x)$
1	0	0.1	-1	0.01	-2
2	0.3010	0.2	-0.6990	0.02	-1.6990
3	0.4771	0.3	-0.5229	0.03	-1.5229
4	0.6021	0.4	-0.3979	0.04	-1.3979
5	0.6990	0.5	-0.3010	0.05	-1.3010
6	0.7782	0.6	-0.2218	0.06	-1.2218
7	0.8451	0.7	-0.1549	0.07	-1.1549
8	0.9031	0.8	-0.0969	0.08	-1.0969
9	0.9542	0.9	-0.0458	0.09	-1.0458

MP.7

12. What pattern(s) can you see in the table from Exercise 11? Write them using logarithmic notation.

    I found the patterns $\log(x) - \log\left(\frac{x}{10}\right) = 1$, *which can be written as* $\log\left(\frac{x}{10}\right) = -1 + \log(x)$, *and* $\log\left(\frac{x}{100}\right) = -2 + \log(x)$.

MP.7

13. **What pattern would you expect to find for $\log\left(\frac{x}{1000}\right)$? Make a conjecture, and test it to see whether or not it appears to be valid.**

I would guess that the values of $\log\left(\frac{x}{1000}\right)$ will all start with -2, and that $\log\left(\frac{x}{1000}\right) = -3 + \log(x)$. This appears to be the case since $\log(0.002) \approx -2.6990$, and $-2.6990 = -3 + 0.3010$; $\log(0.005) \approx -2.3010$, and $-2.3010 = -3 + 0.6990$; $\log(0.008) \approx -2.0969$, and $-2.0969 = -3 + 0.9031$.

14. **Combine your results from Exercises 10 and 12 to make a conjecture about the value of the logarithm for a multiple of a power of 10; that is, find a formula for $\log(10^k \cdot x)$ for any integer k.**

It appears that $\log(10^k \cdot x) = k + \log(x)$, for any integer k.

Discussion (2 minutes)

Ask groups to share the patterns they observed in Exercise 12 and the conjectures they made in Exercises 13 and 14 with the class. Ensure that all students have the correct conjectures recorded in their notebooks or journals before continuing to the next example.

Examples 2–3 (2 minutes)

Lead the class through these calculations. You may decide to let them work on Example 3 either alone or in groups after you have led them through Example 2.

Example 2

Use the logarithm tables and the rules we discovered to calculate $\log(40000)$ to 4 decimal places.

$$\begin{aligned}\log(40000) &= \log(10^4 \cdot 4)\\ &= 4 + \log(4)\\ &\approx 4.6021\end{aligned}$$

Example 3

Use the logarithm tables and the rules we discovered to calculate $\log(0.000004)$ to 4 decimal places.

$$\begin{aligned}\log(0.000004) &= \log(10^{-6} \cdot 4)\\ &= -6 + \log(4)\\ &\approx -5.3979\end{aligned}$$

Closing (2 minutes)

Ask students to summarize the important parts of the lesson, either in writing, to a partner, or as a class. Use this as an opportunity to informally assess understanding of the lesson. The following are some important summary elements:

Lesson Summary

- **The notation $\log(x)$ is used to represent $\log_{10}(x)$.**
- **For integers k, $\log(10^k) = k$.**
- **For integers m and n, $\log(10^m \cdot 10^n) = \log(10^m) + \log(10^n)$.**
- **For integers k and positive real numbers x, $\log(10^k \cdot x) = k + \log(x)$.**

Exit Ticket (4 minutes)

Name ________________________________ Date ________________

Lesson 10: Building Logarithmic Tables

Exit Ticket

1. Use the log table below to approximate the following logarithms to four decimal places. Do not use a calculator.

x	$\log(x)$
1	0.0000
2	0.3010
3	0.4771
4	0.6021
5	0.6990

x	$\log(x)$
6	0.7782
7	0.8451
8	0.9031
9	0.9542
10	1.0000

   a. $\log(500)$

   b. $\log(0.0005)$

2. Suppose that A is a number with $\log(A) = 1.352$.
   a. What is the value of $\log(1000A)$?

   b. Which of the following is true? Explain how you know.
      i. $A < 0$
      ii. $0 < A < 10$
      iii. $10 < A < 100$
      iv. $100 < A < 1000$
      v. $A > 1000$

Exit Ticket Sample Solutions

1. Use the log table below to approximate the following logarithms to four decimal places. Do not use a calculator.

x	$\log(x)$
1	0.0000
2	0.3010
3	0.4771
4	0.6021
5	0.6990

x	$\log(x)$
6	0.7782
7	0.8451
8	0.9031
9	0.9542
10	1.0000

a. $\log(500)$

$$\begin{aligned}\log(500) &= \log(10^2 \cdot 5)\\ &= 2 + \log(5)\\ &\approx 2.6990\end{aligned}$$

b. $\log(0.0005)$

$$\begin{aligned}\log(0.0005) &= \log(10^{-4} \cdot 5)\\ &= -4 + \log(5)\\ &\approx -3.3010\end{aligned}$$

2. Suppose that A is a number with $\log(A) = 1.352$.

a. What is the value of $\log(1000A)$?

$\log(1000A) = \log(10^3 A) = 3 + \log(A) = 4.352$

b. Which of the following statements is true? Explain how you know.

i. $A < 0$

ii. $0 < A < 10$

iii. $10 < A < 100$

iv. $100 < A < 1000$

v. $A > 1000$

Because $\log(A) = 1.352 = 1 + 0.352$, A is greater than 10 and less than 100. Thus, (iii) is true. In fact, from the table above, we can narrow the value of A down to between 20 and 30 because $\log(20) \approx 1.3010$, and $\log(30) \approx 1.4771$.

Problem Set Sample Solutions

These problems should be solved without a calculator.

1. **Complete the following table of logarithms without using a calculator; then, answer the questions that follow.**

x	$\log(x)$
$1,000,000$	6
$100,000$	5
$10,000$	4
1000	3
100	2
10	1

x	$\log(x)$
0.1	-1
0.01	-2
0.001	-3
0.0001	-4
0.00001	-5
0.000001	-6

a. **What is $\log(1)$? How does that follow from the definition of a base 10 logarithm?**

Since $10^0 = 1$, we know that $\log(1) = 0$.

b. **What is $\log(10^k)$ for an integer k? How does that follow from the definition of a base 10 logarithm?**

By the definition of the logarithm, we know that $\log(10^k) = k$.

c. **What happens to the value of $\log(x)$ as x gets really large?**

For any $x > 1$, there exists $k > 0$ so that $10^k \leq x < 10^{k+1}$. As x gets really large, k gets large. Since $k \leq \log(x) < k + 1$, as k gets large, $\log(x)$ gets large.

d. **For $x > 0$, what happens to the value of $\log(x)$ as x gets really close to zero?**

For any $0 < x < 1$, there exists $k > 0$ so that $10^{-k} \leq x < 10^{-k+1}$. Then $-k \leq x < -k + 1$. As x gets closer to zero, k gets larger. Thus, $\log(x)$ is negative, and $|\log(x)|$ gets large as the positive number x gets close to zero.

2. **Use the table of logarithms below to estimate the values of the logarithms in parts (a)–(h).**

x	$\log(x)$
2	0.3010
3	0.4771
5	0.6990
7	0.8451
11	1.0414
13	1.1139

a. $\log(70,000)$

4.8451

b. $\log(0.0011)$

-2.9586

c. $\log(20)$

1.3010

d. $\log(0.00005)$

-4.3010

e. $\log(130,000)$

5.1139

f. $\log(3000)$

3.4771

g. $\log(0.07)$

-1.1549

h. $\log(11,000,000)$

7.0414

3. If $\log(n) = 0.6$, find the value of $\log(10n)$.

$\log(10n) = 1.6$

4. If m is a positive integer and $\log(m) \approx 3.8$, how many digits are there in m? Explain how you know.

Since $3 < \log(m) < 4$, we know $1,000 < m < 10,000$; therefore, m has 4 digits.

5. If m is a positive integer and $\log(m) \approx 9.6$, how many digits are there in m? Explain how you know.

Since $9 < \log(m) < 10$, we know $10^9 < m < 10^{10}$; therefore, m has 10 digits.

6. Vivian says $\log(452,000) = 5 + \log(4.52)$, while her sister Lillian says that $\log(452,000) = 6 + \log(0.452)$. Which sister is correct? Explain how you know.

Both sisters are correct. Since $452,000 = 4.52 \cdot 10^5$, we can write $\log(452,000) = 5 + \log(4.52)$. However, we could also write $452,000 = 0.452 \cdot 10^6$, so $\log(452,000) = 6 + \log(0.452)$. Both calculations give $\log(452,000) \approx 5.65514$.

7. Write the logarithm base 10 of each number in the form $k + \log(x)$, where k is the exponent from the scientific notation, and x is a positive real number.

a. 2.4902×10^4

$4 + \log(2.4902)$

b. 2.58×10^{13}

$13 + \log(2.58)$

c. 9.109×10^{-31}

$-31 + \log(9.109)$

8. For each of the following statements, write the number in scientific notation, and then write the logarithm base 10 of that number in the form $k + \log(x)$, where k is the exponent from the scientific notation, and x is a positive real number.

a. The speed of sound is 1116 ft/s..

$1116 = 1.116 \times 10^3$, *so* $\log(1116) = 3 + \log(1.116)$.

b. The distance from Earth to the Sun is 93 million miles.

$93,000,000 = 9.3 \times 10^7$, *so* $\log(93,000,000) = 7 + \log(9.3)$.

c. The speed of light is $29,980,000,000$ $\mathrm{cm/s}$.

$29,980,000,000 = 2.998 \times 10^{10}$, so $\log(29,980,000,000) = 10 + \log(2.998)$.

d. The weight of the earth is $5,972,000,000,000,000,000,000,000$ kg.

$5,972,000,000,000,000,000,000,000 = 5.972 \times 10^{24}$, so $\log(5,972,000,000,000,000,000,000,000) = 24 + \log(5.972)$.

e. The diameter of the nucleus of a hydrogen atom is 0.00000000000000175 m.

$0.00000000000000175 = 1.75 \times 10^{-15}$, so $\log(0.00000000000000175) = -15 + \log(1.75)$.

f. For each part (a)–(e), you have written each logarithm in the form $k + \log(x)$, for integers k and positive real numbers x. Use a calculator to find the values of the expressions $\log(x)$. Why are all of these values between 0 and 1?

$\log(1.116) \approx 0.047664$

$\log(9.3) \approx 0.968483$

$\log(2.998) \approx 0.476832$

$\log(5.972) \approx 0.77612$

$\log(1.75) \approx 0.243038$

These values are all between 0 and 1 because x is between 1 and 10. We can rewrite $1 < x < 10$ as $10^0 < x < 10^1$. If we write $x = 10^L$ for some exponent L, then $10^0 < 10^L < 10^1$, so $0 < L < 1$. This exponent L is the base 10 logarithm of x.