Lesson 9: Logarithms—How Many Digits Do You Need?

Student Outcomes

- Students use logarithms to determine how many characters are needed to generate unique identification numbers in different scenarios.
- Students understand that logarithms are useful when relating the number of digits in a number to the magnitude of the number and that base 10 logarithms are useful when measuring quantities that have a wide range of values such as the magnitude of earthquakes, volume of sound, and pH levels in chemistry.

Lesson Notes

In this lesson, students learn that logarithms are useful in a wide variety of situations but have extensive application when we want to generate a list of unique identifiers for a population of a given size (**N-Q.A.2**). This application of logarithms is used in computer programming, when determining how many digits are needed in a phone number to have enough unique numbers for a population, and more generally when assigning a scale to any quantity that has a wide range of values.

In this lesson, students make sense of a simple scenario and then see how it can be applied to other real world situations (MP.1 and MP.2). They observe and extend patterns to formulate a model (MP.7 and MP.4). They reason about and make sense of situations in context and use logarithms to draw conclusions regarding different real world scenarios (MP.3 and MP.4).

Classwork

Opening Exercise (2 minutes)

Remind students that the WhatPower expressions are called logarithms, and announce that we will be using logarithms to help us make sense of and solve some real world problems.

Students briefly convert two WhatPower expressions into a logarithmic expression and evaluate the result.

Opening Exercise

a. **Evaluate $\text{WhatPower}_2(8)$. State your answer as a logarithm and evaluate it.**

$\log_2(8) = 3$

b. **Evaluate $\text{WhatPower}_5(625)$. State your answer as a logarithm and evaluate it.**

$\log_5(625) = 4$

If students struggle with these exercises, you may want to plan for some additional practice on problems like those found in Lesson 7.

Exploratory Challenge (15 minutes)

Divide students up into small groups, and give them about ten minutes to work through the questions that follow. As you circulate around the room, encourage students to be systematic when assigning IDs to the club members. If a group is stuck, you can ask questions to help move the group along.

- Remember, we only want to use A's and B's, but two-character IDs only provide enough for four people. Can you provide an example of a three-character ID using only the letters A and B?
  - *A three-character ID might be ABA or AAA.*
- How many different IDs would three characters make? How could you best organize your results?
  - *Three characters would make 8 IDs because I could take the four I already have and add an A or B onto the end. The best way to organize it is to take the existing two-character IDs and add an A and then take those existing two-character IDs again and add a B.*

Exploratory Challenge

Autumn is starting a new club with eight members including herself. She wants everyone to have a secret identification code made up of only A's and B's. For example, using two characters, her ID code could be AB, which also happens to be her initials.

a. Using A's and B's, can Autumn assign each club member a unique two character ID using only A's and B's? Justify your answer. Here's what Autumn has so far.

Club Member Name	Secret ID
Autumn	AA
Kris	
Tia	
Jimmy	

Club Member Name	Secret ID
Robert	
Jillian	
Benjamin	
Scott	

No, she cannot assign a unique 2-character ID to each member. The only codes available are AA, BA, AB, and BB, and there are eight people.

b. Using A's and B's, how many characters would be needed to assign each club member a unique ID code? Justify your answer by showing the IDs you would assign to each club member by completing the table above (adjust Autumn's ID if needed).

You would need three characters in each code. A completed table is shown below. Students could assign one of the unique codes to any club member so this is not the only possible solution.

Club Member Name	Secret ID
Autumn	*AAA*
Kris	*BAA*
Tia	*ABA*
Jimmy	*BBA*

Club Member Name	Secret ID
Robert	*AAB*
Jillian	*BAB*
Benjamin	*ABB*
Scott	*BBB*

MP.7 & MP.4

When the club grew to 16 members, Autumn started noticing a pattern.

Using A's and B's:

i. **Two people could be given a secret ID with 1 letter: A and B.**
ii. **Four people could be given a secret ID with 2 letters: AA, BA, AB, BB.**
iii. **Eight people could be given a secret ID with 3 letters: AAA, BAA, ABA, BBA, AAB, BAB, ABB, BBB.**

c. **Complete the following statement and list the secret IDs for the 16 people.**

16 people could be given a secret ID with _________ letters using A's and B's.

16 people could be given a secret ID with 4 characters. Notice the original members have their original three-character code with an A added to the end. Then, the newer members have the original three-character codes with a B added to the end.

Club Member Name	Secret ID	Club Member Name	Secret ID
Autumn	*AAAA*	Gwen	*AAAB*
Kris	*BAAA*	Jerrod	*BAAB*
Tia	*ABAA*	Mykel	*ABAB*
Jimmy	*BBAA*	Janette	*BBAB*
Robert	*AABA*	Nellie	*AABB*
Jillian	*BABA*	Serena	*BABB*
Benjamin	*ABBA*	Ricky	*ABBB*
Scott	*BBBA*	Mia	*BBBB*

d. **Describe the pattern in words. What type of function could be used to model this pattern?**

The number of people in the club is a power of 2. The number of characters needed to generate a unique ID using only two characters is the exponent of the power of 2.

$$\log_2(2) = 1$$

$$\log_2(4) = 2$$

$$\log_2(8) = 3$$

$$\log_2(16) = 4$$

A logarithm function could be used to model this pattern. For 16 people, you will need a four-character ID.

To debrief this Exploratory Challenge, have different groups explain how they arrived at their solutions. If a group does not demonstrate an efficient way to organize its answers when the club membership increases, be sure to show it to the class. For example, the 16 group member IDs were generated by adding an A onto the end of the original 8 IDs and then adding a B onto the end of the original 8 IDs, as shown in the solutions above.

Exercises 1–2 (3 minutes)

Give students a few minutes to answer these questions individually or in groups. Check to see if students are using logarithms when they explain their solutions. If they are not, be sure to review the answers with the entire class using logarithm notation.

MP.2 & MP.3

Exercises 1–2

In the previous problems, the letters A and B were like the digits in a number. A four-digit ID for Autumn's club could be any four-letter arrangement of A's and B's because in her ID system, the only digits are the letters A and B.

1. **When Autumn's club grows to include more than 16 people, she will need five digits to assign a unique ID to each club member. What is the maximum number of people that could be in the club before she needs to switch to a six-digit ID? Explain your reasoning.**

   Since $\log_2(32) = 5$ and $\log_2(64) = 6$, she will need to switch to a six-digit ID when the club gets more than 32 members.

2. **If Autumn has 256 members in her club, how many digits would she need to assign each club member a unique ID using only A's and B's? Show how you got your answers.**

   She will need 8 digits because $\log_2(256) = 8$.

Discussion (10 minutes)

Computers store keyboard characters, such as 1, 5, X, x, Q, @, /, and &, using an identification system much like Autumn's system called ASCII, which stands for American Standard Code for Information Interchange. We pronounce the acronym ASCII as "as-key." Each character in a font list on the computer is given an ID that a computer can recognize. A computer is essentially a lot of electrical switches, which can be in one of two states, ON or OFF, just like Autumn's A's and B's.

There are usually 256 characters in a font list, so using the solution to Exercise 2 above, a computer needs eight positions or digits to encode each character in a font list.

- For example, the standard ASCII code for uppercase P is 01010000. If A is zero and B is 1, how would uppercase P be encoded using Autumn's code?
  - *Using A's and B's in Autumn's code, this would be ABABAAAA.*
- How would the computer read this code?
  - *The computer reads the code as "on, off, on, off, on, on, on, on."*

If time permits, a quick Internet search for the term *ASCII* will return web pages where you can view the standard code for different keyboard symbols. Each character in the ASCII code is called a *bit*, and the entire 8-character code is called a *byte*. Each byte is made up of eight bits, and each byte describes a unique character in the font list such as a P, p, %, _, 4, etc.

- When computer saves a basic text document and reports that it is 3,242 bytes, what do you think that means?
  - *It means that there would be $3{,}242$ letters, symbols, spaces, punctuation marks, etc. in the document.*

We have seen how to create unique IDs using two letters for both Autumn's secret club and to encode text characters in a way that is readable by a computer using ASCII code. Next, we will examine why a logarithm really is the right operation to describe the number of characters or digits needed to create unique identifiers for people by exploring some real-world situations where people are assigned a number.

Scaffolding:

Use a word wall or a chart to provide a visual reference for English language learners using the academic terms related to computers.

ASCII (American Standard Code for Information Exchange, acronym pronounced "as-key")

Bit (one of eight positions in a byte)

Byte (a unique eight-character identifier for each ASCII symbol in font list)

Scaffolding:

Ask advanced learners to quickly estimate how many digits would be needed to generate ID numbers for the number of students enrolled in your school or in your school district. For example, a school district with more than 10,000 students would need at least a five-digit ID.

Example 1 (5 minutes)

This is a simplified example that switches students to using base 10 logarithms because we are going to be assigning IDs using the digits 0–9. Give students a few minutes to think about this answer and have them discuss their ideas with a partner. Most students will likely say let the first person be 0, the next person be 1, and so on up to 999. Then, tie the solution to logarithms. Since there are 10 symbols (digits), we can use $\log_{10}(1000) = 3$ to find the answer, which just counts the number of digits needed to count to 999.

- How can a logarithm help you determine the solution quickly?
  - *The logarithm counts the number of digits needed because each time we add another digit to our numbers, we are increasing by a factor of 10. For example, $1 = 10^0$, $10 = 10^1$, $100 = 10^2$, etc.*

Example 1

A thousand people are given unique identifiers made up of the digits $0, 1, 2, \dots, 9$. How many digits would be needed for each ID number?

You would just need three digits: $\{000, 001, 002, \dots, 099, 100, 101, \dots 998, 999\}$.

Using logarithms, we need to determine the value of $\log(1000)$, which is 3. This will quickly tell us the number of digits needed to uniquely identify any range of numbers. You can follow up by asking students to extend their thinking.

- When would you need to switch from four to five digits to assign unique numbers to a population?
  - *You could assign up to 10^4 people a four-digit ID, which would be $10{,}000$ people. Once you exceeded that number, you would need five digits to assign each person a unique number.*

Exercises 3–4 (5 minutes)

Students should return to their small groups to work these exercises. Have different groups present their solutions to the whole class after a few minutes. Discuss different approaches, and make sure that students see the power of using a logarithm to help them quickly solve or justify a solution to the problem.

MP.1 & MP.3

Exercises 3–4

3. **There are approximately 317 million people in the United States. Compute and use $\log(100{,}000{,}000)$ and $\log(1{,}000{,}000{,}000)$ to explain why Social Security numbers are 9 digits long.**

   We know that $\log(100{,}000{,}000) = 8$, which is the number of digits needed to assign an ID to 100 million people. We know that $\log(1{,}000{,}000{,}000) = 9$, which is the number of digits needed to assign an ID to 1 billion people.

   The United States government will not need to increase the number of digits in a Social Security number until the United States population reaches one billion.

4. **There are many more telephones than the number of people in the United States because of people having home phones, cell phones, business phones, fax numbers, etc. Assuming we need at most 10 billion phone numbers in the United States, how many digits would be needed so that each phone number is unique? Is this reasonable? Explain.**

   Since $\log(10{,}000{,}000{,}000) = 10$, you would need a ten-digit phone number in order to have ten billion unique numbers. Phone numbers in the United States are 10 digits long. If you divide 10 billion by 317 million (the number of people in the United States), that would allow for approximately 31 phone numbers per person. That is plenty of numbers for individuals to have more than one number, leaving many additional numbers for businesses and the government.

Closing (2 minutes)

Ask students to respond to the following statements in writing or with a partner. Share a few answers to close the lesson before students begin the Exit Ticket. Preview other situations where logarithms are useful, such as the Richter scale for measuring the magnitude of an earthquake.

- To increase the value of $\log_2(x)$ by 1, you would multiply x by 2. To increase the value of $\log_{10}(x)$ by 1, you would multiply x by 10. How does this idea apply to the situations in today's lesson?
  - *We saw that each time the population of Autumn's club doubled, we needed to increase the total number of digits needed for the ID numbers by* 1*. We saw that since the population of the United States was between* 100 *million and* 1 *billion, we only needed* 9 *digits (*$\log(1{,}000{,}000{,}000)$*) to generate a Social Security number.*
- Situations like the ones in today's lesson can be modeled with logarithms. Can you think of a situation besides the ones we discussed today where it would make sense to use logarithms?
  - *Any time a measurement can take on a wide range of values, such as the magnitude of an earthquake or the volume (decibel) level of sounds, a logarithm could be used to model the situation.*

Exit Ticket (3 minutes)

Name ____ Date ____________________

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Exit Ticket

A brand new school district needs to generate ID numbers for its student body. The district anticipates a total enrollment of $75{,}000$ students within the next ten years. Will a five-digit ID number comprising the symbols $0, 1, \ldots, 9$ be enough? Explain your reasoning.

Exit Ticket Sample Solutions

A brand new school district needs to generate ID numbers for its student body. The district anticipates a total enrollment of $75,000$ students within the next ten years. Will a five-digit ID number comprised of the symbols $0, 1, \dots, 9$ be enough? Explain your reasoning.

$\log(10,000) = 4$ and $\log(100,000) = 5$, so 5 digits should be enough. However, students who enter school at the kindergarten level in the tenth year of this numbering scheme would need to keep their IDs for 13 years. Dividing $75,000$ by 13 shows there would be roughly $6,000$ students per grade. Adding that many students per year would take the number of needed IDs at any one time over $100,000$ in just a few more years. The district should probably use a six-digit ID number.

Problem Set Sample Solutions

1. The student body president needs to assign each officially sanctioned club on campus a unique ID number for purposes of tracking expenses and activities. She decides to use the letters A, B, and C to create a unique three-character code for each club.

   a. How many clubs can be assigned a unique ID according to this proposal?

      Since $\log_3(27) = 3$, the president could have 27 clubs according to this proposal.

   b. There are actually over 500 clubs on campus. Assuming the student body president still wants to use the letters A, B, and C, how many characters would be needed to generate a unique ID for each club?

      We need to estimate $\log_3(500)$. Since $3^5 = 243$ and $3^6 = 729$, she could use a six-character combination of letters and have enough unique IDs for up to 729 clubs.

2. Can you use the numbers 1, 2, 3, and 4 in a combination of four digits to assign a unique ID to each of 500 people? Explain your reasoning.

$$\log_4(4) = 1$$
$$\log_4(16) = 2$$
$$\log_4(64) = 3$$
$$\log_4(256) = 4$$
$$\log_4(1024) = 5$$

   No, you would need to use a five-digit ID using combinations of 1s, 2s, 3s and 4s such as 11111 or 12341, or you could use the numbers 1 to 5 in four characters such as 1231, 1232, 1233, 1234, 1235, etc. because $\log_5(625) = 4$.

MP.1

3. Automobile license plates typically have a combination of letters (26) and numbers (10). Over time, the state of New York has used different criteria to assign vehicle license plate numbers.

   a. From 1973 to 1986, the state used a 3-letter and 4-number code where the three letters indicated the county where the vehicle was registered. Essex County had 13 different 3-letter codes in use. How many cars could be registered to this county?

      Since $\log(10,000) = 4$, the 4-digit code could be used to register up to $10,000$ vehicles. Multiply that by 13 different county codes, and up to $130,000$ vehicles could be registered to Essex County.

MP.1

b. Since 2001, the state has used a 3-letter and 4-number code but no longer assigns letters by county. Is this coding scheme enough to register 10 million vehicles?

Since $\log_{26}(x) = 3$ when $x = 26^3 = 17,576$, there are $17,576$ three-letter codes. Since $\log(10000) = 4$, there are $10,000$ four-digit codes. Multiply $17576 \cdot 10,000 = 175,760,000$, and we see that this scheme generates over 100 million license plate numbers.

4. The Richter scale uses common (base 10) logarithms to assign a magnitude to an earthquake that is based on the amount of force released at the earthquake's source as measured by seismographs in various locations.

a. Explain the difference between an earthquake that is assigned a magnitude of 5 versus one assigned a magnitude of 7.

The difference between 5 and 7 is 2, so a magnitude of 7 would be 10^2, or 100, times greater force.

b. A magnitude 2 earthquake can usually be felt by multiple people who are located near the earthquake's origin. The largest recorded earthquake was magnitude 9.5 in Chile in 1960. How many times greater force than a magnitude 2 earthquake was the largest recorded earthquake?

The difference between 2 and 9.5 is 7.5, so it would be about $10^{7.5}$ times the force. This is approximately 31 million times greater force.

c. What would be the magnitude of an earthquake whose force was 1000 times greater than a magnitude 4.3 quake?

Since $10^3 = 1000$, the magnitude would be the sum of 4.3 and 3, which is 7.3.

MP.2 & MP.3

5. Sound pressure level is measured in decibels (dB) according to the formula $L = 10\log\left(\frac{I}{I_0}\right)$, where I is the intensity of the sound and I_0 is a reference intensity that corresponds to a barely perceptible sound.

a. Explain why this formula would assign 0 decibels to a barely perceptible sound.

If we let $I = I_0$, then

$$L = 10\log\left(\frac{I}{I_0}\right)$$
$$L = 10\log(1)$$
$$L = 10 \cdot 0$$
$$L = 0.$$

Therefore, the reference intensity is always 0 dB.

b. Decibel levels above 120 dB can be painful to humans. What would be the intensity that corresponds to this level?

$$120 = 10\log\left(\frac{I}{I_0}\right)$$
$$1.2 = \log\left(\frac{I}{I_0}\right)$$
$$\frac{I}{I_0} = 10^{1.2}$$
$$I \approx 15.8I_0$$

From this equation, we can see that the intensity is about 16 times greater than barely perceptible sound.