



Lesson 6: Euler's Number, e

Student Outcomes

- Students write an exponential function that represents the amount of water in a tank after t seconds if the height of the water doubles every 10 seconds.
- Students discover Euler's number e by numerically approaching the constant for which the height of water in a tank equals the rate of change of the height of the water in the tank.
- Students calculate the average rate of change of a function.

Lesson Notes

Leonhard Euler (pronounced "Oiler"), 1707–1783, was a prolific Swiss mathematician and physicist who made many important contributions to mathematics such as much of our modern terminology and notation, including function notation and popularizing the use of π to represent the circumference of a circle divided by its diameter. Euler also discovered many properties of the irrational number e , which is now known as Euler's number. Euler's number naturally occurs in various applications, and a comparison can be made to π , which also occurs naturally. During the lesson, students determine an explicit expression for the height of water in a water tank from its context (**F-BF.A.1a**) and calculate the average rate of change over smaller and smaller intervals to create a sequence that converges to e (**F-IF.B.6**). It is important to stress that the water tank exploration is a way to *define* e . Yes it is remarkable, but when students discover it, the teacher's reaction should not be "Ta da! It's magic!" Instead, the teacher should stress that students have defined this special constant (similar to how π is defined as the ratio of any circle's circumference to its diameter) that will be used extensively in the near future and occurs in many different applications.

Classwork

Exercises 1–3 (8 minutes)

In these exercises, students find exponential equations that model the increasing height of water in a cylindrical tank as it doubles over a fixed time interval. These preliminary exercises will lead to the discovery of Euler's number, e , at the end of the lesson. As a demonstration, show students the 47-second video in which the height of water in a tank doubles repeatedly until it fills the tank completely; note how long it takes for the height to appear to change at all. Although this situation is contrived, it provides a good visual representation of the power of exponential growth. This is a good time to discuss constraints and how quantities cannot realistically increase exponentially without bound due to physical constraints to the growth. In this case, the water tank has a finite volume and there is only a finite amount of water on the planet. For population growth, the main constraint is availability of such resources as food and land.

After watching the video, students may work individually or in pairs. Point out to the students that the growth shown in the video happened much more quickly than it will in the problems below, but the underlying concept is the same. Students should be prepared to share their solutions with the class.

Exercises 1–3

1. Assume that there is initially 1 cm of water in the tank and the height of the water doubles every 10 seconds. Write an equation that could be used to calculate the height $H(t)$ of the water in the tank at any time t .

The height of the water at time t seconds can be modeled by $H(t) = 2^{t/10}$.

2. How would the equation in Exercise 1 change if ...

- a. The initial depth of water in the tank was 2 cm?

$$H(t) = 2 \cdot 2^{t/10}$$

- b. The initial depth of water in the tank was $\frac{1}{2}$ cm?

$$H(t) = \frac{1}{2} \cdot 2^{t/10}$$

- c. The initial depth of water in the tank was 10 cm?

$$H(t) = 10 \cdot 2^{t/10}$$

- d. The initial depth of water in the tank was A cm, for some positive real number A ?

$$H(t) = A \cdot 2^{t/10}$$

3. How would the equation in Exercise 2, part (d) change if ...

- a. The height tripled every ten seconds?

$$H(t) = A \cdot 3^{t/10}$$

- b. The height doubled every five seconds?

$$H(t) = A \cdot 2^{t/5}$$

- c. The height quadrupled every second?

$$H(t) = A \cdot 4^t$$

- d. The height halved every ten seconds?

$$H(t) = A \cdot (0.5^{t/10})$$

Scaffolding:

Struggling students can create a table of water depths to visualize the accumulation of water. Since the doubling happens every 10 seconds, have them deduce the exponent by asking, “How many times would doubling occur in 30 seconds? One minute?”

Time (s)	Depth (cm)
0	1
10	2
20	4
30	8

Discussion (2 minutes)

Students have worked informally with the average rate of change of a function before in Modules 3 and 4 of Algebra I. For the next examples, we will need the following definition. Go through this definition and post it on the board or in another prominent place before beginning the next example. Students will continue to work with the average rate of change of a function in the problem set.

AVERAGE RATE OF CHANGE: Given a function f whose domain contains the interval of real numbers $[a, b]$ and whose range is a subset of the real numbers, the *average rate of change on the interval $[a, b]$* is defined by the number:

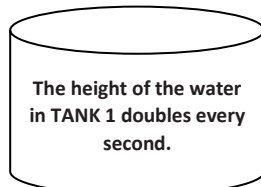
$$\frac{f(b) - f(a)}{b - a}$$

Example 1 (4 minutes)

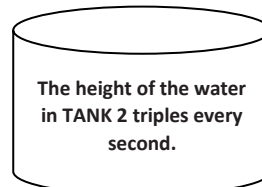
Use this example to model the process of finding the average rate of change of the height of the water that is increasing according to one of the exponential functions in our hypothetical scenario. The students will repeat this calculation in the exercises that follow. The Student Materials contain the images below of the three water tanks, but not the accompanying formulas.

Example 1

1. Consider two identical water tanks, each of which begins with a height of water 1 cm and fills with water at a different rate. Which equations can be used to calculate the height of water in each tank at time t ? Use H_1 for tank 1 and H_2 for tank 2.



$$H_1(t) = 2^t$$



$$H_2(t) = 3^t$$

- a. If both tanks start filling at the same time, which one fills first?

Tank 2 will fill first because the level is rising more quickly.

- b. We want to know the average rate of change of the height of the water in these tanks over an interval that starts at a fixed time T as they are filling up. What is the formula for the average rate of change of a function f on an interval $[a, b]$?

$$\frac{f(b) - f(a)}{b - a}$$

- c. What is the formula for the average rate of change of the function H_1 on an interval $[a, b]$?

$$\frac{H_1(b) - H_1(a)}{b - a}$$

- d. Let's calculate the average rate of change of the function H_1 on the interval $[T, T + 0.1]$, which is an interval one-tenth of a second long starting at an unknown time T .

$$\begin{aligned} \frac{H_1(T + 0.1) - H_1(T)}{T + 0.1 - T} &= \frac{(2^{T+0.1}) - (2^T)}{0.1} \\ &= \frac{2^T \cdot 2^{0.1} - 2^T}{0.1} \\ &= \frac{2^T(2^{0.1} - 1)}{0.1} \\ &\approx 2^T(0.717735) \\ &\approx 0.717735 H_1(T) \end{aligned}$$

- So, the average rate of change of the height function is a multiple of the value of the function. This means that the speed at which the height is changing at time T depends on the depth of water at that time. On average, over the interval $[T, T + 0.1]$, the water in tank 1 rises at a rate of approximately $0.717735 H_1(T)$ centimeters per second.
- Let's say that at time T there is a height of 5 cm of water in the tank. Then after one-tenth of a second, the height of the water would increase by $\frac{1}{10}(0.717735(5)) \approx 0.3589$ cm. But, if there is a height of 20 cm of water in the tank, after one-tenth of a second the height of the water would increase by $\frac{1}{10}(0.717735(20)) \approx 1.4355$ cm.

Exercises 4–5 (10 minutes)

Students will need to use calculators to compute the numerical constants in the exercises below.

Exercises 4–8

4. For the second tank, calculate the average change in the height, H_2 , from time T seconds to $T + 0.1$ seconds. Express the answer as a number times the value of the original function at time T . Explain the meaning of these findings.

$$\begin{aligned} \frac{H_2(T + 0.1) - H_2(T)}{0.1} &= \frac{3^{T+0.1} - 3^T}{0.1} \\ &= \frac{3^T \cdot 3^{0.1} - 3^T}{0.1} \\ &= \frac{3^T(3^{0.1} - 1)}{0.1} \\ &\approx \frac{3^T(0.116123)}{0.1} \\ &\approx 1.16123 \cdot 3^T \\ &\approx 1.16123 \cdot H_2(T) \end{aligned}$$

On average, over the time interval $[T, T + 0.1]$, the water in tank 2 rises at a rate of approximately $1.16123 H_2(T)$ centimeters per second.

5. For each tank, calculate the average change in height from time T seconds to $T + 0.001$ seconds. Express the answer as a number times the value of the original function at time T . Explain the meaning of these findings.

Tank 1:

$$\begin{aligned} \frac{H_1(T + 0.001) - H_1(T)}{0.001} &= \frac{2^{T+0.001} - 2^T}{0.001} \\ &= \frac{2^T \cdot 2^{0.001} - 2^T}{0.001} \\ &= \frac{2^T(2^{0.001} - 1)}{0.001} \\ &\approx \frac{2^T(0.000693)}{0.001} \\ &\approx 0.69339 \cdot 2^T \\ &\approx 0.69339 \cdot H_1(T) \end{aligned}$$

On average, over the time interval $[T, T + 0.001]$, the water in tank 1 rises at a rate of approximately $0.693387 H_1(T)$ centimeters per second.

Tank 2:

$$\begin{aligned}
 \frac{H_2(T + 0.001) - H_2(T)}{0.001} &= \frac{3^{T+0.001} - 3^T}{0.001} \\
 &= \frac{3^T \cdot 3^{0.001} - 3^T}{0.001} \\
 &= \frac{3^T(3^{0.001} - 1)}{0.001} \\
 &\approx \frac{3^T(0.00110)}{0.001} \\
 &\approx 1.09922 \cdot 3^T \\
 &\approx 1.09922 \cdot H_2(T)
 \end{aligned}$$

Over the time interval $[T, T + 0.001]$, the water in tank 2 rises at an average rate of approximately $1.09922H_2(T)$ centimeters per second.

Exercises 6–8 (12 minutes)

The following exercises will lead to discovery of the constant e that occurs naturally in many situations we can model mathematically. Looking at the results of the previous three exercises, if the height of the water doubles, then the expression for the average rate of change contains a factor less than one. If the height of the water triples, then the expression for the average rate of change contains a factor greater than one. Under what conditions will the expression for the average rate of change contain a factor of exactly one? Answering this question leads us to e .

6. In Exercise 5, the average rate of change of the height of the water in tank 1 on the interval $[T, T + 0.01]$ can be described by the expression $c_1 \cdot 2^T$, and the average rate of change of the height of the water in tank 2 on the interval $[T, T + 0.01]$ can be described by the expression $c_2 \cdot 3^T$. What are approximate values of c_1 and c_2 ?
- $c_1 \approx 0.69339$ and $c_2 \approx 1.09922$

7. As an experiment, let's look for a value of b so that if the height of the water can be described by $H(t) = b^t$, then the expression for the average of change on the interval $[T, T + 0.01]$ is $1 \cdot H(T)$.
- a. Write out the expression for the average rate of change of $H(t) = b^t$ on the interval $[T, T + 0.01]$.

$$\frac{H_b(T + 0.001) - H_b(T)}{0.001}$$

- b. Set your expression in part (a) equal to $1 \cdot H(T)$ and reduce to an expression involving a single b .

$$\begin{aligned}
 \frac{H_b(T + 0.001) - H_b(T)}{0.001} &= 1 \cdot H_b(T) \\
 \frac{b^{T+0.001} - b^T}{0.001} &= b^T \\
 \frac{b^T(b^{0.001} - 1)}{0.001} &= b^T \\
 b^{0.001} - 1 &= 0.001 \\
 b^{0.001} &= 1.001
 \end{aligned}$$

- c. Now we want to find the value of b that satisfies the equation you found in part (b), but we do not have a way to explicitly solve this equation. Look back at Exercise 6; which two consecutive integers have b between them?

We are looking for the base of the exponent that produces a rate of change on a small interval near t that is $1 \cdot H(t)$. When that base is 2, the value of the rate is roughly $0.69H(t)$. When the base is 3, the value of the rate is roughly $1.1H$. Since $0.69 < 1 < 1.1$, the base we are looking for is somewhere between 2 and 3.

- d. Use your calculator and a guess-and-check method to find an approximate value of b to 2 decimal places.

Students may choose to use a table such as the following. Make sure that students are maintaining enough decimal places of $b^{0.001}$ to determine which value is closest to 0.001.

b	$b^{0.001}$	b	$b^{0.001}$
2.0	1.00069	2.70	1.000994
2.1	1.00074	2.71	1.000997
2.2	1.00079	2.72	1.001001
2.3	1.00083	2.73	1.001005
2.4	1.00088	2.74	
2.5	1.00092	2.75	
2.6	1.00096	2.76	
2.7	1.00099	2.77	
2.8	1.00103	2.78	
2.9	1.00107	2.79	
3.0	1.00110	2.80	

Then $b \approx 2.72$.

8. Verify that for the value of b found in Exercise 7, $\frac{H_b(T+0.001) - H_b(T)}{0.001} = H_b(T)$, where $H_b(T) = b^T$.

$$\begin{aligned}
 \frac{H_b(T+0.001) - H_b(T)}{0.001} &= \frac{2.72^{T+0.001} - 2.72^{0.001}}{0.001} \\
 &= \frac{2.72^T(2.72^{0.001} - 1)}{0.001} \\
 &\approx \frac{2.72^T(0.001000)}{0.001} \\
 &\approx 1.00 \cdot 2.72^T \\
 &\approx 1.00 \cdot H_b(T)
 \end{aligned}$$

When the height of the water increases by a factor of 2.72 units per second, the height at any time is equal to the rate of change of height at that time.

Discussion (2 minutes)

If there is time, perform the calculation of b several more times, over smaller and smaller time intervals and finding more and more digits of b . If not, then just present students with the fact below.

MP.8

- What happens to the value of b ?
 - If we were to keep finding the average rate of change of the function H_b on smaller and smaller time intervals and solving the equation $H_b(t) = A \cdot b^t$, we would find that the height of the water increases by a factor that gets closer and closer to the number 2.7182818284
- The number that this process leads to is called Euler's number, and is denoted by e . Like π , e is an irrational number so it cannot be accurately represented by a decimal expansion. The approximation of e to 13 decimal places is $e \approx 2.7182818284590$.

- Like π , e is important enough to merit inclusion on scientific calculators. Depending on the calculator, e may appear alone, as the base of an exponential expression e^x , or both. Find the e button on your calculator and experiment with its use. Make sure you can use your calculator to provide an approximation of e and use the button to calculate e^2 and $2e$.

Closing (4 minutes)

Summarize the lesson with the students and ensure the first two points below are addressed. Have students highlight what they think is important about the lesson in writing or with a partner. Use this as an opportunity to informally assess learning.

- We just discovered the number e , which is important in the world of mathematics. It naturally occurred in our water tank exploration. It also occurs naturally in many other applications, such as finance and population growth.
- Just as we can create and use an exponential function $f(x) = 2^x$ or $f(x) = 10^x$, we can also create and use an exponential function $f(x) = e^x$. The interesting thing about the exponential function base e is that the rate of change of this function at a value a is the same as the value of this function at a .
- Euler's number will surface in a variety of different places in your future exposure to mathematics and you will see how it is one of the numbers on which much of the mathematics we practice is based.

Lesson Summary

- Euler's number, e , is an irrational number that is approximately equal to $e \approx 2.7182818284590$.
- AVERAGE RATE OF CHANGE: Given a function f whose domain contains the interval of real numbers $[a, b]$ and whose range is a subset of the real numbers, the *average rate of change on the interval $[a, b]$* is defined by the number

$$\frac{f(b) - f(a)}{b - a}.$$

Exit Ticket (3 minutes)

Name _____

Date _____

Lesson 6: Euler's Number, e

Exit Ticket

1. Suppose that water is entering a cylindrical water tank so that the initial height of the water is 3 cm and the height of the water doubles every 30 seconds. Write an equation of the height of the water at time t seconds.
2. Explain how the number e arose in our exploration of the average rate of change of the height of the water in the water tank.

Exit Ticket Sample Solutions

1. Suppose that water is entering a cylindrical water tank so that the initial height of the water is 3 cm and the height of the water doubles every 30 seconds. Write an equation of the height of the water at time t seconds.

$$H(t) = 3 \left(2^{\frac{t}{30}} \right)$$

2. Explain how the number e arose in our exploration of the average rate of change of the height of the water in the water tank.

We first noticed that if the water level in the tank was doubling every second, then the average rate of change of the height of the water was roughly 0.69 times the height of the water at that time. And if the water level in the tank was tripling every second, then the average rate of change of the height of the water was roughly 1.1 times the height of the water at that time. When we went looking for a base b so that the average rate of change of the height of the water was 1.0 times the height of the water at that time, we found that the base was roughly e . Calculating the average rate of change over shorter intervals gave a better approximation of e .

Problem Set Sample Solutions

Problems 1–5 address other occurrences of e and some fluency practice with the number e , and the remaining problems focus on the average rate of change of a function. The last two problems are extension problems that introduce some ideas of calculus with the familiar formulas for the area and circumference of a circle and the volume and surface area of a sphere.

1. The product $4 \cdot 3 \cdot 2 \cdot 1$ is called 4 *factorial* and is denoted by $4!$. Then $10! = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$, and for any positive integer n , $n! = n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1$.

- a. Complete the following table of factorial values:

n	1	2	3	4	5	6	7	8
$n!$	1	2	6	24	120	720	5040	40320

- b. Evaluate the sum $1 + \frac{1}{1!}$.

2

- c. Evaluate the sum $1 + \frac{1}{1!} + \frac{1}{2!}$.

2.5

- d. Use a calculator to approximate the sum $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!}$ to 7 decimal places. Do not round the fractions before evaluating the sum.

$\frac{8}{3} \approx 2.666667$

- e. Use a calculator to approximate the sum $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!}$ to 7 decimal places. Do not round the fractions before evaluating the sum.

$\frac{65}{24} \approx 2.708333$

- f. Use a calculator to approximate sums of the form $1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{k!}$ to 7 decimal places for $k = 5, 6, 7, 8, 9, 10$. Do not round the fractions before evaluating the sums with a calculator.

If $k = 5$, the sum is $\frac{163}{60} \approx 2.7166667$.

If $k = 6$, the sum is $\frac{1957}{720} \approx 2.7180556$.

If $k = 7$, the sum is $\frac{685}{252} \approx 2.7182540$.

If $k = 8$, the sum is $\frac{109601}{40320} \approx 2.7182788$.

If $k = 9$, the sum is $\frac{98461}{36288} \approx 2.7182815$.

If $k = 10$, the sum is $\frac{9864101}{3628800} \approx 2.7182818$.

- g. Make a conjecture about the sums $1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{k!}$ for positive integers k as k increases in size.

It seems that as k gets larger, the sums $1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{k!}$ get closer to e .

- h. Would calculating terms of this sequence ever yield an exact value of e ? Why or why not?

No. The number e is irrational so it cannot be written as a quotient of integers. Any finite sum $1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{k!}$ can be expressed as a single rational number with denominator $k!$, so the sums are all rational numbers. However, the more terms that are calculated, the closer to e the sum becomes, so these sums provide better and better rational number approximations of e .

2. Consider the sequence given by the function $a_n = \left(1 + \frac{1}{n}\right)^n$, where $n \geq 1$ is an integer.

- a. Use your calculator to approximate the first 5 terms of this sequence to 7 decimal places.

$$a_1 = \left(1 + \frac{1}{1}\right)^1 = 2$$

$$a_2 = \left(1 + \frac{1}{2}\right)^2 = 2.25$$

$$a_3 = \left(1 + \frac{1}{3}\right)^3 \approx 2.3703704$$

$$a_4 = \left(1 + \frac{1}{4}\right)^4 \approx 2.4414063$$

$$a_5 = \left(1 + \frac{1}{5}\right)^5 = 2.4883200$$

- b. Does it appear that this sequence settles near a particular value?

No, the numbers get bigger, but we cannot tell if it keeps getting bigger or settles on or near a particular value.

c. Use a calculator to approximate the following terms of this sequence to 7 decimal places.

i. $a_{100} = 2.7081383$

ii. $a_{1000} = 2.7169239$

iii. $a_{10,000} = 2.7181459$

iv. $a_{100,000} = 2.7182682$

v. $a_{1,000,000} = 2.7182805$

vi. $a_{10,000,000} = 2.7182816$

vii. $a_{100,000,000} = 2.7182818$

d. Does it appear that this sequence settles near a particular value?

Yes, it appears that as n gets really large (at least 100,000,000), the terms a_n of the sequence settle near the value of e .

e. Compare the results of this exercise with the results of Problem 1. What do you observe?

It took about 10 terms of the sum in Problem 1 to see that the sum settled at the value e , but it takes 100,000,000 terms of the sequence in this problem to see that the sum settles at the value e .

3. If $x = 5a^4$ and $a = 2e^3$, express x in terms of e and approximate to the nearest whole number.

If $x = 5a^4$ and $a = 2e^3$, then $x = 5(2e^3)^4$. Rewriting the right side in an equivalent form gives $x = 80e^{12} \approx 13020383$.

4. If $a = 2b^3$ and $b = -\frac{1}{2}e^{-2}$, express a in terms of e and approximate to four decimal places.

If $a = 2b^3$ and $b = -\frac{1}{2}e^{-2}$, then $a = 2\left(-\frac{1}{2}e^{-2}\right)^3$. Rewriting the right side in an equivalent form gives $a = -\frac{1}{4}e^{-6} \approx -0.0006$.

5. If $x = 3e^4$ and $e = \frac{s}{2x^3}$, show that $s = 54e^{13}$ and approximate s to the nearest whole number.

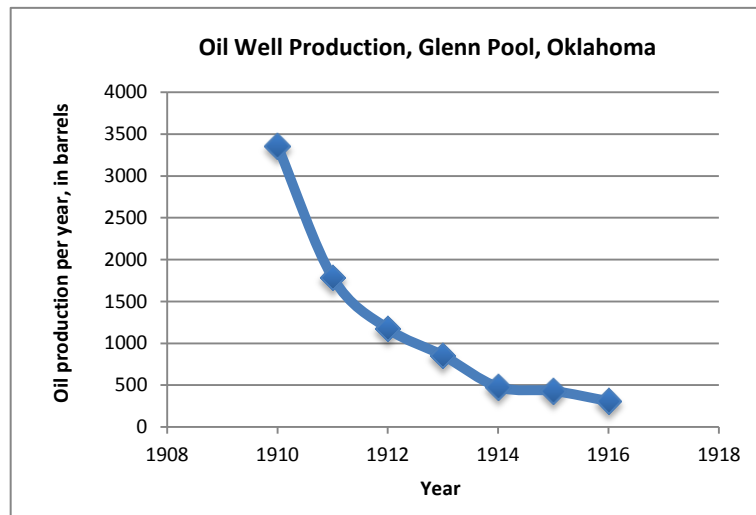
Rewrite the equation $e = \frac{s}{2x^3}$ to isolate the variable s .

$$e = \frac{s}{2x^3}$$

$$2x^3e = s$$

By the substitution property, if $s = 2x^3e$ and $x = 3e^4$, then $s = 2(3e^4)^3 \cdot e$. Rewriting the right side in an equivalent form gives $s = 2 \cdot 27e^{12} \cdot e = 54e^{13} \approx 23890323$.

6. The following graph shows the number of barrels of oil produced by the Glenn Pool well in Oklahoma from 1910 to 1916.



Source: Cutler, Willard W., Jr. Estimation of Underground Oil Reserves by Oil-Well Production Curves, U.S. Department of the Interior, 1924.

- a. Estimate the average rate of change of the amount of oil produced by the well on the interval [1910, 1916] and explain what that number represents.

Student responses will vary based on how they read the points on the graph. Over the interval [1910, 1916], the average rate of change is roughly

$$\frac{300 - 3200}{1916 - 1910} = -\frac{2900}{6} \approx -483.33.$$

This says that the production of the well decreased by an average of about 483 barrels of oil each year between 1910 and 1916.

- b. Estimate the average rate of change of the amount of oil produced by the well on the interval [1910, 1913] and explain what that number represents.

Student responses will vary based on how they read the points on the graph. Over the interval [1910, 1913], the average rate of change is roughly

$$\frac{800 - 3200}{1913 - 1910} = -\frac{2400}{3} = -800.$$

This says that the production of the well decreased by an average of about 800 barrels of oil per year between 1910 and 1913.

- c. Estimate the average rate of change of the amount of oil produced by the well on the interval [1913, 1916] and explain what that number represents.

Student responses will vary based on how they read the points on the graph. Over the interval [1913, 1916], the average rate of change is roughly

$$\frac{300 - 800}{1916 - 1913} = -\frac{500}{3} \approx -166.67.$$

This says that the production of the well decreased by an average of about 166.67 barrels of oil per year between 1913 and 1916.

- d. Compare your results for the rates of change in oil production in the first half and the second half of the time period in question in parts (b) and (c). What do those numbers say about the production of oil from the well?

The production dropped much more rapidly in the first three years than it did in the second three years. Looking at the graph, it looks like the oil in the well might be running out, so less and less can be extracted each year.

- e. Notice that the average rate of change of the amount of oil produced by the well on any interval starting and ending in two consecutive years is always negative. Explain what that means in the context of oil production.

Because the average rate of change of oil production over a one-year period is always negative, the well is producing less oil each year than it did the year before.

7. The following table lists the number of hybrid electric vehicles (HEVs) sold in the United States between 1999 and 2013.

Year	Number of HEVs sold in U.S.	Year	Number of HEVs sold in U.S.
1999	17	2007	352,274
2000	9350	2008	312,386
2001	20,282	2009	290,271
2002	36,035	2010	274,210
2003	47,600	2011	268,752
2004	84,199	2012	434,498
2005	209,711	2013	495,685
2006	252,636		

Source: U.S. Department of Energy, Alternative Fuels and Advanced Vehicle Data Center, 2013

- a. During which one-year interval is the average rate of change of the number of HEVs sold the largest? Explain how you know.

The average rate of change of the number of HEVs sold is largest during [2011, 2012] because the number of HEVs sold increases by the largest amount between those two years.

- b. Calculate the average rate of change of the number of HEVs sold on the interval [2003, 2004] and explain what that number represents.

On the interval [2003, 2004], the average rate of change in sales of HEVs is $\frac{84,199 - 47,600}{2004 - 2003} = 36,599$. This means that during this one-year period, HEVs were selling at a rate of 36,599 vehicles per year.

- c. Calculate the average rate of change of the number of HEVs sold on the interval [2003, 2008] and explain what that number represents.

On the interval [2003, 2008], the average rate of change in sales of HEV is $\frac{312,386 - 47,600}{2008 - 2003} = 52,957.2$. This means that during this five-year period, HEVs were selling at an average rate of 52,957 vehicles per year.

- d. What does it mean if the average rate of change of the number of HEVs sold is negative?

If the average rate of change of the vehicles sold is negative, then the sales are declining. This means that fewer cars were sold than in the previous year.

Extension:

8. The formula for the area of a circle of radius r can be expressed as a function $A(r) = \pi r^2$.

- a. Find the average rate of change of the area of a circle on the interval $[4, 5]$.

$$\frac{A(5) - A(4)}{5 - 4} = \frac{25\pi - 16\pi}{1} = 9\pi$$

- b. Find the average rate of change of the area of a circle on the interval $[4, 4.1]$.

$$\frac{A(4.1) - A(4)}{4.1 - 4} = \frac{16.81\pi - 16\pi}{0.1} = 8.1\pi$$

- c. Find the average rate of change of the area of a circle on the interval $[4, 4.01]$.

$$\frac{A(4.01) - A(4)}{4.01 - 4} = \frac{16.0801\pi - 16\pi}{0.01} = 8.01\pi$$

- d. Find the average rate of change of the area of a circle on the interval $[4, 4.001]$.

$$\frac{A(4.001) - A(4)}{4.001 - 4} = \frac{16.008001\pi - 16\pi}{0.001} = 8.001\pi$$

- e. What is happening to the average rate of change of the area of the circle as the interval gets smaller and smaller?

The average rate of change of the area of the circle appears to be getting close to 8π .

- f. Find the average rate of change of the area of a circle on the interval $[4, 4 + h]$ for some small positive number h .

$$\begin{aligned} \frac{A(4+h) - A(4)}{(4+h) - 4} &= \frac{(4+h)^2\pi - 16\pi}{h} \\ &= \frac{(16 + 8h + h^2)\pi - 16\pi}{h} \\ &= \frac{1}{h}(8h + h^2)\pi \\ &= (8+h)\pi \end{aligned}$$

- g. What happens to the average rate of change of the area of the circle on the interval $[4, 4 + h]$ as $h \rightarrow 0$? Does this agree with your answer to part (d)? Should it agree with your answer to part (e)?

As $h \rightarrow 0$, $8 + h \rightarrow 8$, so as h gets smaller, the average rate of change approaches 8. This agrees with my response to part (e), and it should because as $h \rightarrow 0$, the interval $[4, 4 + h]$ gets smaller.

- h. Find the average rate of change of the area of a circle on the interval $[r_0, r_0 + h]$ for some positive number r_0 and some small positive number h .

$$\begin{aligned} \frac{A(r_0+h) - A(r_0)}{(r_0+h) - r_0} &= \frac{(r_0+h)^2\pi - r_0^2\pi}{h} \\ &= \frac{(r_0^2 + 2r_0h + h^2)\pi - r_0^2\pi}{h} \\ &= \frac{1}{h}(2r_0h + h^2)\pi \\ &= (2r_0+h)\pi \end{aligned}$$

- i. What happens to the average rate of change of the area of the circle on the interval $[r_0, r_0 + h]$ as $h \rightarrow 0$? Do you recognize the resulting formula?

As $h \rightarrow 0$, the expression for the average rate of change becomes $2\pi r_0$, which is the circumference of the circle with radius r_0 .

9. The formula for the volume of a sphere of radius r can be expressed as a function $V(r) = \frac{4}{3}\pi r^3$. As you work through these questions, you will see the pattern develop more clearly if you leave your answers in the form of a coefficient times π . Approximate the coefficient to five decimal places.

- a. Find the average rate of change of the volume of a sphere on the interval $[2, 3]$.

$$\frac{V(3) - V(2)}{3 - 2} = \frac{\frac{4}{3} \cdot 27\pi - \frac{4}{3} \cdot 8\pi}{1} = \frac{4}{3} \cdot 19\pi \approx 25.33333\pi$$

- b. Find the average rate of change of the volume of a sphere on the interval $[2, 2.1]$.

$$\frac{V(2.1) - V(2)}{2.1 - 2} = \frac{\frac{4}{3}\pi(2.1^3 - 8)}{0.1} = 16.81333\pi$$

- c. Find the average rate of change of the volume of a sphere on the interval $[2, 2.01]$.

$$\frac{V(2.01) - V(2)}{2.01 - 2} = \frac{\frac{4}{3}\pi(2.01^3 - 8)}{0.01} = 16.08010\pi$$

- d. Find the average rate of change of the volume of a sphere on the interval $[2, 2.001]$.

$$\frac{V(2.001) - V(2)}{2.001 - 2} = \frac{\frac{4}{3}\pi(2.001^3 - 8)}{0.001} = 16.00800\pi$$

- e. What is happening to the average rate of change of the volume of a sphere as the interval gets smaller and smaller?

The average rate of change of the volume of the sphere appears to be getting close to 16π .

- f. Find the average rate of change of the volume of a sphere on the interval $[2, 2 + h]$ for some small positive number h .

$$\begin{aligned} \frac{V(2+h) - V(2)}{(2+h) - 2} &= \frac{\frac{4}{3}\pi((2+h)^3 - 8)}{h} \\ &= \frac{4}{3}\pi \cdot \frac{1}{h}(8 + 12h + 6h^2 + h^3 - 8) \\ &= \frac{4\pi}{3h}(12h + 6h^2 + h^3) \\ &= \frac{4\pi}{3}(12 + 6h + h^2) \end{aligned}$$

- g. What happens to the average rate of change of the volume of a sphere on the interval $[2, 2 + h]$ as $h \rightarrow 0$? Does this agree with your answer to part (e)? Should it agree with your answer to part (e)?

As $h \rightarrow 0$, the polynomial $12 + 6h + h^2 \rightarrow 12$. Then the average rate of change approaches $\frac{4\pi}{3} \cdot 12 = 16\pi$. This agrees with my response to part (e), and it should because as $h \rightarrow 0$, the interval $[2, 2 + h]$ gets smaller.

- h. Find the average rate of change of the volume of a sphere on the interval $[r_0, r_0 + h]$ for some positive number r_0 and some small positive number h .

$$\begin{aligned}\frac{V(r_0 + h) - V(r_0)}{(r_0 + h) - r_0} &= \frac{\frac{4}{3}\pi((r_0 + h)^3 - r_0^3)}{h} \\ &= \frac{4}{3}\pi \cdot \frac{1}{h}(r_0^3 + 3r_0^2h + 3r_0h^2 + h^3 - r_0^3) \\ &= \frac{4\pi}{3h}(3r_0^2h + 3r_0h^2 + h^3) \\ &= \frac{4\pi}{3}(3r_0^2 + 3r_0h + h^2)\end{aligned}$$

- i. What happens to the average rate of change of the volume of a sphere on the interval $[r_0, r_0 + h]$ as $h \rightarrow 0$? Do you recognize the resulting formula?

As $h \rightarrow 0$, the expression for the average rate of change becomes $4\pi r_0^2$, which is the surface area of the sphere with radius r_0 .