Lesson 5: Irrational Exponents—What Are $2^{\sqrt{2}}$ and 2^{π}?

Student Outcomes

- Students approximate the value of quantities that involve positive irrational exponents.
- Students extend the domain of the function $f(x) = b^x$ for positive real numbers b to all real numbers.

Lesson Notes

Our goal today is to define 2 to an irrational power. We already have a definition for 2 to a rational power $\frac{p}{q}$: $2^{\frac{p}{q}} = \sqrt[q]{2^p}$, but irrational numbers cannot be written as "an integer divided by an integer." By defining 2 to an irrational power, we will be able to state definitively that for any positive real number $b > 0$, the domain of the function $f(x) = b^x$ is all real numbers. This is an important result and one that is necessary for us to proceed to the study of logarithms. The lesson provides a new way to reinforce standard **8.NS.A.2** when students determine a recursive process for calculation from a context (**F-BF.A.1a**) when they use rational approximations of irrational numbers to approximate first $\sqrt{2}$ and then $2^{\sqrt{2}}$. Extending rational exponents to real exponents is an application of **N-RN.A.1**. The foundational work done in this lesson with exponential expressions will be extended to logarithms in later lessons so that logarithmic functions in base 2, 10, and e are well-defined and can be used to solve exponential equations. Understanding the domain of exponential functions will also allow students to correctly graph exponential and logarithmic functions in Topic C. The work done in Lesson 5 will also help demystify irrational numbers, which will ease the introduction to Euler's number, e, in Lesson 6.

Classwork

Opening (5 minutes)

Use the Opening to recall the definitions of rational and irrational numbers and solicit examples and characteristics from the class. Randomly select students to explain what they know about rational and irrational numbers. Then, make a list including examples and characteristics of both. Alternatively, have students give you rational and irrational numbers, make a class list, and then have students generalize characteristics of rational and irrational numbers in their notebooks. Rational and irrational numbers along with some characteristics and examples are described below.

RATIONAL NUMBER: A *rational number* is a number that can be represented as $\frac{p}{q}$ where p and q are integers with $q \neq 0$.

IRRATIONAL NUMBER: An *irrational number* is a real number that cannot be represented as $\frac{p}{q}$ for any integers p and q with $q \neq 0$.

- What are some characteristics of rational numbers?
  - *A rational number can be represented as a finite or repeating decimal; that is, a rational number can be written as a fraction.*
- What are some characteristics of irrational numbers?
  - *An irrational number cannot be represented as a finite or repeating decimal, so it must be represented symbolically or as an infinite, nonrepeating decimal.*

- What are some examples of irrational numbers?
  - $\sqrt{2}$, π, $\sqrt[3]{17}$
- We usually assume that the rules we develop for rational numbers hold true for irrational numbers, but what could something like $2^{\sqrt{2}}$ or 2^{π} mean?
  - *Solicit ideas from the class. Students may consider numbers like this to be between rational exponents or "filling the gaps" from rational exponents.*
- Let's find out more about exponents raised to irrational powers and how we can get a handle on their values.

Exercise 1 (8 minutes)

Have students work on the following exercises independently or in pairs. Students will need to use calculators. After students finish, debrief them with the questions that follow the exercises.

Exercise 1

a. Write the following finite decimals as fractions (you do not need to reduce to lowest terms).

$$\mathbf{1}, \quad \mathbf{1.4}, \quad \mathbf{1.41}, \quad \mathbf{1.414}, \quad \mathbf{1.4142}, \quad \mathbf{1.41421}$$

$1.4 = \dfrac{14}{10}$

$1.41 = \dfrac{141}{100}$

$1.414 = \dfrac{1414}{1000}$

$1.4142 = \dfrac{14142}{10000}$

$1.41421 = \dfrac{141421}{100000}$

b. Write $2^{1.4}$, $2^{1.41}$, $2^{1.414}$, and $2^{1.4142}$ in radical form ($\sqrt[n]{2^m}$).

$2^{1.4} = 2^{14/10} = \sqrt[10]{2^{14}}$

$2^{1.41} = 2^{141/100} = \sqrt[100]{2^{141}}$

$2^{1.414} = 2^{1414/1000} = \sqrt[1000]{2^{1414}}$

$2^{1.4142} = 2^{14142/10000} = \sqrt[10000]{2^{14142}}$

$2^{1.41421} = 2^{141421/100000} = \sqrt[100000]{2^{141421}}$

c. **Compute a decimal approximation to 5 decimal places of the radicals you found in part (b) using your calculator. For each approximation, underline the digits that are also in the previous approximation, starting with 2.00000 done for you below. What do you notice?**

$$2^1 = 2 = 2.00000$$

$2^{1.4} = \sqrt[10]{2^{14}} \approx \underline{2.}63901$

$2^{1.41} = \sqrt[100]{2^{141}} \approx \underline{2.6}5737$

$2^{1.414} = \sqrt[1000]{2^{1414}} \approx \underline{2.6}6474$

$2^{1.4142} = \sqrt[10000]{2^{14142}} \approx \underline{2.66}511$

$2^{1.41421} = \sqrt[100000]{2^{141421}} \approx \underline{2.6651}4$

More and more of the digits are the same.

Note to teacher: Students cannot find 2^{1414} on most calculators due to the number being 426 digits long. They will need to calculate $\left(\sqrt[1000]{2}\right)^{1414}$ instead. At this point it may be a good time to switch to using the decimal approximation within the exponent, reminding students that the calculator is evaluating the decimal by using the radical form, that is, $b^{\frac{m}{n}} = \sqrt[n]{b^m}$. Ideally, a student will suggest using the decimal exponent first. If roots are used, make sure that the root is taken before the exponent for large exponents. Examples and possible solutions throughout the lesson assume that roots are used so the true meaning of rational exponents is emphasized.

MP.8

- Why are more of the digits the same? How are the exponents in each power changing?
  - *A new digit is included in the exponent each time:* 1.4, $1.4\underline{1}$, $1.41\underline{4}$, $1.414\underline{2}$, $1.4142\underline{1}$.
- If we kept including more digits, what do you conjecture will happen to the decimal approximations?
  - *A greater and greater number of digits in each approximation would remain the same.*
- Let's see!

Example 1 (6 minutes)

Students should already be aware that rational exponents are defined using roots and exponents.

Write a decimal approximation for $2^{1.4142135}$.

$$2^{1.4142135} \text{ is the } 10{,}000{,}000\text{th root of } 2^{14142135}$$

Remember to take the root first. We get

$$2^{1.4142135} \approx \underline{2.66514}.$$

- Can anyone tell the class what the exponents $1.4, 1.41, 1.414, \ldots$ approximate?

Hopefully, one student will say $\sqrt{2}$, but if not, ask them to find $\sqrt{2}$ on their calculator and ask again.

- Yes, $\sqrt{2} \approx 1.414213562$.
  The goal of this lesson is to find a meaning for $2^{\sqrt{2}}$. We now know enough to discuss both the problem and solution to defining 2 to an irrational power such as $\sqrt{2}$.

- First, the problem: Each time we took a better finite decimal approximation of the irrational number $2^{\sqrt{2}}$, we needed to take a greater n^{th} root. However, an irrational number has an infinite number of digits in its decimal expansion. We cannot take an ∞^{th} root! In particular, while we have always assumed $2^{\sqrt{2}}$ and 2^{π} existed (because when we show the graph of $f(x) = 2^x$, we drew a solid curve—not one with "holes" at $x = \sqrt{2}, \pi$, etc.), we do not as of yet have a way to define what $2^{\sqrt{2}}$ and 2^{π} really are.
- Fortunately, our beginning exercise suggests a solution using a limit process (much the way we defined the area of a circle in Geometry Module 3 in terms of limits).
- Let a_k stand for the term of the sequence of finite decimal approximations of $\sqrt{2}$ with k digits after the decimal point:
$$\{1, 1.4, 1.41, 1.414, 1.4142, 1.41421, 1.414213, 1.4142135, \ldots\},$$
and label these as $a_0 = 1,\ a_1 = 1.4,\ a_2 = 1.41,\ a_3 = 1.414$. Then define $2^{\sqrt{2}}$ to be the limit of the values of 2^{a_k}. Thus,
$$2^{a_k} \to 2^{\sqrt{2}} \text{ as } k \to \infty.$$

The important point to make to students is that each 2^{a_k} can be computed since each a_k is a rational number and therefore has a well-defined value in terms of n^{th} roots.

This is how calculators and computers are programmed to compute approximations of $2^{\sqrt{2}}$. Try it: The calculator says that $2^{\sqrt{2}} \approx 2.66514414$.

Exercise 2 (5 minutes)

Students should attempt the following exercise independently or in pairs. After the exercise, use the Discussion to debrief and informally assess understanding.

Scaffolding:
- Have advanced students give the most accurate estimate they can for part (b). Most calculators can provide an additional three to four decimal places of π. For reference, $\pi \approx 3.14159265358979323846$.
- Another option for advanced students is to discuss the sequence of upper bounds of π $\{4, 3.2, 3.15, 3.142, 3.1416, \ldots\}$ and whether this will provide an accurate estimate of 2^{π}.

Exercise 2

a. **Write six terms of a sequence that a calculator can use to approximate 2^{π}. (Hint: $\pi = 3.14159\ldots$)**

$\{2^3, 2^{3.1}, 2^{3.14}, 2^{3.141}, 2^{3.1415}, 2^{3.14159}, \ldots\}$

b. **Compute $2^{3.14} = \sqrt[100]{2^{314}}$ and 2^{π} on your calculator. In which digit do they start to differ?**

$2^{3.14} = \sqrt[100]{2^{314}} \approx 8.81524$
$2^{\pi} \approx 8.82497$
They start to differ in the hundredths place.

c. **How could you improve the accuracy of your estimate of 2^{π}?**

Include more digits of the decimal approximation of π in the exponent.

Discussion (10 minutes)

- Why does the sequence $2^3, 2^{3.1}, 2^{3.14}, 2^{3.141}, 2^{3.1415}, \ldots$ get closer and closer to 2^π?

Allow students to make some conjectures, but be sure to go through the reasoning below.

- We can trap 2^π in smaller and smaller intervals, each one contained in the previous interval.

Write the following incomplete inequalities on the board and ask students to help you complete them before continuing. Mention that in this process, we are *squeezing* π between two rational numbers that are each getting closer and closer to the value of π.

$$3 < \pi < 4$$

$$3.1 < \pi < ? \qquad 3.2$$

$$3.14 < \pi < ? \qquad 3.15$$

$$3.141 < \pi < ? \qquad 3.142$$

$$3.1415 < \pi < ? \qquad 3.1416$$

$$\vdots$$

- Since $3 < \pi < 4$, and the function $f(x) = 2^x$ increases, we know that $2^3 < \pi < 2^4$. Likewise, we can use the smaller intervals that contain π to find smaller intervals that contain 2^π. In this way, we can squeeze 2^π between rational powers of 2.

Now, have students use calculators to estimate the endpoints of each interval created by the upper and lower estimates of the values of 2^π and write the numerical approximations of each interval on the board so students can see the endpoints of the intervals getting closer together, squeezing the value of 2^π between them. Record values to four decimal places.

	Decimal Form
$2^3 < 2^\pi < 2^4$	$8.0000 < 2^\pi < 16.0000$
$2^{3.1} < 2^\pi < 2^{3.2}$	$8.5742 < 2^\pi < 9.1896$
$2^{3.14} < 2^\pi < 2^{3.15}$	$8.8152 < 2^\pi < 8.8766$
$2^{3.141} < 2^\pi < 2^{3.142}$	$8.8214 < 2^\pi < 8.8275$
$2^{3.1415} < 2^\pi < 2^{3.1416}$	$8.8244 < 2^\pi < 8.8250$
$\vdots$	$\vdots$

- What is the approximate value of 2^π? How many digits of this number do we know?
  - *Because our upper and lower estimates agree to two decimal places, our best approximation is $2^\pi \approx 8.82$.*
- How could we get a more accurate estimate of 2^π?
  - *Use more and more digits of π as exponents.*

- As the exponents get closer to the value of π, what happens to the size of the interval?
  - *The intervals get smaller; the endpoints of the interval get closer together.*
- What does every interval share in common?
  - *Every interval contains 2^{π}.*
- The only number that is guaranteed to be contained in every interval is 2^{π}. (Emphasize this fact.)
- There was nothing special about our choice of 2 in this discussion, or $\sqrt{2}$ or π. In fact, with a little more work, we could define $\pi^{\sqrt{2}}$ using the same ideas.

Closing (6 minutes)

Ask students to respond to the following questions either in writing or with a partner. Use this as an opportunity to informally assess understanding. The summative point of the lesson is that the domain of an exponential function $f(x) = b^x$ is *all real numbers*, so emphasize the final question below.

- For any positive real number $b > 0$ and any rational number r, how do we define b^r?
  - *If r is rational, then $r = \frac{p}{q}$ for some integers p and q. Then $b^r = \sqrt[q]{b^p}$. For example, $5^{\frac{2}{3}} = \sqrt[3]{5^2}$.*
- For any positive real number $b > 0$ and any irrational number r, how do we define b^r?
  - *If r is irrational, b^r is the limit of the values a^{b_n} where b_n is the finite decimal approximation of b to n decimal places.*
- If b is any positive real number, then consider the function $f(x) = b^x$. How is $f(x)$ defined if x is a rational number?
  - *If x is a rational number, then there are integers p and q so that $x = \frac{p}{q}$. Then $f(x) = b^{\frac{p}{q}} = \sqrt[q]{b^p}$.*
- How is $f(x)$ defined if x is an irrational number?
  - *If x is an irrational number, we find a sequence of rational numbers $\{a_0, a_1, a_2, \ldots\}$ that gets closer and closer to x. Then the sequence $\{b^{a_0}, b^{a_1}, b^{a_2}, \ldots\}$ approaches $f(x)$.*
- What is the domain of the exponential function $f(x) = b^x$?
  - *The domain of the function $f(x) = b^x$ is all real numbers.*

Exit Ticket (5 minutes)

Name ______________________________ Date ______________

Lesson 5: Irrational Exponents—What Are $2^{\sqrt{2}}$ and 2^{π}?

Exit Ticket

Use the process outlined in the lesson to approximate the number $2^{\sqrt{3}}$. Use the approximation $\sqrt{3} \approx 1.7320508$.

a. Find a sequence of five intervals that contain $\sqrt{3}$ whose endpoints get successively closer to $\sqrt{3}$.

b. Find a sequence of five intervals that contain $2^{\sqrt{3}}$ whose endpoints get successively closer to $2^{\sqrt{3}}$. Write your intervals in the form $2^r < 2^{\sqrt{3}} < 2^s$ for rational numbers r and s.

c. Use your calculator to find approximations to four decimal places of the endpoints of the intervals in part (b).

d. Based on your work in part (c) what is your best estimate of the value of $2^{\sqrt{3}}$?

Exit Ticket Sample Solutions

Use the process outlined in the lesson to approximate the number $2^{\sqrt{3}}$. Use the approximation $\sqrt{3} \approx 1.7320508$.

a. Find a sequence of five intervals that contain $\sqrt{3}$ whose endpoints get successively closer to $\sqrt{3}$.

$$1 < \sqrt{3} < 2$$
$$1.7 < \sqrt{3} < 1.8$$
$$1.73 < \sqrt{3} < 1.74$$
$$1.732 < \sqrt{3} < 1.733$$
$$1.7320 < \sqrt{3} < 1.7321$$

b. Find a sequence of five intervals that contain $2^{\sqrt{3}}$ whose endpoints get successively closer to $2^{\sqrt{3}}$. Write your intervals in the form $2^r < 2^{\sqrt{3}} < 2^s$ for rational numbers r and s.

$$2^1 < 2^{\sqrt{3}} < 2^2$$
$$2^{1.7} < 2^{\sqrt{3}} < 2^{1.8}$$
$$2^{1.73} < 2^{\sqrt{3}} < 2^{1.74}$$
$$2^{1.732} < 2^{\sqrt{3}} < 2^{1.733}$$
$$2^{1.7320} < 2^{\sqrt{3}} < 2^{1.7321}$$

c. Use your calculator to find approximations to four decimal places of the endpoints of the intervals in part (b).

$$2.0000 < 2^{\sqrt{3}} < 4.0000$$
$$3.2490 < 2^{\sqrt{3}} < 3.4822$$
$$3.3173 < 2^{\sqrt{3}} < 3.3404$$
$$3.3219 < 2^{\sqrt{3}} < 3.3242$$
$$3.3219 < 2^{\sqrt{3}} < 3.3221$$

d. Based on your work in part (c) what is your best estimate of the value of $2^{\sqrt{3}}$?

$$2^{\sqrt{3}} \approx 3.322$$

Problem Set Sample Solutions

1. Is it possible for a number to be both rational and irrational?

No. Either the number can be written as $\frac{p}{q}$ for integers p and q or it cannot. If it can, the number is rational. If it cannot, the number is irrational.

2. Use properties of exponents to rewrite the following expressions as a number or an exponential expression with only one exponent.

a. $\left(2^{\sqrt{3}}\right)^{\sqrt{3}}$ $= 8$

b. $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$ $= \mathbf{2}$

c. $\left(3^{1+\sqrt{5}}\right)^{1-\sqrt{5}}$ $= \frac{1}{81}$

d. $3^{\frac{1+\sqrt{5}}{2}} \cdot 3^{\frac{1-\sqrt{5}}{2}}$ $= 3$

e. $3^{\frac{1+\sqrt{5}}{2}} \div 3^{\frac{1-\sqrt{5}}{2}}$ $= 3^{\sqrt{5}}$

f. $3^{2\cos^2(x)} \cdot 3^{2\sin^2(x)}$ $= 9$

3.

a. Between what two integer powers of 2 does $2^{\sqrt{5}}$ lie?

$2^2 < 2^{\sqrt{5}} < 2^3$

b. Between what two integer powers of 3 does $3^{\sqrt{10}}$ lie?

$3^3 < 3^{\sqrt{10}} < 3^4$

c. Between what two integer powers of 5 does $5^{\sqrt{3}}$ lie?

$5^1 < 5^{\sqrt{3}} < 5^2$

4. Use the process outlined in the lesson to approximate the number $2^{\sqrt{5}}$. Use the approximation $\sqrt{5} \approx 2.23606798$.

a. Find a sequence of five intervals that contain $\sqrt{5}$ whose endpoints get successively closer to $\sqrt{5}$.

$$2 < \sqrt{5} < 3$$
$$2.2 < \sqrt{5} < 2.3$$
$$2.23 < \sqrt{5} < 2.24$$
$$2.236 < \sqrt{5} < 2.237$$
$$2.2360 < \sqrt{5} < 2.2361$$

b. Find a sequence of five intervals that contain $2^{\sqrt{5}}$ whose endpoints get successively closer to $2^{\sqrt{5}}$. Write your intervals in the form $2^r < 2^{\sqrt{5}} < 2^s$ for rational numbers r and s.

$$2^2 < 2^{\sqrt{5}} < 2^3$$
$$2^{2.2} < 2^{\sqrt{5}} < 2^{2.3}$$
$$2^{2.23} < 2^{\sqrt{5}} < 2^{2.24}$$
$$2^{2.236} < 2^{\sqrt{5}} < 2^{2.237}$$
$$2^{2.2360} < 2^{\sqrt{5}} < 2^{2.2361}$$

c. Use your calculator to find approximations to four decimal places of the endpoints of the intervals in part (b).

$$4.0000 < 2^{\sqrt{5}} < 8.0000$$
$$4.5948 < 2^{\sqrt{5}} < 4.9246$$
$$4.6913 < 2^{\sqrt{5}} < 4.7240$$
$$4.7109 < 2^{\sqrt{5}} < 4.7142$$
$$4.7109 < 2^{\sqrt{5}} < 4.7112$$

d. Based on your work in part (c), what is your best estimate of the value of $2^{\sqrt{5}}$?

$$2^{\sqrt{5}} \approx 4.711$$

e. Can we tell if $2^{\sqrt{5}}$ is rational or irrational? Why or why not?

No. We do not have enough information to determine whether $2^{\sqrt{5}}$ has a repeated pattern in its decimal representation or not.

5. Use the process outlined in the lesson to approximate the number $3^{\sqrt{10}}$. Use the approximation $\sqrt{10} \approx 3.1622777$.

a. Find a sequence of five intervals that contain $3^{\sqrt{10}}$ whose endpoints get successively closer to $3^{\sqrt{10}}$. Write your intervals in the form $3^r < 3^{\sqrt{10}} < 3^s$ for rational numbers r and s.

$$3^3 < 3^{\sqrt{10}} < 3^4$$
$$3^{3.1} < 3^{\sqrt{10}} < 3^{3.2}$$
$$3^{3.16} < 3^{\sqrt{10}} < 3^{3.17}$$
$$3^{3.162} < 3^{\sqrt{10}} < 3^{3.163}$$
$$3^{3.1622} < 3^{\sqrt{10}} < 3^{3.1623}$$

b. Use your calculator to find approximations to four decimal places of the endpoints of the intervals in part (a).

$$9.0000 < 3^{\sqrt{10}} < 81.0000$$
$$30.1353 < 3^{\sqrt{10}} < 33.6347$$
$$32.1887 < 3^{\sqrt{10}} < 32.5443$$
$$32.2595 < 3^{\sqrt{10}} < 32.2949$$
$$32.2666 < 3^{\sqrt{10}} < 32.2701$$

c. Based on your work in part (b), what is your best estimate of the value of $3^{\sqrt{10}}$?

$$3^{\sqrt{10}} \approx 32.27$$

6. Use the process outlined in the lesson to approximate the number $5^{\sqrt{7}}$. Use the approximation $\sqrt{7} \approx 2.64575131$.

a. Find a sequence of seven intervals that contain $5^{\sqrt{7}}$ whose endpoints get successively closer to $5^{\sqrt{7}}$. Write your intervals in the form $5^r < 5^{\sqrt{7}} < 5^s$ for rational numbers r and s.

$$5^2 < 5^{\sqrt{7}} < 5^3$$
$$5^{2.6} < 5^{\sqrt{7}} < 5^{2.7}$$
$$5^{2.64} < 5^{\sqrt{7}} < 5^{2.65}$$
$$5^{2.645} < 5^{\sqrt{7}} < 5^{2.646}$$
$$5^{2.6457} < 5^{\sqrt{7}} < 5^{2.6458}$$
$$5^{2.64575} < 5^{\sqrt{7}} < 5^{2.64576}$$
$$5^{2.645751} < 5^{\sqrt{7}} < 5^{2.645752}$$

b. Use your calculator to find approximations to four decimal places of the endpoints of the intervals in part (a).

$$\mathbf{25.0000 < 5^{\sqrt{7}} < 125.0000}$$
$$\mathbf{65.6632 < 5^{\sqrt{7}} < 77.1292}$$
$$\mathbf{70.0295 < 5^{\sqrt{7}} < 71.1657}$$
$$\mathbf{70.5953 < 5^{\sqrt{7}} < 70.7090}$$
$$\mathbf{70.6749 < 5^{\sqrt{7}} < 70.6862}$$
$$\mathbf{70.6805 < 5^{\sqrt{7}} < 70.6817}$$
$$\mathbf{70.6807 < 5^{\sqrt{7}} < 70.6808}$$

c. Based on your work in part (b), what is your best estimate of the value of $5^{\sqrt{7}}$?

$$\mathbf{5^{\sqrt{7}} \approx 70.681}$$

7. Can the value of an irrational number raised to an irrational power ever be rational?

Yes. For instance, in part (b) above, $\sqrt{2}$ is irrational and the number $\sqrt{2}^{\sqrt{2}}$ is either irrational or rational. If it is rational, then this is an example of an irrational number raised to an irrational power that is rational. If it is not, then $\sqrt{2}^{\sqrt{2}}$ is irrational and part (b) is an example of an irrational number raised to an irrational power that is rational.