Lesson 4: Properties of Exponents and Radicals

Student Outcomes

- Students rewrite expressions involving radicals and rational exponents using the properties of exponents.

Lesson Notes

In Lesson 1, students reviewed the properties of exponents for integer exponents before establishing the meaning of the n^{th} root of a positive real number and how it can be expressed as a rational exponent in Lesson 3. In Lesson 4, students extend properties of exponents that applied to expressions with integer exponents to expressions with rational exponents. In each case, the notation $b^{\frac{1}{n}}$ specifically indicates the principal root (e.g., $2^{\frac{1}{2}}$ is $\sqrt{2}$, as opposed to $-\sqrt{2}$).

This lesson extends students' thinking using the properties of radicals and the definitions from Lesson 3 so that they can see why it makes sense that the properties of exponents hold for any rational exponents (**N-RN.A.1**). Examples and exercises work to establish fluency with the properties of exponents when the exponents are rational numbers and emphasize rewriting expressions and evaluating expressions using the properties of exponents and radicals (**N-RN.A.2**).

Classwork

Opening (2 minutes)

Students revisit the properties of square roots and cube roots studied in Module 1 to remind them that we extended those to any n^{th} root in Lesson 3. So, they are now ready to verify that the properties of exponents hold for rational exponents.

Draw students' attention to a chart posted prominently on the wall or to their notebooks where the properties of exponents and radicals are displayed, including those developed in Lesson 3.

Remind students of the description of exponential expressions of the form $b^{\frac{m}{n}}$, which they will be making use of throughout the lesson:

Let b be any positive real number, and m, n be any integers with $n > 0$; then $b^{\frac{m}{n}} = \sqrt[n]{b^m}$ and $b^{\frac{m}{n}} = \left(\sqrt[n]{b}\right)^m$.

Scaffolding:

- Throughout the lesson, remind students of past properties of integer exponents and radicals either through an anchor chart posted on the wall or by writing relevant properties as they come up. Included is a short list of previous properties used in this module.
- For all real numbers a, $b > 0$, and all integers m, n:

$$b^m \cdot b^n = b^{m+n}$$
$$(b^m)^n = b^{mn}$$
$$(ab)^m = a^m \cdot b^m$$
$$b^{-m} = \frac{1}{b^m}$$
$$\sqrt[n]{b} = b^{\frac{1}{n}}$$
$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$$
$$\sqrt[n]{b^n} = \left(\sqrt[n]{b}\right)^n = b$$
$$\sqrt[n]{b^m} = \left(\sqrt[n]{b}\right)^m = b^{\frac{m}{n}}.$$

Opening Exercise (5 minutes)

These exercises briefly review content from Module 1 and the last lesson.

Opening Exercise

Write each exponent as a radical, and then use the definition and properties of radicals to write that expression as an integer.

a. $7^{\frac{1}{2}} \cdot 7^{\frac{1}{2}}$

$\sqrt{7} \cdot \sqrt{7} = \sqrt{49} = 7$

b. $3^{\frac{1}{3}} \cdot 3^{\frac{1}{3}} \cdot 3^{\frac{1}{3}}$

$\sqrt[3]{3} \cdot \sqrt[3]{3} \cdot \sqrt[3]{3} = \sqrt[3]{9} \cdot \sqrt[3]{3} = \sqrt[3]{27} = 3$

c. $12^{\frac{1}{2}} \cdot 3^{\frac{1}{2}}$

$\sqrt{12} \cdot \sqrt{3} = \sqrt{12 \cdot 3} = \sqrt{36} = 6$

d. $\left(64^{\frac{1}{3}}\right)^{\frac{1}{2}}$

$\sqrt{\sqrt[3]{64}} = \sqrt{4} = 2$

To transition from the Opening Exercise to Example 1, ask students to write the first two problems above in exponent form. Then, ask them to discuss with a partner whether or not it would be true in general that $b^{\frac{m}{n}} \cdot b^{\frac{p}{q}} = b^{\frac{m}{n}+\frac{p}{q}}$ for positive real numbers b where m, n, p, and q are integers with $n \neq 0$ and $q \neq 0$.

MP.3 & MP.8

- How could you write the $\sqrt{7} \cdot \sqrt{7} = 7$ with rational exponents? How about $\sqrt[3]{3} \cdot \sqrt[3]{3} \cdot \sqrt[3]{3} = 3$?
  - $7^{\frac{1}{2}} \cdot 7^{\frac{1}{2}} = 7^1$ *and* $3^{\frac{1}{3}} \cdot 3^{\frac{1}{3}} \cdot 3^{\frac{1}{3}} = 3^1$
- Based on these examples, is the exponent property $b^m \cdot b^n = b^{m+n}$ valid when m and n are rational numbers? Explain how you know.
  - *Since the exponents on the left side of each statement add up to the exponents on the right side, it appears to be true. However, the right side exponent was always* 1. *If we work with* $\sqrt[3]{8} \cdot \sqrt[3]{8} = \sqrt[3]{64}$ *and write it in exponent form, we would get (after noting* $\sqrt[3]{64} = \sqrt[3]{8^2}$*) that* $8^{\frac{1}{3}} \cdot 8^{\frac{1}{3}} = 8^{\frac{2}{3}}$. *So, it appears to be true in general. Note that examples alone do not prove that a mathematical statement is always true.*

In the rest of this lesson, we will make sense of these observations in general to extend the properties of exponents to rational numbers by applying the definition of the n root *of* b and the properties of radicals introduced in Lesson 3.

Examples 1–3 (10 minutes)

In the previous lesson, we assumed that the exponent property $b^m b^n = b^{m+n}$ for positive real numbers b and integers m and n would also hold for rational exponents when the exponents were of the form $\frac{1}{n}$, where n was a positive integer. This example will help students see that the property below makes sense for any rational exponent:

$$b^{\frac{m}{n}} \cdot b^{\frac{p}{q}} = b^{\frac{m}{n}+\frac{p}{q}}, \text{ where } m, n, p, \text{ and } q \text{ are integers with } n > 0 \text{ and } q > 0.$$

Perhaps model Example 1 below and then have students work with a partner on Example 2. Make sure students include a justification for each step in the problem. When you get to Example 3, be sure to use the following discussion questions to guide students.

MP.7 & MP.8

- How can we write these expressions using radicals?
  - *In Lesson 3, we learned that $b^{\frac{1}{n}} = \sqrt[n]{b}$ and $b^{\frac{m}{n}} = \sqrt[n]{b^m}$ for positive real numbers b and positive integers m and n.*
- Which properties help us to write the expression as a single radical?
  - *The property of radicals that states $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$ for positive real numbers a and b and positive integer n, and the property of exponents that states $b^m \cdot b^n = b^{m+n}$ for positive real numbers b and integers m and n.*
- How do we rewrite this expression in exponent form?
  - *In Lesson 3, we related radicals and rational exponents by $b^{\frac{m}{n}} = \sqrt[n]{b^m}$.*
- What makes Example 3 different from Examples 1 and 2?
  - *The exponents have different denominators, so when we write the expression in radical form, the roots are not the same, and we cannot apply the property that $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$.*
- Can you think of a way to rewrite the problem so it looks like the first two problems?
  - *We can write the exponents as equivalent fractions with the same denominator.*

Examples 1–3

Write each expression in the form $b^{\frac{m}{n}}$ for positive real numbers b and integers m and n with $n > 0$ by applying the properties of radicals and the definition of n^{th} root.

1. $b^{\frac{1}{4}} \cdot b^{\frac{1}{4}}$

By the definition of nth root,

$b^{\frac{1}{4}} \cdot b^{\frac{1}{4}} = \sqrt[4]{b} \cdot \sqrt[4]{b}$.

$= \sqrt[4]{b \cdot b}$ *By the properties of radicals and properties of exponents*

$= \sqrt[4]{b^2}$

$= b^{\frac{2}{4}}$ *By the definition of $b^{\frac{m}{n}}$*

The rational number $\frac{2}{4}$ is equal to $\frac{1}{2}$. Thus,

$b^{\frac{1}{4}} \cdot b^{\frac{1}{4}} = b^{\frac{1}{2}}$.

Scaffolding:

- Throughout the lesson, you can create parallel problems to demonstrate that these problems work with numerical values as well.
- For example, in part (a) substitute 4 for b.
- In part (b), substitute a perfect cube such as 8 or 27 for b.

2. $b^{\frac{1}{3}} \cdot b^{\frac{4}{3}}$

$b^{\frac{1}{3}} \cdot b^{\frac{4}{3}} = \sqrt[3]{b} \cdot \sqrt[3]{b^4}$ *By the definition of* $b^{\frac{1}{n}}$ *and* $b^{\left(\frac{m}{n}\right)}$

$= \sqrt[3]{b \cdot b^4}$ *By the properties of radicals and properties of exponents*

$= \sqrt[3]{b^5}$

$= b^{\frac{5}{3}}$ *By the definition of* $b^{\frac{m}{n}}$

Thus,

$b^{\frac{1}{3}} \cdot b^{\frac{4}{3}} = b^{\frac{5}{3}}.$

3. $b^{\frac{1}{5}} \cdot b^{\frac{3}{4}}$

Write the exponents as equivalent fractions with the same denominator.

$$b^{\frac{1}{5}} \cdot b^{\frac{3}{4}} = b^{\frac{4}{20}} \cdot b^{\frac{15}{20}}$$

Rewrite in radical form.

$$= \sqrt[20]{b^4} \cdot \sqrt[20]{b^{15}}$$

Rewrite as a single radical expression.

$$= \sqrt[20]{b^4 \cdot b^{15}}$$
$$= \sqrt[20]{b^{19}}$$

Rewrite in exponent form using the definition.

$$= b^{\frac{19}{20}}$$

Thus,

$$b^{\frac{1}{5}} \cdot b^{\frac{3}{4}} = b^{\frac{19}{20}}.$$

- Now add the exponents in each example. What is $\frac{1}{4}+\frac{1}{4}$? $\frac{1}{3}+\frac{4}{3}$? $\frac{1}{5}+\frac{3}{4}$?
  - $\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$, $\frac{1}{3}+\frac{4}{3}=\frac{5}{3}$, *and* $\frac{1}{5}+\frac{3}{4}=\frac{19}{20}$.
- What do you notice about these sums and the value of the exponent when we rewrote each expression?
  - *The sum of the exponents was equal to the exponent of the answer.*

Based on these examples, particularly the last one, it seems reasonable to extend the properties of exponents to hold when the exponents are any rational number. Thus, we can state the following property.

- For any integers m, n, p and q, with $n > 0$ and $q > 0$, and any real numbers b so that $b^{\frac{1}{n}}$ and $b^{\frac{1}{q}}$ are defined,

$$b^{\frac{m}{n}} \cdot b^{\frac{p}{q}} = b^{\frac{m}{n}+\frac{p}{q}}.$$

Have students copy this property into their notes along with the ones listed below. You can also write these properties on a piece of chart paper and display them in your classroom. These properties are listed in the lesson summary.

In a similar fashion, the other properties of exponents can be extended to hold for any rational exponents as well.

- For any integers m, n, p, and q, with $n > 0$ and $q > 0$, and any real numbers a and b so that $a^{\frac{1}{n}}$, $b^{\frac{1}{n}}$, and $b^{\frac{1}{q}}$ are defined,

$$b^{\frac{m}{n}} = \sqrt[n]{b^m}$$
$$\left(b^{\frac{1}{n}}\right)^n = b$$
$$(b^n)^{\frac{1}{n}} = b$$
$$(ab)^{\frac{m}{n}} = a^{\frac{m}{n}} \cdot b^{\frac{m}{n}}$$
$$\left(b^{\frac{m}{n}}\right)^{\frac{p}{q}} = b^{\frac{mp}{nq}}$$
$$b^{-\frac{m}{n}} = \frac{1}{b^{\frac{m}{n}}}.$$

At this point, you might have your class look at the opening exercise again and ask them which property could be used to simplify each problem.

For advanced learners, a derivation of the property we explored in Example 1 is provided below.

Rewrite $b^{\frac{m}{n}}$ and $b^{\frac{p}{q}}$ as equivalent exponential expressions in which the exponents have the same denominator, and apply the definition of the $b^{\frac{m}{n}}$as the n^{th} root.

By the definition of $b^{\frac{m}{n}}$ and then using properties of algebra, we can rewrite the exponent to be $\frac{m}{n} + \frac{\mathrm{p}}{\mathrm{q}}$.

$$\begin{aligned}
b^{\frac{m}{n}} \cdot b^{\frac{p}{q}} &= b^{\frac{mq}{nq}} b^{\frac{np}{nq}} \\
&= \sqrt[nq]{b^{mq}} \cdot \sqrt[nq]{b^{np}} \\
&= \sqrt[nq]{b^{mq} \cdot b^{np}} \\
&= \sqrt[nq]{b^{mq+np}} \\
&= b^{\frac{mq+np}{nq}} \\
&= b^{\frac{mq}{nq}+\frac{np}{nq}} \\
&= b^{\frac{m}{n}+\frac{p}{q}}
\end{aligned}$$

Exercises 1–4 (6 minutes)

Have students work with a partner or in small groups to complete these exercises. Students are rewriting expressions with rational exponents using the properties presented above. As students work, emphasize that we do not need to write these expressions using radicals since we have just established that we believe that the properties of exponents hold for rational numbers. In the last two exercises, students will have to use their knowledge of radicals to rewrite the answers without exponents.

Scaffolding:

- When students get to these exercises, you may need to remind them that it is often easier to rewrite $b^{\frac{m}{n}}$ as $\left(\sqrt[n]{b}\right)^m$ when you are trying to evaluate radical expressions.
- For students who struggle with arithmetic, you can provide a scientific calculator, but be sure to encourage them to show the steps using the properties.

Exercises 1–4

Write each expression in the form $b^{\frac{m}{n}}$. If a numeric expression is a rational number, then write your answer without exponents.

1. $b^{\frac{2}{3}} \cdot b^{\frac{1}{2}}$

   $b^{\frac{2}{3}+\frac{1}{2}} = b^{\frac{7}{6}}$

2. $\left(b^{-\frac{1}{5}}\right)^{\frac{2}{3}}$

   $b^{-\frac{1}{5}\cdot\frac{2}{3}} = b^{-\frac{2}{15}}$

3. $64^{\frac{1}{3}} \cdot 64^{\frac{3}{2}}$

   $$\begin{aligned} 64^{\frac{1}{3}+\frac{3}{2}} &= 64^{\frac{11}{6}} \\ &= \left(\sqrt[6]{64}\right)^{11} \\ &= 2^{11} \\ &= 2048 \end{aligned}$$

4. $\left(\frac{9^3}{4^2}\right)^{\frac{3}{2}}$

   $$\begin{aligned} \left(\frac{9^3}{4^2}\right)^{\frac{3}{2}} &= \frac{9^{\frac{9}{2}}}{4^3} \\ &= \frac{\left(\sqrt[2]{9}\right)^9}{64} \\ &= \frac{3^9}{64} \\ &= \frac{19683}{64} \end{aligned}$$

Example 4 (5 minutes)

- We can rewrite radical expressions using properties of exponents. There are other methods for rewriting radical expressions, but this example models using the properties of exponents. Often, textbooks and exams give directions to simplify an expression, which is vague unless we specify what it means. We want students to develop fluency in applying the properties, so the directions here say to rewrite in a specific fashion.

Example 4

Rewrite the radical expression $\sqrt{48x^5y^4z^2}$ so that no perfect square factors remain inside the radical.

$$\begin{aligned}\sqrt{48\cdot x^5\cdot y^4\cdot z^2} &= \left(4^2\cdot 3\cdot x^5\cdot y^4\cdot z^2\right)^{\frac{1}{2}}\\ &= 4^{\frac{2}{2}}\cdot 3^{\frac{1}{2}}\cdot x^{\frac{5}{2}}\cdot y^{\frac{4}{2}}\cdot z^{\frac{2}{2}}\\ &= 4\cdot 3^{\frac{1}{2}}\cdot x^{2+\frac{1}{2}}\cdot y^2\cdot z\\ &= 4x^2y^2z\cdot(3x)^{\frac{1}{2}}\\ &= 4x^2y^2z\sqrt{3x}\end{aligned}$$

- Although this process may seem drawn out, once it has been practiced, most of the steps can be internalized and expressions are quickly rewritten using this technique.

Exercise 5 (5 minutes)

Students should work individually or in pairs on this exercise.

Exercise 5

5. **If $x = 50$, $y = 12$, and $z = 3$, the following expressions are difficult to evaluate without using properties of radicals or exponents (or a calculator). Use the definition of rational exponents and properties of exponents to rewrite each expression in a form where it can be easily evaluated, and then use that exponential expression to find the value.**

   a. $\sqrt{8x^3y^2}$

   $$\begin{aligned}\sqrt{8x^3y^2} &= 2^{\frac{3}{2}}x^{\frac{3}{2}}y^{\frac{2}{2}}\\ &= 2xy\cdot(2x)^{\frac{1}{2}}\end{aligned}$$

   Evaluating, we get $2(50)(12)(2\cdot 50)^{\frac{1}{2}} = 100\cdot 12\cdot 10 = 12{,}000.$

   b. $\sqrt[3]{54y^7z^2}$

   $$\begin{aligned}\sqrt[3]{54y^7z^2} &= 27^{\frac{1}{3}}\cdot 2^{\frac{1}{3}}\cdot y^{\frac{7}{3}}\cdot z^{\frac{2}{3}}\\ &= 3y^2\cdot\left(2yz^2\right)^{\frac{1}{3}}\end{aligned}$$

   Evaluating, we get $3(12)^2(2\cdot 12\cdot 3^2)^{\frac{1}{3}} = 3(144)(216)^{\frac{1}{3}} = 3\cdot 144\cdot 6 = 2592.$

Exercise 6 (5 minutes)

This exercise will remind students that rational numbers can be represented in decimal form and will give them a chance to work on their numeracy skills. Students should work on this exercise with a partner or in their groups to encourage dialogue and debate. Have a few students demonstrate their results to the entire class. There is more than one possible approach, so when you debrief, try to share different approaches that show varied reasoning. Conclude with one or two strong arguments. Students can confirm their reasoning using a calculator.

MP.3

Exercise 6

6. **Order these numbers from smallest to largest. Explain your reasoning.**

$16^{2.5}$ $\quad$ $9^{3.1}$ $\quad$ $32^{1.2}$

$16^{2.5}$ is between 256 and $4{,}096$. We can rewrite $16^{2.5} = (2^4)^{2.5}$, which is 2^{10}.

$32^{1.2}$ is between 32 and $1{,}024$. We can rewrite $32^{1.2} = (2^5)^{1.2}$, which is 2^6.

$9^{3.6}$ is larger than $9^3 = 729$.

Thus, $32^{1.2}$ is clearly the smallest number, but we need to determine if $9^{3.6}$ is greater than or less than $1{,}024$. To do this, we know that $9^{3.1} = 9^{3+0.6} = 9^3 \cdot 9^{0.6}$. This means that $9^{3.6} > 9^3 \cdot 9^{0.5}$, and $9^3 \cdot 9^{0.5} = 729 \cdot 3$, which is greater than $1{,}024$.

Thus, the numbers in order from smallest to largest are $32^{1.2}$, $16^{2.5}$, and $9^{3.6}$.

Closing (2 minutes)

Have students summarize the definition and properties of rational exponents and any important ideas from the lesson by creating a list of what they have learned so far about the properties of exponents and radicals. Circulate around the classroom to informally assess understanding. Reinforce the properties of exponents listed below.

Lesson Summary

The properties of exponents developed in Grade 8 for integer exponents extend to rational exponents.

That is, for any integers m, n, p, and q, with $n > 0$ and $q > 0$, and any real numbers a and b so that $a^{\frac{1}{n}}$, $b^{\frac{1}{n}}$, and $b^{\frac{1}{q}}$ are defined, we have the following properties of exponents:

$$b^{\frac{m}{n}} \cdot b^{\frac{p}{q}} = b^{\frac{m}{n}+\frac{p}{q}}$$
$$b^{\frac{m}{n}} = \sqrt[n]{b^m}$$
$$\left(b^{\frac{1}{n}}\right)^n = b$$
$$(b^n)^{\frac{1}{n}} = b$$
$$(ab)^{\frac{m}{n}} = a^{\frac{m}{n}} \cdot b^{\frac{m}{n}}$$
$$\left(b^{\frac{m}{n}}\right)^{\frac{p}{q}} = b^{\frac{mp}{nq}}$$
$$b^{-\frac{m}{n}} = \frac{1}{b^{\frac{m}{n}}}.$$

Exit Ticket (5 minutes)

Name ______________________________ Date ____________________

Lesson 4: Properties of Exponents and Radicals

Exit Ticket

1. Find the exact value of $9^{\frac{11}{10}} \cdot 9^{\frac{2}{5}}$ without using a calculator.

2. Justify that $\sqrt[3]{8} \cdot \sqrt[3]{8} = \sqrt{16}$ using the properties of exponents in at least two different ways.

Exit Ticket Sample Solutions

1. Find the exact value of $9^{\frac{11}{10}} \cdot 9^{\frac{2}{5}}$ without using a calculator.

$$\begin{aligned} 9^{\frac{11}{10}} \cdot 9^{\frac{2}{5}} &= 9^{\frac{11}{10}+\frac{2}{5}} \\ &= 9^{\frac{15}{10}} \\ &= 9^{\frac{3}{2}} \\ &= \left(\sqrt[2]{9}\right)^3 \\ &= 27 \end{aligned}$$

2. Justify that $\sqrt[3]{8} \cdot \sqrt[3]{8} = \sqrt{16}$ using the properties of exponents in at least two different ways.

$$\begin{aligned} 8^{\frac{1}{3}} \cdot 8^{\frac{1}{3}} &= 8^{\frac{2}{3}} \\ &= (2^3)^{\frac{2}{3}} \\ &= 2^2 \\ &= 2^{\frac{4}{2}} \\ &= (2^4)^{\frac{1}{2}} \\ &= \sqrt{16} \end{aligned}$$

$$\begin{aligned} 16^{\frac{1}{2}} &= (4 \cdot 4)^{\frac{1}{2}} \\ &= 4^{\frac{1}{2}} \cdot 4^{\frac{1}{2}} \\ &= 2 \cdot 2 \\ &= 8^{\frac{1}{3}} \cdot 8^{\frac{1}{3}} \\ &= \sqrt[3]{8} \cdot \sqrt[3]{8} \end{aligned}$$

Problem Set Sample Solutions

1. Evaluate each expression if $a = 27$ and $b = 64$.

a. $\sqrt[3]{a}\sqrt{b}$

$\sqrt[3]{27} \cdot \sqrt{64} = 3 \cdot 8 = 24$

b. $\left(3\sqrt[3]{a}\sqrt{b}\right)^2$

$(3 \cdot 3 \cdot 8)^2 = 5184$

c. $\left(\sqrt[3]{a} + 2\sqrt{b}\right)^2$

$(3 + 2 \cdot 8)^2 = 361$

d. $a^{-\frac{2}{3}} + b^{\frac{3}{2}}$

$\dfrac{1}{\left(\sqrt[3]{27}\right)^2} + \left(\sqrt{64}\right)^3 = \dfrac{1}{9} + 512 = 512\dfrac{1}{9}$

e. $\left(a^{-\frac{2}{3}} \cdot b^{\frac{3}{2}}\right)^{-1}$

$\left(\dfrac{1}{9} \cdot 512\right)^{-1} = \dfrac{9}{512}$

f. $\left(a^{-\frac{2}{3}} - \dfrac{1}{8}b^{\frac{3}{2}}\right)^{-1}$

$\left(\dfrac{1}{9} - \dfrac{1}{8} \cdot 512\right)^{-1} = \left(-\dfrac{575}{9}\right)^{-1} = -\dfrac{9}{575}$

2. Rewrite each expression so that each term is in the form kx^n, where k is a real number, x is a positive real number, and n is a rational number.

a. $x^{-\frac{2}{3}} \cdot x^{\frac{1}{3}}$

$x^{-\frac{1}{3}}$

b. $2x^{\frac{1}{2}} \cdot 4x^{-\frac{5}{2}}$

$8x^{-2}$

c. $\dfrac{10x^{\frac{1}{3}}}{2x^2}$

$5x^{-\frac{5}{3}}$

d. $\left(3x^{\frac{1}{4}}\right)^{-2}$

$\frac{1}{9}x^{-\frac{1}{2}}$

e. $x^{\frac{1}{2}}\left(2x^2 - \frac{4}{x}\right)$

$2x^{\frac{5}{2}} - 4x^{-\frac{1}{2}}$

f. $\sqrt[3]{\dfrac{27}{x^6}}$

$3x^{-2}$

g. $\sqrt[3]{x} \cdot \sqrt[3]{-8x^2} \cdot \sqrt[3]{27x^4}$

$-6x^{\frac{7}{3}}$

h. $\dfrac{2x^4 - x^2 - 3x}{\sqrt{x}}$

$2x^{\frac{7}{2}} - x^{\frac{3}{2}} - 3x^{\frac{1}{2}}$

i. $\dfrac{\sqrt{x} - 2x^{-3}}{4x^2}$

$\frac{1}{4}x^{-\frac{3}{2}} - \frac{1}{2}x^{-5}$

3. Show that $\left(\sqrt{x} + \sqrt{y}\right)^2$ is not equal to $x^1 + y^1$ when $x = 9$ and $y = 16$.

When $x = 9$ and $y = 16$, the two expressions are $\left(\sqrt{9} + \sqrt{16}\right)^2$ and $9 + 16$. The first expression simplifies to 49, and the second simplifies to 25. The two expressions are not equal.

4. Show that $\left(x^{\frac{1}{2}} + y^{\frac{1}{2}}\right)^{-1}$ is not equal to $\dfrac{1}{x^{\frac{1}{2}}} + \dfrac{1}{y^{\frac{1}{2}}}$ when $x = 9$ and $y = 16$.

When $x = 9$ and $y = 16$, the two expressions are $\left(\sqrt{9} + \sqrt{16}\right)^{-1}$ and $\frac{1}{\sqrt{9}} + \frac{1}{\sqrt{16}}$. The first expression is $\frac{1}{7}$, and the second one is $\frac{1}{3} + \frac{1}{4} = \frac{7}{12}$. The two expressions are not equal.

5. From these numbers, select (a) one that is negative, (b) one that is irrational, (c) one that is not a real number, and (d) one that is a perfect square:

$$3^{\frac{1}{2}} \cdot 9^{\frac{1}{2}},\ 27^{\frac{1}{3}} \cdot 144^{\frac{1}{2}},\ 64^{\frac{1}{3}} - 64^{\frac{2}{3}}, \text{ and } \left(4^{-\frac{1}{2}} - 4^{\frac{1}{2}}\right)^{\frac{1}{2}}.$$

The first number, $3^{\frac{1}{2}} \cdot 9^{\frac{1}{2}}$, is irrational, the second number, $27^{\frac{1}{3}} \cdot 144^{\frac{1}{2}}$, is a perfect square, the third number, $64^{\frac{1}{3}} - 64^{\frac{2}{3}}$, is negative, and the last number, $\left(4^{-\frac{1}{2}} - 4^{\frac{1}{2}}\right)^{\frac{1}{2}}$, is not a real number.

6. Show that the expression $2^n \cdot 4^{n+1} \cdot \left(\frac{1}{8}\right)^n$ is equal to 4.

$2^n \cdot 2^{2n+2} \cdot 2^{-3n} = 2^2 = 4$

7. Express each answer as a power of 10.

a. Multiply 10^n by 10.

$10^n \cdot 10 = 10^{n+1}$

b. Multiply $\sqrt{10}$ by 10^n.

$10^{\frac{1}{2}} \cdot 10^n = 10^{\frac{1}{2}+n}$

c. Square 10^n.

$(10^n)^2 = 10^{2n}$

d. Divide $100 \cdot 10^n$ by 10^{2n}.

$\frac{100 \cdot 10^n}{10^{2n}} = 10^{2+n-2n} = 10^{2-n}$

e. Show that $10^n = 11 \cdot 10^n - 10^{n+1}$.

$$\begin{aligned} 11 \cdot 10^n - 10^{n+1} &= 11 \cdot 10^n - 10 \cdot 10^n \\ &= 10^n(11-10) \\ &= 10^n \cdot 1 \\ &= 10^n \end{aligned}$$

8. Rewrite each of the following radical expressions as an equivalent exponential expression in which each variable occurs no more than once.

a. $\sqrt{8x^2y}$

$$\begin{aligned} \sqrt{8x^2y} &= 2^{\frac{2}{2}}x^{\frac{2}{2}}(2y)^{\frac{1}{2}} \\ &= 2x \cdot (2y)^{\frac{1}{2}} \\ &= 2^{\frac{3}{2}}\, x\, y^{\frac{1}{2}} \end{aligned}$$

b. $\sqrt[5]{96x^3y^{15}z^6}$

$$\begin{aligned} \sqrt[5]{96x^3y^{15}z^6} &= (32 \cdot 3 \cdot x^3 \cdot y^{15} \cdot z^6)^{\frac{1}{5}} \\ &= 32^{\frac{1}{5}} \cdot 3^{\frac{1}{5}} \cdot x^{\frac{3}{5}} \cdot y^{\frac{15}{5}} \cdot z^{\frac{6}{5}} \\ &= 2 \cdot 3^{\frac{1}{5}} \cdot x^{\frac{3}{5}} \cdot y^3 \cdot z^{\frac{6}{5}} \end{aligned}$$

9. Use properties of exponents to find two integers that are upper and lower estimates of the value of $4^{1.6}$.

$$4^{1.5} < 4^{1.6} < 4^2$$

$$4^{1.5} = 2^3 = 8 \textit{ and } 4^2 = 16, \textit{ so } 8 < 4^{1.6} < 16$$

10. **Use properties of exponents to find two integers that are upper and lower estimates of the value of $8^{2.3}$.**

$$8^2 < 8^{2.3} < 8^{2+\frac{1}{3}}$$

$$8^2 = 64 \textit{ and } 8^{\frac{1}{3}} = 2\textit{, so } 8^{2+\frac{1}{3}} = 8^2 \cdot 8^{\frac{1}{3}} = 128. \textit{ Thus, } 64 < 8^{2.3} < 128.$$

11. **Kepler's third law of planetary motion relates the average distance, a, of a planet from the Sun to the time, t, it takes the planet to complete one full orbit around the Sun according to the equation $t^2 = a^3$. When the time, t, is measured in Earth years, the distance, a, is measured in astronomical units (AU). (One AU is equal to the average distance from Earth to the Sun.)**

a. **Find an equation for t in terms of a and an equation for a in terms of t.**

$$t^2 = a^3$$
$$t = a^{\frac{3}{2}}$$
$$a = t^{\frac{2}{3}}$$

b. **Venus takes about 0.616 Earth years to orbit the Sun. What is its average distance from the Sun?**

$a = (0.616)^{\frac{2}{3}} \approx 0.724 \text{ AU}$

c. **Mercury is an average distance of 0.387 AU from the Sun. About how long is its orbit in Earth years?**

$t = (0.387 \text{ AU})^{\frac{3}{2}} \approx 0.241 \text{ year}$