Lesson 4: Properties of Exponents and Radicals

Classwork

Opening Exercise

Write each exponent as a radical, and then use the definition and properties of radicals to write that expression as an integer.

a. $7^{\frac{1}{2}} \cdot 7^{\frac{1}{2}}$

b. $3^{\frac{1}{3}} \cdot 3^{\frac{1}{3}} \cdot 3^{\frac{1}{3}}$

c. $12^{\frac{1}{2}} \cdot 3^{\frac{1}{2}}$

d. $\left(64^{\frac{1}{3}}\right)^{\frac{1}{2}}$

Examples 1–3

Write each expression in the form $b^{\frac{m}{n}}$ for positive real numbers b and integers m and n with $n > 0$ by applying the properties of radicals and the definition of n^{th} root.

1. $b^{\frac{1}{4}} \cdot b^{\frac{1}{4}}$

2. $b^{\frac{1}{3}} \cdot b^{\frac{4}{3}}$

3. $b^{\frac{1}{5}} \cdot b^{\frac{3}{4}}$

Exercises 1–4

Write each expression in the form $b^{\frac{m}{n}}$. If a numeric expression is a rational number, then write your answer without exponents.

1. $b^{\frac{2}{3}} \cdot b^{\frac{1}{2}}$

2. $\left(b^{-\frac{1}{5}}\right)^{\frac{2}{3}}$

3. $64^{\frac{1}{3}} \cdot 64^{\frac{3}{2}}$

4. $\left(\frac{9^3}{4^2}\right)^{\frac{3}{2}}$

Example 4

Rewrite the radical expression $\sqrt{48x^5y^4z^2}$ so that no perfect square factors remain inside the radical.

Exercise 5

5. If $x = 50$, $y = 12$, and $z = 3$, the following expressions are difficult to evaluate without using properties of radicals or exponents (or a calculator). Use the definition of rational exponents and properties of exponents to rewrite each expression in a form where it can be easily evaluated, and then use that exponential expression to find the value.

   a. $\sqrt{8x^3y^2}$

   b. $\sqrt[3]{54y^7z^2}$

Exercise 6

6. Order these numbers from smallest to largest. Explain your reasoning.

   $16^{2.5}$ $\quad 9^{3.1}$ $\quad 32^{1.2}$

Lesson Summary

The properties of exponents developed in Grade 8 for integer exponents extend to rational exponents.

That is, for any integers m, n, p, and q, with $n > 0$ and $q > 0$ and any real numbers a and b so that $a^{\frac{1}{n}}$, $b^{\frac{1}{n}}$, and $b^{\frac{1}{q}}$ are defined, we have the following properties of exponents:

$$b^{\frac{m}{n}} \cdot b^{\frac{p}{q}} = b^{\frac{m}{n}+\frac{p}{q}}$$

$$b^{\frac{m}{n}} = \sqrt[n]{b^m}$$

$$\left(b^{\frac{1}{n}}\right)^n = b$$

$$(b^n)^{\frac{1}{n}} = b$$

$$(ab)^{\frac{m}{n}} = a^{\frac{m}{n}} \cdot b^{\frac{m}{n}}$$

$$\left(b^{\frac{m}{n}}\right)^{\frac{p}{q}} = b^{\frac{mp}{nq}}$$

$$b^{-\frac{m}{n}} = \frac{1}{b^{\frac{m}{n}}}$$

Problem Set

1. Evaluate each expression if $a = 27$ and $b = 64$.

   a. $\sqrt[3]{a}\sqrt{b}$

   b. $\left(3\sqrt[3]{a}\sqrt{b}\right)^2$

   c. $\left(\sqrt[3]{a} + 2\sqrt{b}\right)^2$

   d. $a^{-\frac{2}{3}} + b^{\frac{3}{2}}$

   e. $\left(a^{-\frac{2}{3}} \cdot b^{\frac{3}{2}}\right)^{-1}$

   f. $\left(a^{-\frac{2}{3}} - \frac{1}{8}b^{\frac{3}{2}}\right)^{-1}$

2. Rewrite each expression so that each term is in the form kx^n, where k is a real number, x is a positive real number, and n is a rational number.

   a. $x^{-\frac{2}{3}} \cdot x^{\frac{1}{3}}$

   b. $2x^{\frac{1}{2}} \cdot 4x^{-\frac{5}{2}}$

   c. $\frac{10x^{\frac{1}{3}}}{2x^2}$

   d. $\left(3x^{\frac{1}{4}}\right)^{-2}$

   e. $x^{\frac{1}{2}}\left(2x^2 - \frac{4}{x}\right)$

   f. $\sqrt[3]{\frac{27}{x^6}}$

   g. $\sqrt[3]{x} \cdot \sqrt[3]{-8x^2} \cdot \sqrt[3]{27x^4}$

   h. $\frac{2x^4 - x^2 - 3x}{\sqrt{x}}$

   i. $\frac{\sqrt{x} - 2x^{-3}}{4x^2}$

3. Show that $(\sqrt{x}+\sqrt{y})^2$ is not equal to $x^1 + y^1$ when $x = 9$ and $y = 16$.

4. Show that $\left(x^{\frac{1}{2}} + y^{\frac{1}{2}}\right)^{-1}$ is not equal to $\frac{1}{x^{\frac{1}{2}}} + \frac{1}{y^{\frac{1}{2}}}$ when $x = 9$ and $y = 16$.

5. From these numbers, select (a) one that is negative, (b) one that is irrational, (c) one that is not a real number, and (d) one that is a perfect square:

$$3^{\frac{1}{2}} \cdot 9^{\frac{1}{2}},\ 27^{\frac{1}{3}} \cdot 144^{\frac{1}{2}},\ 64^{\frac{1}{3}} - 64^{\frac{2}{3}},\text{ and } \left(4^{-\frac{1}{2}} - 4^{\frac{1}{2}}\right)^{\frac{1}{2}}$$

6. Show that the expression $2^n \cdot 4^{n+1} \cdot \left(\frac{1}{8}\right)^n$ is equal to 4.

7. Express each answer as a power of 10.
   a. Multiply 10^n by 10.
   b. Multiply $\sqrt{10}$ by 10^n.
   c. Square 10^n.
   d. Divide $100 \cdot 10^n$ by 10^{2n}.
   e. Show that $10^n = 11 \cdot 10^n - 10^{n+1}$

8. Rewrite each of the following radical expressions as an equivalent exponential expression in which each variable occurs no more than once.
   a. $\sqrt{8x^2y}$
   b. $\sqrt[5]{96x^3y^{15}z^6}$

9. Use properties of exponents to find two integers that are upper and lower estimates of the value of $4^{1.6}$.

10. Use properties of exponents to find two integers that are upper and lower estimates of the value of $8^{2.3}$.

11. Kepler's third law of planetary motion relates the average distance, a, of a planet from the Sun to the time t it takes the planet to complete one full orbit around the Sun according to the equation $t^2 = a^3$. When the time, t, is measured in Earth years, the distance, a, is measured in astronomical units (AU). (One AU is equal to the average distance from Earth to the Sun.)
    a. Find an equation for t in terms of a and an equation for a in terms of t.
    b. Venus takes about 0.616 Earth years to orbit the Sun. What is its average distance from the Sun?
    c. Mercury is an average distance of 0.387 AU from the Sun. About how long is its orbit in Earth years?