Lesson 3: Rational Exponents—What are $2^{\frac{1}{2}}$ and $2^{\frac{1}{3}}$?

Student Outcomes

- Students will calculate quantities that involve positive and negative rational exponents.

Lesson Notes

Students extend their understanding of integer exponents to rational exponents by examining the graph of $f(x) = 2^x$ and estimating the values of $2^{\frac{1}{2}}$ and $2^{\frac{1}{3}}$. The lesson establishes the meaning of these numbers in terms of radical expressions and these form the basis of how we define expressions of the form $b^{\frac{1}{n}}$ before generalizing further to expressions of the form $b^{\frac{m}{n}}$, where b is a positive real number and m and n are integers, with $n \neq 0$ (**N-RN.A.1**). The lesson and Problem Set provide fluency practice in applying the properties of exponents to expressions containing rational exponents and radicals (**N-RN.A.2**). In the following lesson, students will verify that the definition of an expression with rational exponents, $b^{\frac{m}{n}} = \sqrt[n]{b^m}$, is consistent with the remaining exponential properties. The lesson begins with students creating a graph of a simple exponential function, $f(x) = 2^x$, which they studied in Module 3 of Algebra I. In that module, students also learned about geometric sequences and their relationship to exponential functions, which is a concept that we revisit at the end of this module. In Algebra I, students worked with exponential functions with integer domains. This lesson, together with the subsequent Lessons 4 and 5, helps students understand why the domain of an exponential function $f(x) = b^x$, where b is positive number and $b \neq 1$, is the set of real numbers. To do so we must establish what it means to raise b to a rational power and, in Lesson 5, to any real number power.

Classwork

Opening (1 minutes)

In Algebra I, students worked with geometric sequences and simple exponential functions. Remind them that in Lesson 1, we created formulas based on repeatedly doubling and halving a number when we modeled folding a piece of paper. We reviewed how to use the properties of exponents for expressions that had integer exponents.

Scaffolding:

- Encourage students to create a table of values to help them construct the graph.
- For advanced learners, have them repeat these exercises with the function $f(x) = 3^x$ and ask them to estimate $3^{\frac{1}{2}}$ and $3^{\frac{1}{3}}$.

Opening Exercise (5 minutes)

Have students graph $f(x) = 2^x$ for each integer x from $x = -2$ to $x = 5$ without using a graphing utility or calculator on the axes provided. Discuss the pattern of points and ask students to connect the points in a way that produces a smooth curve. Students should work these two problems independently and check their solutions with a partner if time permits before you lead a whole class discussion to review the solutions.

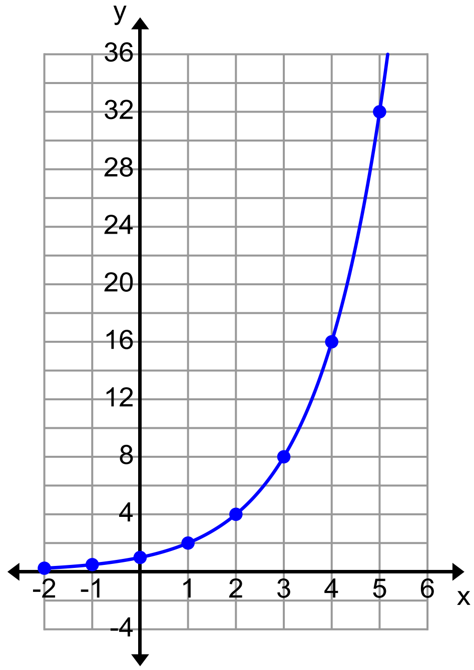
MP.3

Opening Exercise

a. **What is the value of $2^{\frac{1}{2}}$? Justify your answer.**

A possible student response follows: I think it will be around 1.5 because $2^0 = 1$ and $2^1 = 2$.

b. **Graph $f(x) = 2^x$ for each integer x from $x = -2$ to $x = 5$. Connect the points on your graph with a smooth curve.**



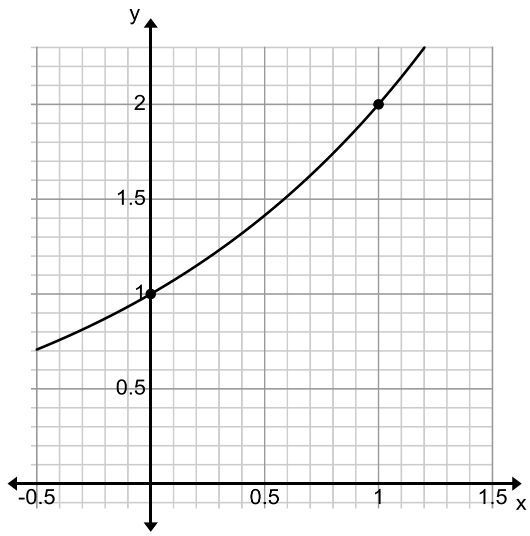
Ask for a few volunteers to explain their reasoning for their answers to Opening Exercise part (a). Then, debrief these two exercises by leading a short discussion.

- The directions in the Opening Exercise said to connect the points with a smooth curve. What does it imply about the domain of a function when we connect points that we have plotted?
  - *That the domain of the function includes the points between the points that we plotted.*
- How does your graph support or refute your answer to the first exercise? Do you need to modify your answer to the question: What is the value of $2^{\frac{1}{2}}$? Why or why not?
  - *If the domain is all real numbers, then the value of $2^{\frac{1}{2}}$ will be the y-coordinate of the point on the graph where $x = \frac{1}{2}$. From the graph it looks like the value is somewhere between 1 and 2. The scaling on this graph is not detailed enough for me to accurately refine my answer yet.*

Transition to the next set of exercises by telling the students that we can better estimate the value of $2^{\frac{1}{2}}$ by looking at a graph that is scaled more precisely. Have students proceed to the next questions. They should work in small groups or with a partner on these exercises.

The graph of $f(x) = 2^x$ shown below for the next exercises will appear in the student materials but you could also display the graph using technology. Have students estimate the value of $2^{\frac{1}{2}}$ and $2^{\frac{1}{3}}$ from the graph. Students should work by themselves or in pairs to work through the following questions without using a calculator.

The graph at right shows a close-up view of $f(x) = 2^x$ for $-0.5 < x < 1.5$.



c. **Find two consecutive integers that are over and underestimates of the value of $2^{\frac{1}{2}}$.**

$2^0 < 2^{\frac{1}{2}} < 2^1$
$1 < 2^{\frac{1}{2}} < 2$

d. **Does it appear that $2^{\frac{1}{2}}$ is halfway between the integers you specified in Exercise 1?**

No, it looks like $2^{\frac{1}{2}}$ is a little less than halfway between 1 and 2.

e. **Use the graph of $f(x) = 2^x$ to estimate the value of $2^{\frac{1}{2}}$.**

$2^{\frac{1}{2}} \approx 1.4$

f. **Use the graph of $f(x) = 2^x$ to estimate the value of $2^{\frac{1}{3}}$.**

$2^{\frac{1}{3}} \approx 1.25$

Discussion (9 minutes)

Before getting into the point of this lesson, which is to connect rational exponents to radical expressions, revisit the initial question with students.

> *Scaffolding:*
> - If needed, you can demonstrate the argument using perfect squares. For example, use a base of 4 instead of a base of 2.
>
> Show that $\left(4^{\frac{1}{2}}\right)^2 = 4$ and $\left(\sqrt{4}\right)^2 = 4$.

MP.6

- What is the value of $2^{\frac{1}{2}}$? Does anyone want to adjust their initial guess?
  - *Our initial guess was a little too big. It seems like 1.4 might be a better answer.*
- How could we make a better guess?
  - *We could look at the graph with even smaller increments for the scale using technology.*

If time permits, you can zoom in further on the graph of $f(x) = 2^x$ using a graphing calculator or other technology either by examining a graph or a table of values of x closer and closer to $\frac{1}{2}$.

Next, we will make the connection that $2^{\frac{1}{2}} = \sqrt{2}$. Walk students through the following questions, providing guidance as needed. Students proved that there was only one positive number that squared to 2 in Geometry, Module 2. You may need to remind them of this with a bit more detail if they are struggling to follow this argument.

- Assume for the moment that whatever $2^{\frac{1}{2}}$ means that it satisfies $b^m \cdot b^n = b^{m+n}$. Working with this assumption what is the value of $2^{\frac{1}{2}} \cdot 2^{\frac{1}{2}}$?
  - *It would be 2 because $2^{\frac{1}{2}} \cdot 2^{\frac{1}{2}} = 2^{\frac{1}{2}+\frac{1}{2}} = 2^1 = 2$.*
- What unique positive number squares to 2? That is, what is the only positive number that when multiplied by itself is equal to 2?
  - *By definition, we call the unique positive number that squares to 2 the square root of 2 and we write $\sqrt{2}$.*

MP.7

Write the following statements on the board and ask your students to compare them and think about what they must tell us about the meaning of $2^{\frac{1}{2}}$.

$$2^{\frac{1}{2}} \cdot 2^{\frac{1}{2}} = 2 \text{ and } \sqrt{2} \cdot \sqrt{2} = 2$$

- What do these two statements tell us about the meaning of $2^{\frac{1}{2}}$.
  - *Since both statements involve multiplying a number by itself and getting 2, and we know that there is only one number that does that, we can conclude that $2^{\frac{1}{2}} = \sqrt{2}$.*

At this point, you can have students confirm these results by using a calculator to approximate both $2^{\frac{1}{2}}$ and $\sqrt{2}$ to several decimal places. In the opening, we approximated $2^{\frac{1}{2}}$ graphically, and now we have shown it to be an irrational number.

Next ask students to think about the meaning of $2^{\frac{1}{3}}$ using a similar line of reasoning.

- Assume that whatever $2^{\frac{1}{3}}$ means it will satisfy $b^m \cdot b^n = b^{m+n}$. What is the value of $\left(2^{\frac{1}{3}}\right)\left(2^{\frac{1}{3}}\right)\left(2^{\frac{1}{3}}\right)$?
  - *The value is 2 because $\left(2^{\frac{1}{3}}\right)\left(2^{\frac{1}{3}}\right)\left(2^{\frac{1}{3}}\right) = 2^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = 2^1 = 2$.*

MP.7

- What is the value of $\sqrt[3]{2} \cdot \sqrt[3]{2} \cdot \sqrt[3]{2}$?
  - *The value is 2 because $\sqrt[3]{2} \cdot \sqrt[3]{2} \cdot \sqrt[3]{2} = \left(\sqrt[3]{2}\right)^3 = 2$.*
- What appears to be the meaning of $2^{\frac{1}{3}}$?
  - *Since both the exponent expression and the radical expression involve multiplying a number by itself three times and the result is equal to 2, we know that $2^{\frac{1}{3}} = \sqrt[3]{2}$.*

> *Scaffolding:*
> - If needed, you can demonstrate the argument using perfect cubes. For example, use a base of 8 instead of a base of 2.
>
> Show $\left(8^{\frac{1}{3}}\right)^3 = 8$ and $\left(\sqrt[3]{8}\right)^3 = 8$.

Students can also confirm using a calculator that the decimal approximations of $2^{\frac{1}{3}}$ and $\sqrt[3]{2}$ are the same. Next, we ask them to generalize their findings.

- Can we generalize this relationship? Does $2^{\frac{1}{4}} = \sqrt[4]{2}$? Does $2^{\frac{1}{10}} = \sqrt[10]{2}$? What is $2^{\frac{1}{n}}$, for any positive integer n? Why?
  - $2^{\frac{1}{n}} = \sqrt[n]{2}$ *because*

MP.8

$$\left(2^{\frac{1}{n}}\right)^n = \underbrace{\left(2^{\frac{1}{n}}\right)\left(2^{\frac{1}{n}}\right)\cdots\left(2^{\frac{1}{n}}\right)}_{n\text{ times}} = 2^{\overbrace{\frac{1}{n}+\frac{1}{n}+\cdots+\frac{1}{n}}^{n\text{ times}}} = 2^1 = 2.$$

MP.8 Have students confirm these using a calculator as well checking to see if the decimal approximations of $2^{\frac{1}{n}}$ and $\sqrt[n]{2}$ are the same for different values of n such as $4, 5, 6, 10, \ldots$. Be sure to share the generalization shown above on the board to help students understand why it makes sense to define $2^{\frac{1}{n}}$ to be $\sqrt[n]{2}$.

However, be careful to not stop here; there is a problem with our reasoning if we do not define $\sqrt[n]{2}$. In previous courses, only square roots and cube roots were defined.

We first need to define the nth root of a number; there may be more than one, as in the case where $2^2 = 4$ and $(-2)^2 = 4$. We say that both -2 and 2 are square roots of 4. However, we give priority to the positive-valued square root, and we say that 2 is the *principal square root* of 4. We often just refer to *the* square root of 4 when we mean the *principal* square root of 4. The definition of n^{th} root presented below is consistent with allowing complex n^{th} roots, which students will encounter in college if they pursue engineering or higher mathematics. If we allow complex n^{th} roots, there are three cube roots of 2: $\sqrt[3]{2}$, $\sqrt[3]{2}\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$, and $\sqrt[3]{2}\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)$, and we refer to the real number $\sqrt[3]{2}$ as the principal cube root of 2. There is no need to discuss this with any but the most advanced students.

The $\sqrt[n]{2}$ is the positive real number a such that $a^n = 2$. In general, if a is positive, then the n^{th} root of a exists for any positive integer n, and if a is negative, then the n^{th} root of a exists only for odd integers n. This even/odd condition is handled subtly in the definition below; the n^{th} root exists only if there is already an exponential relationship $b = a^n$.

Present the following definitions to students and have them record them in their notes.

> **n^{TH} ROOT OF A NUMBER:** Let a and b be numbers, and let $n \geq 2$ be a positive integer. If $b = a^n$,
>
> then a is an *n^{th} root of b*. If $n = 2$, then the root is a called a *square root*. If $n = 3$, then the root is called a *cube root*.
>
> **PRINCIPAL n^{TH} ROOT OF A NUMBER:** Let b be a real number that has at least one real n^{th} root. The *principal n^{th} root of b* is the real n^{th} root that has the same sign as b and is denoted by a radical symbol: $\sqrt[n]{b}$.
>
> Every positive number has a unique principal n^{th} root. We often refer to the principal n^{th} root of b as just the *n^{th} root of b*. The n^{th} root of 0 is 0.

Students have already learned about square and cube roots in previous grades. In Module 1 and at the beginning of this lesson, students worked with radical expressions involving cube and square roots. Explain that the n^{th} roots of a number satisfy the same properties of radicals learned previously. Have students record these properties in their notes.

> If $a \geq 0$, $b \geq 0$ ($b \neq 0$ when b is a denominator) and n is a positive integer, then
>
> $$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b} \quad \text{and} \quad \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}.$$

Background information regarding n^{th} roots and their uniqueness is provided below. You may wish to share this with your advanced learners or the entire class if you extend this lesson to an additional day.

The existence of the principal n^{th} root of a real number b is a consequence of the fundamental theorem of algebra: Consider the polynomial function $f(x) = x^n - b$. When n is odd, we know that f has at least one real zero because the graph of f must cross the x-axis. That zero is a positive number which (after showing that it is the *ONLY* real zero) is the n^{th} root. The case for when n is even follows a similar argument.

To show uniqueness of the n^{th} root, suppose there are two n^{th} roots of a number b, x, and y such that $x > 0$, $y > 0$, $x^n = b$, and $y^n = b$. Then $x^n - y^n = b - b = 0$. The expression $x^n - y^n$ factors (see Lesson 7 in Module 1).

$$\begin{aligned} 0 &= x^n - y^n \\ &= (x-y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + xy^{n-2} + y^{n-1}) \end{aligned}$$

Since both x and y are positive, the second factor is never zero. Thus, for $x^n - y^n = 0$, we must have $x - y = 0$ and it follows that $x = y$. Thus, there is only one n^{th} root of b.

A proof of the first radical property is shown below for teacher background information. You may wish to share this proof with your advanced learners or the entire class if you extend this lesson to an additional day.

Prove that $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$.

Let $x \geq 0$ be the number such that $x^n = a$ and let $y \geq 0$ be the number such that $y^n = b$, so that $x = \sqrt[n]{a}$ and $y = \sqrt[n]{b}$. Then, by a property of exponents, $(xy)^n = x^n y^n = ab$. Thus, xy must be the n^{th} root of ab. Writing this using our notation gives

$$\sqrt[n]{ab} = xy = \sqrt[n]{a} \cdot \sqrt[n]{b}.$$

After students have recorded this information in their notes, you can proceed with Example 1 and Exercise 1.

Example 1 (3 minutes)

This example will familiarize students with the wording in the definition presented above.

Example 1

a. **What is the 4^{th} root of 16?**

$x^4 = 16$ *when* $x = 2$ *because* $2^4 = 16$. *Thus,* $\sqrt[4]{16} = 2$.

b. **What is the cube root of 125?**

$x^3 = 125$ *when* $x = 5$ *because* $5^3 = 125$. *Thus,* $\sqrt[3]{125} = 5$.

c. **What is the 5^{th} root of $100,000$?**

$x^5 = 100,000$ *when* $x = 10$ *because* $10^5 = 100,000$. *Thus,* $\sqrt[5]{100,000} = 10$.

Exercise 1 (2 minutes)

In these brief exercises, students will work with definition of n^{th} roots and the multiplication property presented above. Have students check their work with a partner and briefly discuss as a whole class any questions that arise.

Exercise 1

1. **Evaluate each expression.**

 a. $\sqrt[4]{81}$

 3

 b. $\sqrt[5]{32}$

 2

 c. $\sqrt[3]{9} \cdot \sqrt[3]{3}$

 $\sqrt[3]{27} = 3$

 d. $\sqrt[4]{25} \cdot \sqrt[4]{100} \cdot \sqrt[4]{4}$

 $\sqrt[4]{10,000} = 10$

Scaffolding:

If needed, continue to support students that struggle with abstraction by including additional numeric examples.

Discussion (8 minutes)

Return to the question posted in the title of this lesson: What are $2^{\frac{1}{2}}$ and $2^{\frac{1}{3}}$? Now we know the answer, $2^{\frac{1}{2}} = \sqrt{2}$ and $2^{\frac{1}{3}} = \sqrt[3]{2}$. So far, we have given meaning to $2^{\frac{1}{n}}$ by equating $2^{\frac{1}{n}} = \sqrt[n]{2}$. Ask students if they believe these results extend to any base $b > 0$.

- We just did some exercises where the b-value was a number different from 2. In our earlier work, was there something special about using 2 as the base of our exponential expression? Would these results generalize to expressions of the form $3^{\frac{1}{n}}$? $7^{\frac{1}{n}}$? $10^{\frac{1}{n}}$? $b^{\frac{1}{n}}$, for any positive real number b?
  - *There is nothing inherently special about the base 2 in the above discussion. These results should generalize to expressions of the form $b^{\frac{1}{n}}$ for any positive real number base b because we have defined an n^{th} root for any positive base b.*

Now that we know what the n^{th} root of a number b, $\sqrt[n]{b}$, means, the work we did earlier with base 2 suggests that $b^{\frac{1}{n}}$ should also be defined as the n^{th} root of b. Discuss this definition with your class. If the class is unclear on the definition, do some numerical examples: $(-32)^{\frac{1}{5}} = -2$ because $(-2)^5 = -32$, but $(-16)^{\frac{1}{4}}$ does not exist because there is no principal 4^{th} root of a negative number.

For a real number b and a positive integer n, define $b^{\frac{1}{n}}$ to be the principal n^{th} root of b when it exists. That is, $b^{\frac{1}{n}} = \sqrt[n]{b}$.

Note that this definition holds for any real number b if n is an odd integer and for positive real numbers b if n is an even integer. You may choose to emphasize this with your class. Thus, when b is negative and n is an odd integer, the expression $b^{\frac{1}{n}}$ will be negative. If n is an even integer, then we must restrict b to positive real numbers only, and $b^{\frac{1}{n}}$ will be positive.

In the next lesson, we see that with this definition $b^{\frac{1}{n}}$ satisfies all the usual properties of exponents so it makes sense to define it in this way.

At this point, you can also revisit our original question with the students one more time.

- What is the value of $2^{\frac{1}{2}}$? What does it mean? What is the value of $b^{\frac{1}{2}}$ for any positive number b? How are radicals related to rational exponents?
  - *We now know that $2^{\frac{1}{2}}$ is equal to $\sqrt{2}$. In general $b^{\frac{1}{2}} = \sqrt{b}$ for any positive real number b.*
    A number that contains a radical can be expressed using rational exponents in place of the radical.

As you continue the discussion, we extend the definition of exponential expressions with exponents of the form $\frac{1}{n}$ to any positive rational exponent. We begin by considering an example: What do we mean by $2^{\frac{3}{4}}$? Give students a few minutes to respond individually in writing to this question on their student pages and then have them discuss their reasoning with a partner. Make sure to correct any blatant misconceptions and to clarify incomplete thinking as you lead the discussion that follows.

MP.3

Discussion

If $2^{\frac{1}{2}} = \sqrt{2}$ and $2^{\frac{1}{3}} = \sqrt[3]{2}$, what does $2^{\frac{3}{4}}$ equal? Explain your reasoning.

Student solutions and explanations will vary. One possible solution would be $2^{\frac{3}{4}} = \left(2^{\frac{1}{4}}\right)^3$, so it must mean that $2^{\frac{3}{4}} = \left(\sqrt[4]{2}\right)^3$. Since the properties of exponents and the meaning of an exponent made sense with integers and now for rational numbers in the form $\frac{1}{n}$, it would make sense that they would work all rational numbers too.

Now that we have a definition for exponential expressions of the form $b^{\frac{1}{n}}$, use the discussion below to define $b^{\frac{m}{n}}$, where m and n are both integers, $n \neq 0$, and b is a positive real number. Make sure students understand that our interpretation of $b^{\frac{m}{n}}$ must be consistent with the exponent properties (which hold for integer exponents) and the definition of $b^{\frac{1}{n}}$.

- How can we rewrite the exponent of $2^{\frac{3}{4}}$ using integers and rational numbers in the form $\frac{1}{n}$?
  - *We can write $2^{\frac{3}{4}} = \left(2^{\frac{1}{4}}\right)^3$ or we can write $2^{\frac{3}{4}} = (2^3)^{\frac{1}{4}}$.*
- Now apply our definition of $b^{\frac{1}{n}}$.
  - $2^{\frac{3}{4}} = \left(2^{\frac{1}{4}}\right)^3 = \left(\sqrt[4]{2}\right)^3$ *or* $2^{\frac{3}{4}} = (2^3)^{\frac{1}{4}} = \sqrt[4]{2^3} = \sqrt[4]{8}$

- Does this make sense? If $2^{\frac{3}{4}} = \sqrt[4]{8}$, then if we raise $2^{\frac{3}{4}}$ to the fourth power, we should get 8. Does this happen?
  - $\left(2^{\frac{3}{4}}\right)^4 = \left(2^{\frac{3}{4}}\right)\left(2^{\frac{3}{4}}\right)\left(2^{\frac{3}{4}}\right)\left(2^{\frac{3}{4}}\right) = 2^{\left(4\cdot\frac{3}{4}\right)} = 2^3 = 8.$
  - *So, 8 is the product of four equal factors, which we denote by $2^{\frac{3}{4}}$. Thus, $2^{\frac{3}{4}} = \sqrt[4]{8}$.*

Take a few minutes to allow students to think about generalizing their work above to $2^{\frac{m}{n}}$ and then to $b^{\frac{m}{n}}$. Have them write a response to the following questions and share it with a partner before proceeding as a whole class.

- Can we generalize this result? How would you define $2^{\frac{m}{n}}$, for positive integers m and n?
  - *Conjecture: $2^{\frac{m}{n}} = \sqrt[n]{2^m}$, or equivalently, $2^{\frac{m}{n}} = \left(\sqrt[n]{2}\right)^m$.*
- Can we generalize this result to any positive real number base b? What is $3^{\frac{m}{n}}$? $7^{\frac{m}{n}}$? $10^{\frac{m}{n}}$? $b^{\frac{m}{n}}$?
  - *There is nothing inherently special about the base 2 in the above Discussion. These results should generalize to expressions of the form $b^{\frac{m}{n}}$ for any positive real number base b.*
  - *Then we are ready to define $b^{\frac{m}{n}} = \sqrt[n]{b^m}$ for positive integers m and n and positive real numbers b.*

This result is summarized in the box below.

For any positive integers m and n, and any real number b for which $b^{\frac{1}{n}}$ exists, we define

$$b^{\frac{m}{n}} = \sqrt[n]{b^m}, \text{ which is equivalent to } b^{\frac{m}{n}} = \left(\sqrt[n]{b}\right)^m$$

Note that this property holds for any real number b if n is an odd integer. You may choose to emphasize this with your class. When b is negative and n is an odd integer, the expression $b^{\frac{m}{n}}$ will be negative. If n is an even integer then we must restrict b to positive real numbers only.

Exercises 2–8 (4 minutes)

In these exercises, students use the definitions above to rewrite and evaluate expressions. Have students check their work with a partner and briefly discuss as a whole class any questions that arise.

Exercises 2–12

Rewrite each exponential expression as an n^{th} root.

2. $3^{\frac{1}{2}}$

$3^{\frac{1}{2}} = \sqrt{3}$

3. $11^{\frac{1}{5}}$

$11^{\frac{1}{5}} = \sqrt[5]{11}$

4. $\left(\frac{1}{4}\right)^{\frac{1}{5}}$

$\left(\frac{1}{4}\right)^{\frac{1}{5}} = \sqrt[5]{\frac{1}{4}}$

5. $6^{\frac{1}{10}}$

$6^{\frac{1}{10}} = \sqrt[10]{6}$

Rewrite the following exponential expressions as equivalent radical expressions. If the number is rational, write it without radicals or exponents.

6. $2^{\frac{3}{2}}$

$2^{\frac{3}{2}} = \sqrt{2^3} = 2\sqrt{2}$

7. $4^{\frac{5}{2}}$

$4^{\frac{5}{2}} = \sqrt{4^5} = \left(\sqrt{4}\right)^5 = 2^5 = 32$

8. $\left(\frac{1}{8}\right)^{\frac{5}{3}}$

$\left(\frac{1}{8}\right)^{\frac{5}{3}} = \sqrt[3]{\left(\frac{1}{8}\right)^5} = \left(\sqrt[3]{\frac{1}{8}}\right)^5 = \left(\frac{1}{2}\right)^5 = \frac{1}{32}$

Exercise 9 (3 minutes)

In this exercise, we ask students to consider a negative rational exponent. Have students work directly with a partner and ask them to use thinking similar to the preceding discussion. Correct and extend student thinking as you review the solution.

9. **Show why the following statement is true:**

$2^{-\frac{1}{2}} = \frac{1}{2^{\frac{1}{2}}}$

Student solutions and explanations will vary. One possible solution would be

$2^{-\frac{1}{2}} = \left(2^{\frac{1}{2}}\right)^{-1} = \left(\sqrt{2}\right)^{-1} = \frac{1}{\sqrt{2}} = \frac{1}{2^{\frac{1}{2}}}$

Since $\sqrt{2}$ is a real number the properties of exponents hold and we have defined $b^{\frac{1}{n}}$. We can show these two expressions are the same.

Share the following property with your class and show how the work they did in the previous exercises supports this conclusion. Should you choose to you can verify these properties using an argument similar to the ones presented earlier for the meaning of $b^{\frac{m}{n}}$.

For any positive integers m and n, and any real number b for which $b^{\frac{1}{n}}$ exists, we define

$$b^{-\frac{m}{n}} = \frac{1}{\sqrt[n]{b^m}}$$

or, equivalently,

$$b^{-\frac{m}{n}} = \frac{1}{\left(\sqrt[n]{b}\right)^m}.$$

Exercises 10–12 (3 minutes)

Rewrite the following exponential expressions as equivalent radical expressions. If the number is rational, write it without radicals or exponents.

10. $4^{-\frac{3}{2}}$

$$4^{-\frac{3}{2}} = \frac{1}{\sqrt{4^3}} = \frac{1}{\left(\sqrt{4}\right)^3} = \frac{1}{8}$$

11. $27^{-\frac{2}{3}}$

$$27^{-\frac{2}{3}} = \frac{1}{27^{\frac{2}{3}}} = \frac{1}{\left(\sqrt[3]{27}\right)^2} = \frac{1}{3^2} = \frac{1}{9}$$

12. $\left(\frac{1}{4}\right)^{-\frac{1}{2}}$

We have $\left(\frac{1}{4}\right)^{-\frac{1}{2}} = \left(\sqrt{\frac{1}{4}}\right)^{-1} = \left(\frac{1}{2}\right)^{-1} = 2$. *Alternatively,* $\left(\frac{1}{4}\right)^{-\frac{1}{2}} = \left(\left(\frac{1}{4}\right)^{-1}\right)^{\frac{1}{2}} = (4)^{\frac{1}{2}} = \sqrt{4} = 2$.

Closing (3 minutes)

Have students summarize the key points of the lesson in writing. Circulate around the classroom to informally assess understanding and provide assistance. Their work should reflect the summary provided below.

Lesson Summary

n^{TH} ROOT OF A NUMBER: Let a and b be numbers, and let $n \geq 2$ be a positive integer. If $b = a^n$, then a is an n^{th} *root of* b. If $n = 2$, then the root is a called a square root. If $n = 3$, then the root is called a cube root.

PRINCIPAL n^{TH} ROOT OF A NUMBER: Let b be a real number that has at least one real n^{th} root. The *principal* n^{th} *root of* b is the real n^{th} root that has the same sign as b, and is denoted by a radical symbol: $\sqrt[n]{b}$.

Every positive number has a unique principal n^{th} root. We often refer to the principal n^{th} root of b as just *the* n^{th} *root of* b. The n^{th} root of 0 is 0.

For any positive integers m and n, and any real number b for which the principal n^{th} root of b exists, we have

$$b^{\frac{1}{n}} = \sqrt[n]{b}$$

$$b^{\frac{m}{n}} = \sqrt[n]{b^m} = \left(\sqrt[n]{b}\right)^m$$

$$b^{-\frac{m}{n}} = \frac{1}{\sqrt[n]{b^m}}.$$

Exit Ticket (4 minutes)

Name ____ Date ____________________

Lesson 3: Rational Exponents—What Are $2^{\frac{1}{2}}$ and $2^{\frac{1}{3}}$?

Exit Ticket

1. Write the following exponential expressions as equivalent radical expressions.

   a. $2^{\frac{1}{2}}$

   b. $2^{\frac{3}{4}}$

   c. $3^{-\frac{2}{3}}$

2. Rewrite the following radical expressions as equivalent exponential expressions.

   a. $\sqrt{5}$

   b. $2\sqrt[4]{3}$

   c. $\frac{1}{\sqrt[3]{16}}$

3. Provide a written explanation for each question below.

   a. Is it true that $\left(4^{\frac{1}{2}}\right)^3 = (4^3)^{\frac{1}{2}}$? Explain how you know.

   b. Is it true that $\left(1000^{\frac{1}{3}}\right)^3 = (1000^3)^{\frac{1}{3}}$? Explain how you know.

   c. Suppose that m and n are positive integers and b is a real number so that the principal n^{th} root of b exists. In general does $\left(b^{\frac{1}{n}}\right)^m = (b^m)^{\frac{1}{n}}$? Provide at least one example to support your claim.

Exit Ticket Sample Solutions

1. **Rewrite the following exponential expressions as equivalent radical expressions.**

 a. $2^{\frac{1}{2}}$

 $2^{\frac{1}{2}} = \sqrt{2}$

 b. $2^{\frac{3}{4}}$

 $2^{\frac{3}{4}} = \sqrt[4]{2^3} = \sqrt[4]{8}$

 c. $3^{-\frac{2}{3}}$

 $3^{-\frac{2}{3}} = \frac{1}{\sqrt[3]{3^2}} = \frac{1}{\sqrt[3]{9}}$

2. **Rewrite the following radical expressions as equivalent exponential expressions.**

 a. $\sqrt{5}$

 $\sqrt{5} = 5^{\frac{1}{2}}$

 b. $2\sqrt[4]{3}$

 $2\sqrt[4]{3} = \sqrt[4]{2^4 \cdot 3} = \sqrt[4]{48} = 48^{\frac{1}{4}}$

 c. $\frac{1}{\sqrt[3]{16}}$

 $\frac{1}{\sqrt[3]{16}} = (2^4)^{-\frac{1}{3}} = 2^{-\frac{4}{3}}$

 $\frac{1}{\sqrt[3]{16}} = (16)^{-\frac{1}{3}}$

MP.3 & MP.7

3. **Provide a written explanation for each question below.**

 a. **Is it true that $\left(4^{\frac{1}{2}}\right)^3 = (4^3)^{\frac{1}{2}}$? Explain how you know.**

 $\left(4^{\frac{1}{2}}\right)^3 = \left(\sqrt{4}\right)^3 = 2^3 = 8$

 $(4^3)^{\frac{1}{2}} = 64^{\frac{1}{2}} = \sqrt{64} = 8$

 So the first statement is true.

 b. **Is it true that $\left(1000^{\frac{1}{3}}\right)^3 = (1000^3)^{\frac{1}{3}}$? Explain how you know.**

 Similarly the left and right sides of the second statement are equal to one another.

 $\left(1000^{\frac{1}{3}}\right)^3 = \left(\sqrt[3]{1000}\right)^3 = 10^3 = 1000$

 $(1000^3)^{\frac{1}{3}} = (1000000000)^{\frac{1}{3}} = 1000$

c. Suppose that m and n are positive integers and b is a real number so that the principal n^{th} root of b exists. In general does $\left(b^{\frac{1}{n}}\right)^m = (b^m)^{\frac{1}{n}}$? Provide at least one example to support your claim.

Thus, it appears the statement is true and it also holds for when the exponents are integers. Based on our other work in this lesson, this property should extend to rational exponents as well.

Here is another example,

$$\left(8^{\frac{1}{3}}\right)^2 = (2)^2 = 4$$

and

$$(8^2)^{\frac{1}{3}} = (64)^{\frac{1}{3}} = 4.$$

Thus,

$$\left(8^{\frac{1}{3}}\right)^2 = (8^2)^{\frac{1}{3}}.$$

Problem Set Sample Solutions

1. Select the expression from (A), (B), and (C) that correctly completes the statement.

		(A)	(B)	(C)
a.	$x^{\frac{1}{3}}$ is equivalent to ________.	$\frac{1}{3}x$	$\sqrt[3]{x}$	$\frac{3}{x}$
b.	$x^{\frac{2}{3}}$ is equivalent to ________.	$\frac{2}{3}x$	$\sqrt[3]{x^2}$	$(\sqrt{x})^3$
c.	$x^{-\frac{1}{4}}$ is equivalent to ________.	$-\frac{1}{4}x$	$\frac{4}{x}$	$\frac{1}{\sqrt[4]{x}}$
d.	$\left(\frac{4}{x}\right)^{\frac{1}{2}}$ is equivalent to ________.	$\frac{2}{x}$	$\frac{4}{x^2}$	$\frac{2}{\sqrt{x}}$

a. *(B)*

b. *(B)*

c. *(C)*

d. *(C)*

2. Identify which of the expressions (A), (B), and (C) are equivalent to the given expression.

		(A)	(B)	(C)
a.	$16^{\frac{1}{2}}$	$\left(\frac{1}{16}\right)^{-\frac{1}{2}}$	$8^{\frac{2}{3}}$	$64^{\frac{3}{2}}$
b.	$\left(\frac{2}{3}\right)^{-1}$	$-\frac{3}{2}$	$\left(\frac{9}{4}\right)^{\frac{1}{2}}$	$\frac{27^{\frac{1}{3}}}{6}$

a. *(A) and (B)*

b. *(B) only*

3. Rewrite in radical form. If the number is rational, write it without using radicals.

a. $6^{\frac{3}{2}}$

$\sqrt{216}$

b. $\left(\frac{1}{2}\right)^{\frac{1}{4}}$

$\sqrt[4]{\frac{1}{2}}$

c. $3(8)^{\frac{1}{3}}$

$3\sqrt[3]{8} = 6$

d. $\left(\frac{64}{125}\right)^{-\frac{2}{3}}$

$\left(\sqrt[3]{\frac{125}{16}}\right)^2 = \frac{25}{16}l$

e. $81^{-\frac{1}{4}}$

$\frac{1}{\sqrt[4]{81}} = \frac{1}{3}$

4. Rewrite the following expressions in exponent form.

a. $\sqrt{5}$

$5^{\frac{1}{2}}$

b. $\sqrt[3]{5^2}$

$5^{\frac{2}{3}}$

c. $\sqrt{5^3}$

$5^{\frac{3}{2}}$

d. $\left(\sqrt[3]{5}\right)^2$

$5^{\frac{2}{3}}$

5. Use the graph of $f(x) = 2^x$ shown to the right to estimate the following powers of 2.

a. $2^{\frac{1}{4}}$ ≈ 1.2

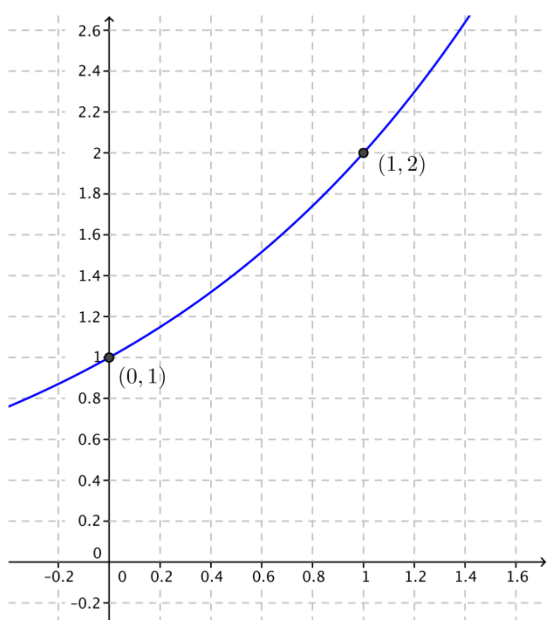
b. $2^{\frac{2}{3}}$ ≈ 1.6

c. $2^{\frac{3}{4}}$ ≈ 1.7

d. $2^{0.2}$ ≈ 1.15

e. $2^{1.2}$ ≈ 2.3

f. $2^{-\frac{1}{5}}$ ≈ 0.85



6. Rewrite each expression in the form kx^n, where k is a real number, x is a positive real number, and n is rational.

a. $\sqrt[4]{16x^3}$

$2x^{\frac{3}{4}}$

b. $\frac{5}{\sqrt{x}}$

$5x^{-\frac{1}{2}}$

c. $\sqrt[3]{1/x^4}$

$x^{-\frac{4}{3}}$

d. $\frac{4}{\sqrt[3]{8x^3}}$

$2x^{-1}$

e. $\frac{27}{\sqrt{9x^4}}$

$9x^{-2}$

f. $\left(\frac{125}{x^2}\right)^{-\frac{1}{3}}$

$\frac{1}{5}x^{\frac{2}{3}}$

7. Find a value of x for which $2x^{\frac{1}{2}} = 32$.

256

8. Find a value of x for which $x^{\frac{4}{3}} = 81$.

27

9. If $x^{\frac{3}{2}} = 64$, find the value of $4x^{-\frac{3}{4}}$.

$x = 16$, *so* $4(16)^{-\frac{3}{4}} = 4(2)^{-3} = \frac{4}{8} = \frac{1}{2}$.

10. If $b = \frac{1}{9}$, evaluate the following expressions.

a. $b^{-\frac{1}{2}}$

$\left(\frac{1}{9}\right)^{-\frac{1}{2}} = 9^{\frac{1}{2}} = 3$

b. $b^{\frac{5}{2}}$

$\left(\frac{1}{9}\right)^{\frac{5}{2}} = \left(\frac{1}{3}\right)^{5} = \frac{1}{243}$

c. $\sqrt[3]{3b^{-1}}$

$\sqrt[3]{3\left(\frac{1}{9}\right)^{-1}} = \sqrt[3]{27} = 3$

11. Show that each expression is equivalent to $2x$. Assume x is a positive real number.

a. $\sqrt[4]{16x^4}$

$\sqrt[4]{16} \cdot \sqrt[4]{x^4} = 2x$

b. $\dfrac{\left(\sqrt[3]{8x^3}\right)^2}{\sqrt{4x^2}}$

$\dfrac{(2x)^2}{(4x^2)^{\frac{1}{2}}} = \dfrac{4x^2}{2x} = 2x$

c. $\dfrac{6x^3}{\sqrt[3]{27x^6}}$

$\dfrac{6x^3}{3x^{\frac{6}{3}}} = \dfrac{6x^3}{3x^2} = 2x$

MP.3

12. Yoshiko said that $16^{\frac{1}{4}} = 4$ because 4 is one-fourth of 16. Use properties of exponents to explain why she is or is not correct.

Yoshiko's reasoning is not correct. By our exponent properties, $\left(16^{\frac{1}{4}}\right)^4 = 16^{\left(\frac{1}{4}\right)\cdot 4} = 16^1 = 16$*, but* $4^4 = 256$*. Since* $\left(16^{\frac{1}{4}}\right)^4 \neq 4^4$*, we know that* $16^{\frac{1}{4}} \neq 4$.

13. Jefferson said that $8^{\frac{4}{3}} = 16$ because $8^{\frac{1}{3}} = 2$ and $2^4 = 16$. Use properties of exponents to explain why he is or is not correct.

Jefferson's reasoning is correct. We know that $8^{\frac{4}{3}} = \left(8^{\frac{1}{3}}\right)^4$*, so* $8^{\frac{4}{3}} = 2^4$*, and thus* $8^{\frac{4}{3}} = 16$.

14. Rita said that $8^{\frac{2}{3}} = 128$ because $8^{\frac{2}{3}} = 8^2 \cdot 8^{\frac{1}{3}}$, so $8^{\frac{2}{3}} = 64 \cdot 2$, and then $8^{\frac{2}{3}} = 128$. Use properties of exponents to explain why she is or is not correct.

Rita's reasoning is not correct because she did not apply the properties of exponents correctly. She should also realize that raising 8 *to a positive power less than* 1 *will produce a number less than* 8*. The correct calculation is below.*

$$\begin{aligned} 8^{\frac{2}{3}} &= \left(8^{\frac{1}{3}}\right)^2 \\ &= 2^2 \\ &= 4 \end{aligned}$$

15. Suppose for some positive real number a that $\left(a^{\frac{1}{4}} \cdot a^{\frac{1}{2}} \cdot a^{\frac{1}{4}}\right)^2 = 3$.

 a. What is the value of a?

$$\left(a^{\frac{1}{4}} \cdot a^{\frac{1}{2}} \cdot a^{\frac{1}{4}}\right)^2 = 3$$
$$\left(a^{\frac{1}{4}+\frac{1}{2}+\frac{1}{4}}\right)^2 = 3$$
$$(a^1)^2 = 3$$
$$a^2 = 3$$
$$a = \sqrt{3}$$

 b. Which exponent properties did you use to find your answer to part (a)?

 We used the properties $b^n \cdot b^m = b^{m+n}$ and $(b^m)^n = b^{mn}$.

16. In the lesson, you made the following argument:

$$\left(2^{\frac{1}{3}}\right)^3 = 2^{\frac{1}{3}} \cdot 2^{\frac{1}{3}} \cdot 2^{\frac{1}{3}}$$
$$= 2^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}}$$
$$= 2^1$$
$$= 2.$$

Since $\sqrt[3]{2}$ is a number so that $\left(\sqrt[3]{2}\right)^3 = 2$ and $2^{\frac{1}{3}}$ is a number so that $\left(2^{\frac{1}{3}}\right)^3 = 2$, you concluded that $2^{\frac{1}{3}} = \sqrt[3]{2}$. Which exponent property was used to make this argument?

We used the property $b^n \cdot b^m = b^{m+n}$. (Students may also mention the uniqueness of n^{th} roots.)