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**Lesson 3: Rational Exponents—What are and ?**

**Student Outcomes**

* Students will calculate quantities that involve positive and negative rational exponents.

**Lesson Notes**

Students extend their understanding of integer exponents to rational exponents by examining the graph of and estimating the values of and . The lesson establishes the meaning of these numbers in terms of radical expressions and these form the basis of how we define expressions of the form before generalizing further to expressions of the form , where is a positive real number and and are integers, with (**N-RN.A.1**). The lesson and Problem Set provide fluency practice in applying the properties of exponents to expressions containing rational exponents and radicals (**N-RN.A.2**). In the following lesson, students will verify that the definition of an expression with rational exponents, , is consistent with the remaining exponential properties. The lesson begins with students creating a graph of a simple exponential function, , which they studied in Module of Algebra I. In that module, students also learned about geometric sequences and their relationship to exponential functions, which is a concept that we revisit at the end of this module. In Algebra I, students worked with exponential functions with integer domains. This lesson, together with the subsequent Lessons 4 and 5, helps students understand why the domain of an exponential function , where is positive number and , is the set of real numbers. To do so we must establish what it means to raise to a rational power and, in Lesson 5, to any real number power.

**Classwork**

**Opening (1 minutes)**

*Scaffolding:*

* Encourage students to create a table of values to help them construct the graph.
* For advanced learners, have them repeat these exercises with the function and ask them to estimate and .

In Algebra I, students worked with geometric sequences and simple exponential functions. Remind them that in Lesson 1, we created formulas based on repeatedly doubling and halving a number when we modeled folding a piece of paper. We reviewed how to use the properties of exponents for expressions that had integer exponents.

**Opening Exercise (5 minutes)**

Have students graph for each integer from to without using a graphing utility or calculator on the axes provided. Discuss the pattern of points and ask students to connect the points in a way that produces a smooth curve. Students should work these two problems independently and check their solutions with a partner if time permits before you lead a whole class discussion to review the solutions.

**Opening Exercise**

* 1. **What is the value of ? Justify your answer.**

**MP.3**

A possible student response follows: I think it will be around because and .

* 1. **Graph for each integer from to . Connect the points on your graph with a smooth curve.**



Ask for a few volunteers to explain their reasoning for their answers to Opening Exercise part (a). Then, debrief these two exercises by leading a short discussion.

* The directions in the Opening Exercise said to connect the points with a smooth curve. What does it imply about the domain of a function when we connect points that we have plotted?
  + *That the domain of the function includes the points between the points that we plotted.*
* How does your graph support or refute your answer to the first exercise? Do you need to modify your answer to the question: What is the value of ? Why or why not?
  + *If the domain is all real numbers, then the value of will be the -coordinate of the point on the graph where . From the graph it looks like the value is somewhere between and . The scaling on this graph is not detailed enough for me to accurately refine my answer yet.*

Transition to the next set of exercises by telling the students that we can better estimate the value of by looking at a graph that is scaled more precisely. Have students proceed to the next questions. They should work in small groups or with a partner on these exercises.

The graph of shown below for the next exercises will appear in the student materials but you could also display the graph using technology. Have students estimate the value of and from the graph. Students should work by themselves or in pairs to work through the following questions without using a calculator.

**The graph at right shows a close-up view of for .**

* 1. **Find two consecutive integers that are over and underestimates of the value of .**
  2. **Does it appear that is halfway between the integers you specified in Exercise 1?**

***No, it looks like is a little less than halfway between and .***

* 1. **Use the graph of to estimate the value of .**
  2. **Use the graph of to estimate the value of .**

**Discussion (9 minutes)**

Before getting into the point of this lesson, which is to connect rational exponents to radical expressions, revisit the initial question with students.

*Scaffolding:*

* If needed, you can demonstrate the argument using perfect squares. For example, use a base of instead of a base of .

Show that   
and .

* What is the value of ? Does anyone want to adjust their initial guess?

**MP.6**

* + *Our initial guess was a little too big. It seems like might be a better answer.*
* How could we make a better guess?
  + *We could look at the graph with even smaller increments for the scale using technology.*

If time permits, you can zoom in further on the graph of using a graphing calculator or other technology either by examining a graph or a table of values of closer and closer to .

Next, we will make the connection that . Walk students through the following questions, providing guidance as needed. Students proved that there was only one positive number that squared to in Geometry, Module 2. You may need to remind them of this with a bit more detail if they are struggling to follow this argument.

* Assume for the moment that whatever means that it satisfies . Working with this assumption what is the value of ?

**MP.7**

* + *It would be because .*
* What unique positive number squares to ? That is, what is the only positive number that when multiplied by itself is equal to ?
  + *By definition, we call the unique positive number that squares to the square root of and we write .*

Write the following statements on the board and ask your students to compare them and think about what they must tell us about the meaning of .

and

* What do these two statements tell us about the meaning of .
  + *Since both statements involve multiplying a number by itself and getting , and we know that there is only one number that does that, we can conclude that .*

At this point, you can have students confirm these results by using a calculator to approximate both and to several decimal places. In the opening, we approximated graphically, and now we have shown it to be an irrational number.

Next ask students to think about the meaning of using a similar line of reasoning.

*Scaffolding:*

* If needed, you can demonstrate the argument using perfect cubes. For example, use a base of instead of a base of .

Show   
and

* Assume that whatever means it will satisfy . What is the value of ?
  + *The value is because .*

**MP.7**

* What is the value of ?
  + *The value is because .*
* What appears to be the meaning of ?
  + *Since both the exponent expression and the radical expression involve multiplying a number by itself three times and the result is equal to , we know that .*

Students can also confirm using a calculator that the decimal approximations of and are the same. Next, we ask them to generalize their findings.

* Can we generalize this relationship? Does ? Does ? What is , for any positive integer ? Why?

**MP.8**

* + *because*

Have students confirm these using a calculator as well checking to see if the decimal approximations of and are the same for different values of such as . Be sure to share the generalization shown above on the board to help students understand why it makes sense to define to be .

**MP.8**

However, be careful to not stop here; there is a problem with our reasoning if we do not define . In previous courses, only square roots and cube roots were defined.

We first need to define the th root of a number; there may be more than one, as in the case where and . We say that both and are square roots of . However, we give priority to the positive-valued square root, and we say that is the *principal square root* of . We often just refer to *the* square root of when we mean the *principal* square root of . The definition of th root presented below is consistent with allowing complex th roots, which students will encounter in college if they pursue engineering or higher mathematics. If we allow complex th roots, there are three cube roots of : , and , and we refer to the real number as the principal cube root of . There is no need to discuss this with any but the most advanced students.

The is the positive real number such that . In general, if is positive, then the th root of exists for any positive integer , and if is negative, then the th root of exists only for odd integers . This even/odd condition is handled subtly in the definition below; the th root exists only if there is already an exponential relationship .

Present the following definitions to students and have them record them in their notes.

Students have already learned about square and cube roots in previous grades. In Module 1 and at the beginning of this lesson, students worked with radical expressions involving cube and square roots. Explain that the th roots of a number satisfy the same properties of radicals learned previously. Have students record these properties in their notes.

If , ( when is a denominator) and is a positive integer, then

**th root of a number:** Let and be numbers, and let be a positive integer. If ,

then is an *th root of .* If , then the root is a called a *square root*. If , then the root is called a *cube root*.

**Principal th Root of a number:**  Let be a real number that has at least one real th root. The *principal th root of* is the real th root that has the same sign as and is denoted by a radical symbol: .

Every positive number has a unique principal th root. We often refer to the principal th root of as just the *th root of .* The th root of is .

Background information regarding th roots and their uniqueness is provided below. You may wish to share this with your advanced learners or the entire class if you extend this lesson to an additional day.

The existence of the principal th root of a real number is a consequence of the fundamental theorem of algebra: Consider the polynomial function . When is odd, we know that has at least one real zero because the graph of must cross the -axis. That zero is a positive number which (after showing that it is the *ONLY* real zero) is the th root. The case for when is even follows a similar argument.

To show uniqueness of the th root, suppose there are two th roots of a number , , and such that , , , and . Then . The expression factors (see Lesson 7 in Module 1).

Since both and are positive, the second factor is never zero. Thus, for , we must have   
 and it follows that . Thus, there is only one th root of .

A proof of the first radical property is shown below for teacher background information. You may wish to share this proof with your advanced learners or the entire class if you extend this lesson to an additional day.

Prove that .

Let be the number such that and let be the number such that , so that and . Then, by a property of exponents, . Thus, must be the th root of . Writing this using our notation gives

After students have recorded this information in their notes, you can proceed with Example 1 and Exercise 1.

Example 1 (3 minutes)

This example will familiarize students with the wording in the definition presented above.

**Example 1**

* 1. **What is the th root of ?**

when because . ***Thus,* .**

* 1. **What is the cube root of ?**

when because . ***Thus,* .**

* 1. **What is the th root of ?**

when because . Thus, **.**

**Exercise 1 (2 minutes)**

In these brief exercises, students will work with definition of th roots and the multiplication property presented above. Have students check their work with a partner and briefly discuss as a whole class any questions that arise.

**Exercise 1**

1. **Evaluate each expression.**

*Scaffolding:*

If needed, continue to support students that struggle with abstraction by including additional numeric examples.



**Discussion (8 minutes)**

Return to the question posted in the title of this lesson: What are and ? Now we know the answer, and . So far, we have given meaning to by equating . Ask students if they believe these results extend to any base .

* We just did some exercises where the -value was a number different from . In our earlier work, was there something special about using as the base of our exponential expression? Would these results generalize to expressions of the form ? ? ? , for any positive real number ?
  + *There is nothing inherently special about the base in the above discussion. These results should generalize to expressions of the form for any positive real number base because we have defined an th root for any positive base .*

Now that we know what the th root of a number means, the work we did earlier with base suggests that should also be defined as the th root of . Discuss this definition with your class. If the class is unclear on the definition, do some numerical examples: because , but does not exist because there is no principal th root of a negative number.

For a real number and a positive integer , define  to be the principal th root of when it exists. That is,

Note that this definition holds for any real number if is an odd integer and for positive real numbers if is an even integer. You may choose to emphazise this with your class. Thus, when is negative and is an odd integer, the expression will be negative. If is an even integer, then we must restrict to positive real numbers only, and will be positive.

In the next lesson, we see that with this definition satisfies all the usual properties of exponents so it makes sense to define it in this way.

At this point, you can also revisit our original question with the students one more time.

* What is the value of ? What does it mean? What is the value of for any positive number ? How are radicals related to rational exponents?
  + *We now know that is equal to . In general for any positive real number .   
    A number that contains a radical can be expressed using rational exponents in place of the radical.*

As you continue the discussion, we extend the definition of exponential expressions with exponents of the form to any positive rational exponent. We begin by considering an example: What do we mean by ? Give students a few minutes to respond individually in writing to this question on their student pages and then have them discuss their reasoning with a partner. Make sure to correct any blatant misconceptions and to clarify incomplete thinking as you lead the discussion that follows.

Discussion

**If and , what does equal? Explain your reasoning.**

**MP.3**

Student solutions and explanations will vary. One possible solution would be ***,*** so it must mean that   
. Since the properties of exponents and the meaning of an exponent made sense with integers and now for rational numbers in the form , it would make sense that they would work all rational numbers too.

Now that we have a definition for exponential expressions of the form , use the discussion below to define , where and are both integers, , and is a positive real number. Make sure students understand that our interpretation of must be consistent with the exponent properties (which hold for integer exponents) and the definition of .

* How can we rewrite the exponent of using integers and rational numbers in the form ?
  + *We can write or we can write*
* Now apply our definition of .
  + *or*
* Does this make sense? If , then if we raise to the fourth power, we should get Does this happen?
  + .
  + *So, is the product of four equal factors, which we denote by Thus, .*

Take a few minutes to allow students to think about generalizing their work above to and then to . Have them write a response to the following questions and share it with a partner before proceeding as a whole class.

* Can we generalize this result? How would you define , for positive integers and ?
  + *Conjecture: , or equivalently, .*
* Can we generalize this result to any positive real number base ? What is ? ? ? ?
  + *There is nothing inherently special about the base in the above Discussion. These results should generalize to expressions of the form for any positive real number base .*
  + *Then we are ready to define for positive integers and and positive real numbers .*

This result is summarized in the box below.

Note that this property holds for any real number if is an odd integer. You may choose to emphasize this with your class. When is negative and is an odd integer, the expression will be negative. If is an even integer then we must restrict to positive real numbers only.

For any positive integers and , and any real number for which exists, we define

, which is equivalent to

**Exercises 2–8 (4 minutes)**

In these exercises, students use the definitions above to rewrite and evaluate expressions. Have students check their work with a partner and briefly discuss as a whole class any questions that arise.

**Exercises 2–12**

**Rewrite each exponential expression as an th root.**



**Rewrite the following exponential expressions as equivalent radical expressions. If the number is rational, write it without radicals or exponents.**



**Exercise 9 (3 minutes)**

In this exercise, we ask students to consider a negative rational exponent. Have students work directly with a partner and ask them to use thinking similar to the preceding discussion. Correct and extend student thinking as you review the solution.

1. **Show why the following statement is true:**

***Student solutions and explanations will vary. One possible solution would be***

***Since is a real number the properties of exponents hold and we have defined . We can show these two expressions are the same.***

Share the following property with your class and show how the work they did in the previous exercises supports this conclusion. Should you choose to you can verify these properties using an argument similar to the ones presented earlier for the meaning of .

**Exercises 10–12 (3 minutes)**

For any positive integers and , and any real number for which exists, we define   
or, equivalently,

**Rewrite the following exponential expressions as equivalent radical expressions. If the number is rational, write it without radicals or exponents.**

We have . Alternatively,.

**Closing (3 minutes)**

Have students summarize the key points of the lesson in writing. Circulate around the classroom to informally assess understanding and provide assistance. Their work should reflect the summary provided below.

Lesson Summary

th root of a number: Let and be numbers, and let be a positive integer. If , then is an *th root of .* If , then the root is a called a square root. If , then the root is called a cube root.

Principal th root of a number: Let be a real number that has at least one real th root. The *principal th root of* is the real th root that has the same sign as , and is denoted by a radical symbol: .

Every positive number has a unique principal th root. We often refer to the principal th root of as just *the th root of .* The th root of is .

For any positive integers and , and any real number for which the principal th root of exists, we have

**Exit Ticket (4 minutes)**

Name Date

**Lesson 3: Rational Exponents—What Are and ?**

**Exit Ticket**

1. Write the following exponential expressions as equivalent radical expressions.
2. Rewrite the following radical expressions as equivalent exponential expressions.
3. Provide a written explanation for each question below.

Is it true that ? Explain how you know.

Is it true that ? Explain how you know.

Suppose that and are positive integers and is a real number so that the principal th root of exists. In general does ? Provide at least one example to support your claim.

**Exit Ticket Sample Solutions**

1. **Rewrite the following exponential expressions as equivalent radical expressions.**
2. **Rewrite the following radical expressions as equivalent exponential expressions.**
3. **Provide a written explanation for each question below.**
   1. **Is it true that ? Explain how you know.**

**MP.3**

**&**

**MP.7**

***So the first statement is true.***

* 1. **Is it true that ? Explain how you know.**

***Similarly the left and right sides of the second statement are equal to one another.***

* 1. **Suppose that and are positive integers and is a real number so that the principal th root of exists. In general does ? Provide at least one example to support your claim.**

Thus, it appears the statement is true and it also holds for when the exponents are integers. Based on our other work in this lesson, this property should extend to rational exponents as well.

***Here is another example,***

***and***

***Thus,***

**Problem Set Sample Solutions**

1. **Select the expression from (A), (B), and (C) that correctly completes the statement.**

**(A) (B) (C)**

* 1. **is equivalent to .**

***(B)***

* 1. **is equivalent to .**

***(B)***

* 1. **is equivalent to .**

***(C)***

* 1. **is equivalent to .**

***(C)***

1. **Identify which of the expressions (A), (B), and (C) are equivalent to the given expression.**

**(A) (B) (C)**



***(A) and (B)***



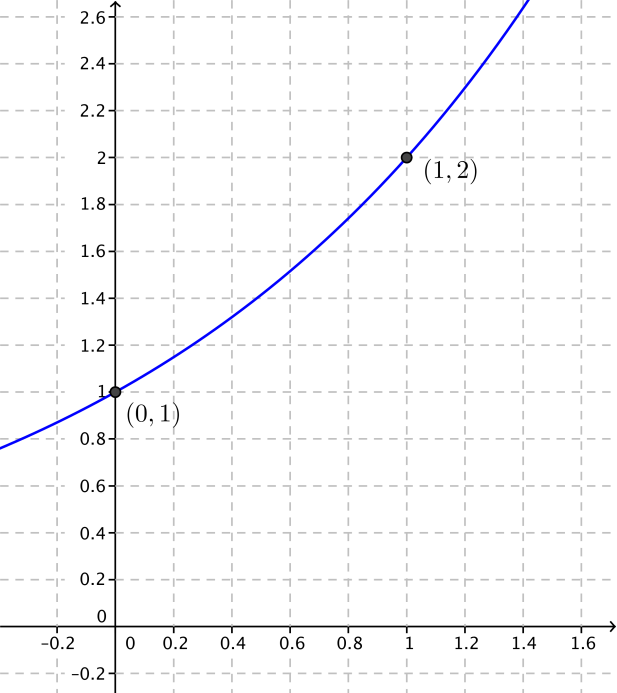
***(B) only***

1. **Rewrite in radical form. If the number is rational, write it without using radicals.**

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1. **Rewrite the following expressions in exponent form.**

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1. **Use the graph of shown to the right to estimate the following powers of .**
2. **Rewrite each expression in the form , where is a real number, is a positive real number, and is rational.**

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1. **Find a value of for which .**
2. **Find a value of for which .**
3. **If , find the value of .**

***, so .***

1. **If , evaluate the following expressions.**
2. **Show that each expression is equivalent to . Assume is a positive real number.**
3. **Yoshiko said that because is one-fourth of . Use properties of exponents to explain why she is or is not correct.**

**MP.3**

***Yoshiko’s reasoning is not correct. By our exponent properties, , but . Since , we know that .***

1. **Jefferson said that because and . Use properties of exponents to explain why he is or is not correct.**

***Jefferson’s reasoning is correct. We know that , so , and thus***

1. **Rita said that because , so and then . Use properties of exponents to explain why she is or is not correct.**

Rita’s reasoning is not correct because she did not apply the properties of exponents correctly. She should also realize that raising to a positive power less than will produce a number less than . The correct calculation is below.

1. **Suppose for some positive real number that .** 
   1. **What is the value of ?**
   2. **Which exponent properties did you use to find your answer to part (a)?**

We used the properties and .

1. **In the lesson, you made the following argument:**

**Since is a number so that and is a number so that , you concluded that Which exponent property was used to make this argument?**

We used the property . (Students may also mention the uniqueness of th roots.)