



## Topic B:

## Logarithms

N-Q.A.2, A-CED.A.1, F-BF.A.1a, F-LE.A.4

<b>Focus Standards:</b>	N-Q.A.2	Define appropriate quantities for the purpose of descriptive modeling.*
	A-CED.A.1	Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions.*
	F-BF.A.1a	Write a function that describes a relationship between two quantities.* a. Determine an explicit expression, a recursive process, or steps for calculation from a context.
	F-LE.A.4	For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where $a$ , $c$ , and $d$ are numbers and the base $b$ is 2, 10, or $e$ ; evaluate the logarithm using technology.*

**Instructional Days:** 9

- Lesson 7:** Bacteria and Exponential Growth (S)<sup>1</sup>
- Lesson 8:** The “WhatPower” Function (P)
- Lesson 9:** Logarithms—How Many Digits Do You Need? (E)
- Lesson 10:** Building Logarithmic Tables (P)
- Lesson 11:** The Most Important Property of Logarithms (P)
- Lesson 12:** Properties of Logarithms (P)
- Lesson 13:** Changing the Base (P)
- Lesson 14:** Solving Logarithmic Equations (P)
- Lesson 15:** Why Were Logarithms Developed? (P)

<sup>1</sup> Lesson Structure Key: **P**-Problem Set Lesson, **M**-Modeling Cycle Lesson, **E**-Exploration Lesson, **S**-Socratic Lesson

The lessons covered in Topic A familiarize students with the laws and properties of real-valued exponents. In Topic B, students extend their work with exponential functions to include solving exponential equations numerically and to develop an understanding of the relationship between logarithms and exponentials. To model the growth of bacteria and populations, Lesson 7 introduces students to simple exponential equations whose solutions do not follow from equating exponential terms of similar bases. To solve those equations, an algorithmic numerical approach is employed (**F-BF.A.1a**). Students work to develop progressively better approximations for the solutions to equations whose solutions are irrational numbers. In doing this, students increase their understanding of the real number system and truly begin to understand what it means for a number to be irrational. Students learn that some simple exponential equations can be solved exactly without much difficulty but that we lack mathematical tools to solve other equations whose solutions must be approximated numerically.

Lesson 8 begins with the logarithmic function disguised as the more intuitive “WhatPower” function, whose behavior is studied as a means of introducing how the function works and what it does to expressions. Students find the power needed to raise a base  $b$  in order to produce a given number. The lesson ends with students defining the term *logarithm base  $b$* . Lesson 8 is just a first introduction to logarithms in preparation for solving exponential equations per **F-LE.A.4**; students neither use tables nor look at graphs in this lesson. Instead, they simply develop the ideas and notation of logarithmic expressions, leaving many ideas to be explored later in the module.

Just as population growth is a natural example to use to give context to exponential growth, Lesson 9 gives context to logarithmic calculation through the example of assigning unique identification numbers to a group of people. In this lesson, students consider the meaning of the logarithm in the context of calculating the number of digits needed to create student ID numbers, phone numbers, and social security numbers, in accordance with **N-Q.A.2**. This gives students a real-world context for the abstract idea of a logarithm; in particular, students observe that a base 10 logarithm provides a way to keep track of the number of digits used in a number in our base 10 system.

Lessons 10–15 develop both the theory of logarithms and procedures for solving various forms of exponential and logarithmic equations. In Lessons 10 and 11, students discover the logarithmic properties by completing carefully structured logarithmic tables and answering sets of directed questions. Throughout these two lessons, students look for the structure in the table and use that structure to extract logarithmic properties (MP.7). Using the structure of the logarithmic expression together with the logarithmic properties to rewrite an expression aligns with the foundational standard **A-SSE.A.2**. While the logarithmic properties are not themselves explicitly listed in the standards, standard **F-LE.A.4** cannot be adequately met without an understanding of how to apply logarithms to solve exponential equations, and the seemingly odd behavior of graphs of logarithmic functions (**F-IF.C.7e**) cannot be adequately explained without an understanding of the properties of logarithms. In particular, in Lesson 11, students discover the “most important property of logarithms”: for positive real numbers  $x$  and  $y$ ,  $\log(xy) = \log(x) + \log(y)$ . Students also discover the pattern  $\log_b\left(\frac{1}{x}\right) = -\log_b(x)$  that leads to conjectures about additional properties of logarithms.

Lesson 12 continues the consideration of properties of the logarithm function, while remaining focused solely on base 10 logarithms. Its centerpiece is the demonstration of basic properties of logarithms such as the power, product, and quotient properties, which allows students to practice MP.3 and **A-SSE.A.2**, providing justification in terms of the definition of logarithm and the properties already developed. In this lesson, students begin to learn how to solve exponential equations, beginning with base 10 exponential equations that can be solved by taking the common logarithm of both sides of the equation.

Lesson 13 again focuses on the structure of expressions (**A-SSE.A.2**), as students change logarithms from one base to another. It begins by showing students how they can make that change and then develops properties of logarithms for the general base  $b$ . The students are introduced to the use of a calculator instead of a table in finding logarithms, and then *natural logarithms* are defined:  $\ln(x) = \log_e(x)$ . One goal of the lesson, in addition to introducing the base  $e$  for logarithms, is to explain why, for finding logarithms to any base, the calculator has only LOG and LN keys. In this lesson, students learn to solve exponential equations with any base by the application of an appropriate logarithm. Lessons 12 and 13 both address **F-LE.A.4**, solving equations of the form  $ab^{ct} = d$ , as do later lessons in the module.

Lesson 14 includes the first introduction to solving logarithmic equations. In this lesson, students apply the definition of the logarithm to rewrite logarithmic equations in exponential form, so the equations must first be rewritten in the form  $\log_b(X) = c$ , for an algebraic expression  $X$  and some constant  $c$ . Solving equations in this way requires that students think deeply about the definition of the logarithm and how logarithms interact with exponential expressions. Although solving logarithmic equations is not listed explicitly in the standards, this skill is implicit in standard **A-REI.D.11**, which has students solve equations of the form  $f(x) = g(x)$  where  $f$  and  $g$  can be logarithmic functions. Additionally, logarithmic equations provide a greater context in which to study both the properties of logarithms and the definition, both of which are needed to solve the equations listed in **F-LE.A.4**.

Topic B concludes with Lesson 15, in which students learn a bit of the history of how and why logarithms first appeared. The materials for this lesson contain a base 10 logarithm table that should be copied and distributed to the class. Although modern technology has made logarithm tables functionally obsolete, there is still value in understanding the historical development of logarithms. Logarithms were critical to the development of astronomy and navigation in the days before computing machines, and this lesson presents a rationale for the pre-technological advantage afforded to scholars by the use of logarithms. In this lesson, the case is finally made that logarithm functions are one-to-one (without explicitly using that terminology): if  $\log_b(X) = \log_b(Y)$ , then  $X = Y$ . In alignment with **A-SSE.A.2**, this fact not only validates the use of tables to look up anti-logarithms, but also allows exponential equations to be solved with logarithms on both sides of the equation.