Lesson 1: Integer Exponents

Student Outcomes

- Students review and practice applying the properties of exponents for integer exponents.
- Students model a real-world scenario involving exponential growth and decay.

Lesson Notes

To fully understand exponential functions and their use in modeling real-world situations, students must be able to extend the properties of integer exponents to rational and real numbers. In previous grades, students established the properties of exponents for integer exponents and worked with radical expressions and irrational numbers such as $\sqrt{2}$.

In this module, we use the properties of exponents to show that for any positive real number b, the domain of the exponential function $f(x) = b^x$ is all real numbers. In Algebra I, students primarily worked with exponential functions where the domain was limited to a set of integers. In the latter part of this module, students are introduced to logarithms, which allow them to find real number solutions to exponential equations. Students come to understand how logarithms simplify computation, particularly with different measuring scales.

Much of the work in this module relies on the ability to reason quantitatively and abstractly (MP.2), to make use of structure (MP.7), and to model with mathematics (MP.4). Lesson 1 begins with a challenge problem where students are asked to fold a piece of paper in half 10 times and construct exponential functions based on their experience (**F-LE.A.2**). It is physically impossible to fold a sheet of notebook paper in half more than seven or eight times; the difficulty lies in the thickness of the paper compared to the resulting area when the paper is folded. To fold a piece of paper in half more than seven or eight times requires either a larger piece of paper, a very thin piece of paper, or a different folding scheme, such as accordion folding. In 2001, a high school student, Britney Gallivan, successfully folded a very large piece of paper in half 12 times and derived a mathematical formula to determine how large a piece of paper would be needed to successfully accomplish this task (http://pomonahistorical.org/12times.htm). Others have tried the problem as well, including the hosts of the television show Mythbusters (https://www.youtube.com/watch?v=kRAEBbotuIE) and students at St. Mark's High School in Massachusetts (http://www.newscientist.com/blogs/nstv/2012/01/paper-folding-limits-pushed.html). You may wish to share one of these resources with your class at the close of the lesson or after they have completed the Exploratory Challenge. After this challenge, which reintroduces students to exponential growth, students review the properties of exponents for integer exponents and apply them to the rewriting of algebraic expressions (**N-RN.A.2**). The lesson concludes with fluency practice where students apply properties of exponents to rewrite expressions in a specified form.

You may want to have the following materials on hand in case students want to explore this problem in more detail: access to the internet, chart paper, cash register tape, a roll of toilet paper, origami paper, tissue paper or facial tissues, and rulers.

Classwork

Students begin this lesson by predicting whether they can fold a piece of paper in half 10 times, how tall the folded paper will be, and whether or not the area of paper showing on top is smaller or larger than a postage stamp. They will explore the validity of their predictions in the Exploratory Challenge that follows.

Opening Exercise (3 minutes)

Give students a short amount of time to think about and write individual responses. Lead a short discussion with the entire class to poll students on their responses. Record solutions on the board or chart paper for later reference. At this point, most students will probably say that they *can* fold the paper in half 10 times. The sample responses shown below are NOT correct and represent possible initial student responses. Some students may be familiar with this challenge, having seen it discussed on a television program or on the Internet and, consequently, will say that you cannot fold a piece of notebook paper in half 10 times. Accept all responses and avoid excessive explaining or justifying of answers at this point.

Scaffolding:

- Demonstrate folding the paper once or twice during the discussion to illustrate what the questions are asking and what is meant by *how thick* the folded paper will be.
- Ask advanced learners to provide justification for their claim about the thickness of the folded paper.

Opening Exercise

Can you fold a piece of notebook paper in half 10 times?

Answers will vary. Although incorrect, many students may initially answer "Yes."

How thick will the folded paper be?

Answers will vary. The following is a typical student guess: It will be about 1 cm.

Will the area of the paper on the top of the folded stack be larger or smaller than a postage stamp?

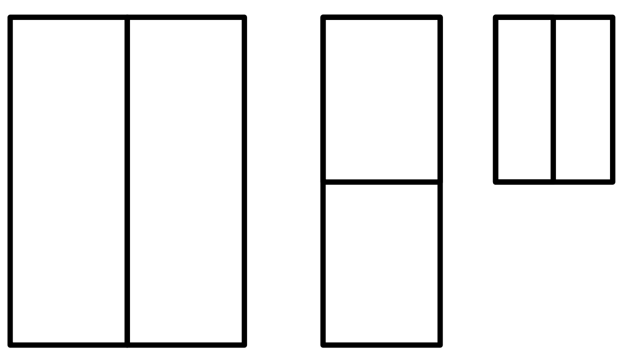
It will be smaller because I will be dividing the rectangle in half 10 times, and since a piece of paper is about 8.5 in. by 11 in., it will be very small when divided in half that many times.

Discussion (2 minutes)

MP.1

Students should brainstorm ideas for further exploring these questions to come up with a more precise answer to these questions. At this point, some students will likely be folding a piece of notebook paper. On chart paper, record ideas for additional information needed to answer the original questions more precisely.

- How can you be sure of your answers?
  - *We could actually fold a piece of paper and measure the height and area on top of the stack. We could determine the thickness of a sheet of notebook paper and then multiply it by the number of folds. We could find the area of the original paper and divide it by 2 successively for each fold.*



MP.1

- What additional information is needed to solve this problem?
  - *We need to know the thickness of the paper, the dimensions of the original piece of paper, and a consistent way to fold the paper.*
  - *We need to know the size of a postage stamp.*
- How will you organize your work?
  - *We can make a table.*

Exploratory Challenge (20 minutes)

Students should work on this challenge problem in small groups. Groups can use the suggested scaffolding questions on the student pages, or you can simply give each group a piece of chart paper and access to appropriate tools such as a ruler, different types of paper, a computer for researching the thickness of a sheet of paper, etc., and start them on the task. Have them report their results on their chart paper. Student solutions will vary from group to group. Sample responses have been provided below for a standard 8.5 in. by 11 in. piece of paper that has been folded in half as shown below. The size of a small postage stamp is $\frac{7}{8}$ in. by 1 in.

Exploratory Challenge

a. What are the dimensions of your paper?

The dimensions are 8.5 in. by 11 in.

b. How thick is one sheet of paper? Explain how you decided on your answer.

A ream of paper is 500 sheets. It is about 2 in. high. Dividing 2 by 500 would give a thickness of a piece of paper to be approximately 0.004 in.

c. Describe how you folded the paper.

First, we folded the paper in half so that it was 8.5 in. by 5.5 in.; then, we rotated the paper and folded it again so that it was 5.5 in. by 4.25 in.; then, we rotated the paper and folded it again, and so on.

d. Record data in the following table based on the size and thickness of your paper.

Number of Folds	0	1	2	3	4
Thickness of the Stack (in.)	0.004	0.008	0.016	0.032	0.064
Area of the Top of the Stack (sq. in.)	93.5	46.75	23.375	11.6875	5.84375

Number of Folds	5	6	7	8	9	10
Thickness of the Stack (in.)	0.128	0.256	0.512	1.024	2.048	4.096
Area of the Top of the Stack (sq. in.)	2.921875	1.461	0.730	0.365	0.183	0.091

Answers are rounded to three decimal places after the fifth fold.

e. Were you able to fold a piece of notebook paper in half 10 times? Why or why not?

No. It got too small and too thick for us to continue folding it.

Debrief after part (e) by having groups present their solutions so far. At this point, students should realize that it is impossible to fold a sheet of notebook paper in half 10 times. If groups wish to try folding a larger piece of paper, such as a piece of chart paper, or a different thickness of paper, such as a facial tissue, or using a different folding technique, such as an accordion fold, then allow them to alter their exploration. You may wish to have tissue paper, facial tissues, a roll of toilet paper, chart paper, or cash register tape on hand for student groups to use in their experiments.

Scaffolding:

If students are struggling to develop the formulas in their groups, you can complete the rest of this challenge as a whole class. Students will have many opportunities to model using exponential functions later in this module. You can write the height and thickness as products of repeated twos to help students see the pattern. For example, after three folds, the height would be $T \cdot 2 \cdot 2 \cdot 2$, where T is the thickness of the paper.

After students have made adjustments to their model and tested it, have them write formulas to predict the height and area after 10 folds and explain how these answers compare to their original predictions. Students worked with exponential functions and geometric sequences in Algebra I. Since this situation involves doubling or halving, most groups should be able to write a formula. When debriefing this next section with the entire class, help students to write a well-defined formula. Did they specify the meaning of their variables? Did they specify a domain if they used function notation? They may not have used the same variables shown in the solutions below and should be using specific values for the thickness and area of the paper based on their assumptions during modeling. Students will likely be surprised by these results.

- How thick would the stack be if you could fold it 10 times?
- Is the area of the top of the stack smaller or larger than a postage stamp?
- How do these answers compare to your prediction?

MP.4

f. **Create a formula that approximates the height of the stack after n folds.**

Our formula is $H(n) = T(2)^n$, where T is the thickness of the paper, and $H(n)$ is the height after n folds. In this case, $T = 0.004 \text{ in.}$

g. **Create a formula that will give you the approximate area of the top after n folds.**

Our formula is $A(n) = A_0\left(\frac{1}{2}\right)^n$, where A_0 is the area of the original piece of paper, and $A(n)$ is the area of the top after n folds. In this case, $A_0 = 93.5 \text{ sq. in.}$

h. **Answer the original questions from the Opening Exercise. How do the actual answers compare to your original predictions?**

It was impossible to fold the paper more than 7 times. Using our model, if we could fold the paper 10 times, it would be just over 4 in. thick and less than $\frac{1}{10} \text{ sq. in.}$, which is much smaller than the area of a postage stamp. Our predictions were inaccurate because we did not consider how drastically the sizes change when successively doubling or halving measurements.

Student groups should present their solutions again. If it did not come up earlier, ask students to consider how they might increase the likelihood that they could fold a piece of paper in half more than seven or eight times.

- What are some ways to increase the likelihood that you could successfully fold a piece of paper in half more than seven or eight times?
    - *You could use a thinner piece of paper. You could use a larger piece of paper. You could try different ways of folding the paper.*

Brittney Gallivan, the high school student who solved this problem in 2001, first folded a very thin sheet of gold foil in half over seven times and then successfully folded an extremely large piece of paper in half 12 times at a local shopping mall. In 2011, students at St. Mark's High School in Massachusetts folded miles of taped together toilet paper in half 13 times.

Example 1 (5 minutes): Using the Properties of Exponents to Rewrite Expressions

In this example, you will show students how to represent their expressions using powers of 2. Model this using the folding of a 10 in. by 10 in. square sheet of gold foil. The thickness of gold foil is 0.28 millionths of a meter. The information in this problem is based on the first task Britney accomplished: folding a sheet of gold foil in half twelve times. Of course, her teacher then modified the task and required her to actually use paper. (http://www.abc.net.au/science/articles/2005/12/21/1523497.htm). The goal of this example is to remind students of the meaning of integer exponents.

> *Scaffolding:*
>
> Use a vocabulary notebook or an anchor chart posted on the wall to remind students of vocabulary words associated with exponents.
>
> n is the exponent
>
> b^n
>
> b is the base

Many students will likely see the area sequence as successive divisions by two. For students who are struggling to make sense of the meaning of a negative exponent, you can model rewriting the expressions in the last column as follows:

$$\frac{100}{2} = 100\left(\frac{1}{2}\right) = 100\left(\frac{1}{2}\right)^1 = 100 \cdot 2^{-1};$$

$$\frac{100}{4} = 100\left(\frac{1}{4}\right) = 100\left(\frac{1}{2}\right)^2 = 100 \cdot 2^{-2};$$

$$\frac{100}{8} = 100\left(\frac{1}{8}\right) = 100\left(\frac{1}{2}\right)^3 = 100 \cdot 2^{-3}.$$

Example 1: Using the Properties of Exponents to Rewrite Expressions

The table below displays the thickness and area of a folded square sheet of gold foil. In 2001, Britney Gallivan, a California high school junior, successfully folded this size sheet of gold foil in half 12 times to earn extra credit in her mathematics class.

Rewrite each of the table entries as a multiple of a power of 2.

Number of Folds	Thickness of the Stack (Millionths of a Meter)	Thickness Using a Power of 2	Area of the Top (Square Inches)	Area Using a Power of 2
0	0.28	$0.28 \cdot 2^0$	100	$100 \cdot 2^0$
1	0.56	$0.28 \cdot 2^1$	50	$100 \cdot 2^{-1}$
2	1.12	$0.28 \cdot 2^2$	25	$100 \cdot 2^{-2}$
3	2.24	$0.28 \cdot 2^3$	12.5	$100 \cdot 2^{-3}$
4	4.48	$0.28 \cdot 2^4$	6.25	$100 \cdot 2^{-4}$
5	8.96	$0.28 \cdot 2^5$	3.125	$100 \cdot 2^{-5}$
6	17.92	$0.28 \cdot 2^6$	1.5625	$100 \cdot 2^{-6}$

As you model this example with students, take the opportunity to discuss the fact that exponentiation with positive integers can be thought of as repeated multiplication by the base, whereas exponentiation with negative integers can be thought of as repeated division by the base. For example,

$$4^{24} = \underbrace{4 \cdot 4 \cdot 4 \cdot \dots \cdot 4}_{24 \text{ times}} \quad \text{and} \quad 4^{-24} = \frac{1}{\underbrace{4 \cdot 4 \cdot 4 \cdot \dots \cdot 4}_{24 \text{ times}}}.$$

Alternatively, you can describe the meaning of a negative exponent when the exponent is an integer as repeated multiplication by the reciprocal of the base. For example,

$$4^{-24} = \underbrace{\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\cdots\left(\frac{1}{4}\right)}_{24 \text{ times}}.$$

Interpreting exponents as repeated multiplication or division only makes sense for integer exponents. However, the properties of exponents do apply for any real number exponent.

Example 2 (5 minutes): Applying the Properties of Exponents to Rewrite Expressions

Transition into this example by explaining that many times when we work with algebraic and numeric expressions that contain exponents, it is advantageous to rewrite them in different forms (MP.7). One obvious advantage of exponents is that they shorten the length of an expression involving repeated multiplication. Imagine always having to write out ten 2s if you wanted to express the number 1024 as a power of 2. While the exponent notation gives us a way to express repeated multiplication succinctly, the properties of exponents also provide a way to make computations with exponents more efficient. Share the properties of exponents below. Have students record these in their math notebooks.

> The Properties of Exponents
>
> For nonzero real numbers x and y and all integers a and b, the following properties hold.
>
> $$x^a \cdot x^b = x^{a+b}$$
> $$(x^a)^b = x^{ab}$$
> $$(xy)^a = x^a y^a$$
> $$\frac{x^a}{x^b} = x^{a-b}$$

(Note: Most cases of the properties listed above hold when $x = 0$ or $y = 0$. The only cases that cause problems are when values of the variables result in the expression 0^0 or division by 0.) Ask students to discuss with a partner different ways to rewrite the following expressions in the form of kx^n, where k is a real number and n is an integer. Have students share out their responses with the entire class. For each problem, model both approaches. As you model, make sure to have students verbalize the connections between the methods. Ask volunteers to explain why the rules hold.

Scaffolding:

Students needing additional practice with exponents can use the following numeric examples that mirror the algebraic expressions in Example 1. Have students work them side-by-side to help see the structure of the expressions.

- Write each expression in the form kb^n, where k is a real number and b and n are integers.

$$(5 \cdot 2^7)(-3 \cdot 2^2)$$
$$\frac{3 \cdot 4^5}{(2 \cdot 4)^4}$$
$$\frac{3}{(5^2)^{-3}}$$
$$\frac{3^{-3} \cdot 3^4}{3^8}$$

Example 2: Applying the Properties of Exponents to Rewrite Expressions

Rewrite each expression in the form of kx^n, where k is a real number, n is an integer, and x is a nonzero real number.

a. $5x^5 \cdot -3x^2$

Method 1: Apply the definition of an exponent and properties of algebra.

$$5x^5 \cdot -3x^2 = 5 \cdot -3 \cdot x^5 \cdot x^2 = -15 \cdot (x \cdot x \cdot x \cdot x \cdot x) \cdot (x \cdot x) = -15\, x^7$$

Method 2: Apply the rules of exponents and the properties of algebra.

$$5x^5 \cdot -3x^2 = 5 \cdot -3 \cdot x^5 \cdot x^2 = -15 \cdot x^{5+2} = -15\, x^7$$

b. $\dfrac{3x^5}{(2x)^4}$

Method 1: Apply the definition of an exponent and properties of algebra.

$$\frac{3x^5}{(2x)^4} = \frac{3 \cdot x \cdot x \cdot x \cdot x \cdot x}{2x \cdot 2x \cdot 2x \cdot 2x} = \frac{3 \cdot x \cdot x \cdot x \cdot x \cdot x}{2 \cdot 2 \cdot 2 \cdot 2 \cdot x \cdot x \cdot x \cdot x} = \frac{3x}{16} \cdot \frac{x}{x} \cdot \frac{x}{x} \cdot \frac{x}{x} \cdot \frac{x}{x} = \frac{3x}{16} \cdot 1 \cdot 1 \cdot 1 \cdot 1 = \frac{3}{16}x$$

Method 2: Apply the rules of exponents and the properties of algebra.

$$\frac{3x^5}{2^4x^4} = \frac{3}{16}x^{5-4} = \frac{3}{16}x$$

c. $\dfrac{3}{(x^2)^{-3}}$

Method 1: Apply the definition of an exponent and properties of algebra.

$$\frac{3}{(x^2)^{-3}} = \frac{3}{\left(\frac{1}{x^2}\right)\left(\frac{1}{x^2}\right)\left(\frac{1}{x^2}\right)} = \frac{3}{\frac{1}{x} \cdot \frac{1}{x} \cdot \frac{1}{x} \cdot \frac{1}{x} \cdot \frac{1}{x} \cdot \frac{1}{x}} = \frac{3}{\frac{1}{x^6}} = 3 \cdot x^6 = 3x^6$$

Method 2: Apply the properties of exponents.

$$\frac{3}{(x^2)^{-3}} = \frac{3}{x^{2 \cdot -3}} = \frac{3}{x^{-6}} = 3x^6$$

d. $\dfrac{x^{-3}x^4}{x^8}$

Method 1: Apply the definition of an exponent and properties of algebra.

$$\frac{x^{-3}x^4}{x^8} = \frac{\frac{1}{x} \cdot \frac{1}{x} \cdot \frac{1}{x} \cdot x \cdot x \cdot x \cdot x}{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x} = \frac{\frac{x}{x} \cdot \frac{x}{x} \cdot \frac{x}{x} \cdot x}{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x} = \frac{x}{x} \cdot \frac{1}{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot} = x^{-7}$$

Method 2: Apply the properties of exponents.

$$\frac{x^{-3}x^4}{x^8} = \frac{x^{-3+4}}{x^8} = \frac{x^1}{x^8} = x^{1-8} = x^{-7}$$

After seeing these examples, students should begin to understand why the properties of exponents are so useful for working with exponential expressions. Show both methods as many times as necessary in order to reinforce the properties with students so that they will ultimately rewrite most expressions like these by inspection, using the properties rather than expanding exponential expressions.

Exercises 1–5 (5 minutes)

The point of these exercises is to force students to use the properties of exponents to rewrite expressions in the form kx^n. Typical high school textbooks ask students to write expressions with non-negative exponents. However, in advanced mathematics classes, students need to be able to fluently rewrite expressions in different formats. Students who continue on to study higher-level mathematics such as calculus will need to rewrite expressions in this format in order to quickly apply a common derivative rule. The last two exercises are not feasible to work out by expanding the exponent. Use them to assess whether or not students are able to apply the rules fluently even when larger numbers or variables are involved.

Exercises 1–5

Rewrite each expression in the form of kx^n, where k is a real number and n is an integer. Assume $x \neq 0$.

1. $2x^5 \cdot x^{10}$

   $2x^{15}$

2. $\dfrac{1}{3x^8}$

   $\dfrac{1}{3}x^{-8}$

3. $\dfrac{6x^{-5}}{x^{-3}}$

   $6x^{-5-(-3)} = 6x^{-2}$

4. $\left(\dfrac{3}{x^{-22}}\right)^{-3}$

   $(3x^{22})^{-3} = 3^{-3}x^{-66} = \dfrac{1}{27}x^{-66}$

5. $(x^2)^n \cdot x^3$

   $x^{2n} \cdot x^3 = x^{2n+3}$

Closing (2 minutes)

Have students respond individually in writing or with a partner to the following questions.

- How can the properties of exponents help us to rewrite expressions?
  - *They make the process less tedious, especially when the exponents are very large or very small integers.*
- Why are the properties of exponents useful when working with large or small numbers?
  - *You can quickly rewrite expressions without having to rewrite each power of the base in expanded form.*

Lesson Summary

The Properties of Exponents
For real numbers x and y with $x \neq 0$, $y \neq 0$, and all integers a and b, the following properties hold.

$$x^a \cdot x^b = x^{a+b}$$

$$(x^a)^b = x^{ab}$$

$$(xy)^a = x^a y^a$$

$$\frac{1}{x^a} = x^{-a}$$

$$\frac{x^a}{x^b} = x^{a-b}$$

$$\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$$

$$x^0 = 1$$

Exit Ticket (3 minutes)

Name ____ Date ________________

Lesson 1: Integer Exponents

Exit Ticket

The following formulas for paper folding were discovered by Britney Gallivan in 2001 when she was a high school junior. The first formula determines the minimum width, W, of a square piece of paper of thickness T needed to fold it in half n times, alternating horizontal and vertical folds. The second formula determines the minimum length, L, of a long rectangular piece of paper of thickness T needed to fold it in half n times, always folding perpendicular to the long side.

$$W = \pi \cdot T \cdot 2^{\frac{3(n-1)}{2}} \qquad L = \frac{\pi T}{6}(2^n + 4)(2^n - 1)$$

1. Notebook paper is approximately 0.004 in. thick. Using the formula for the width W, determine how wide a square piece of notebook paper would need to be to successfully fold it in half 13 times, alternating horizontal and vertical folds.

2. Toilet paper is approximately 0.002 in. thick. Using the formula for the length L, how long would a continuous sheet of toilet paper have to be to fold it in half 12 times, folding perpendicular to the long edge each time?

3. Use the properties of exponents to rewrite each expression in the form kx^n. Then evaluate the expression for the given value of x.

   a. $2x^3 \cdot \frac{5}{4}x^{-1}$; $x = 2$

   b. $\frac{9}{(2x)^{-3}}$; $x = -\frac{1}{3}$

Exit Ticket Sample Solutions

The following formulas for paper folding were discovered by Britney Gallivan in 2001 when she was a high school junior. The first formula determines the minimum width, W, of a square piece of paper of thickness T needed to fold it in half n times, alternating horizontal and vertical folds. The second formula determines the minimum length, L, of a long rectangular piece of paper of thickness T needed to fold it in half n times, always folding perpendicular to the long side.

$$W = \pi \cdot T \cdot 2^{\frac{3(n-1)}{2}} \qquad L = \frac{\pi T}{6}(2^n + 4)(2^n - 1)$$

1. Notebook paper is approximately 0.004 in. thick. Using the formula for the width W, determine how wide a square piece of notebook paper would need to be to successfully fold it in half 13 times.

   The paper would need to be approximately 3294.2 in. *wide:* $W = \pi\, T\, 2^{\frac{3(13-1)}{2}} = \pi(0.004)2^{18} \approx 3294.199.$

2. Toilet paper is approximately 0.002 in. thick. Using the formula for the length L, how long would a continuous sheet of toilet paper have to be to fold it in half 12 times?

   The paper would have to be approximately $17{,}581.92$ in. *long, which is approximately* 0.277 mi.*:*

   $$L = \left(\frac{\pi(0.002)}{6}\right)(2^{12} + 4)(2^{12} - 1) = \pi\left(\frac{1}{300}\right)(4100)(4095) = 55965\,\pi \approx 17{,}581.92.$$

3. Use the properties of exponents to rewrite each expression in the form kx^n. Then evaluate the expression for the given value of x.

   a. $2x^3 \cdot \frac{5}{4}x^{-1};\ x = 2$

   $2\left(\frac{5}{4}\right)x^3x^{-1} = \frac{5}{2}x^2$

   When $x = 2$*, the expression has the value* $\frac{5}{2}(2)^2 = 10.$

   b. $\frac{9}{(2x)^{-3}};\ x = -\frac{1}{3}$

   $\frac{9}{2^{-3}x^{-3}} = 72x^3$

   When $x = -\frac{1}{3}$*, the expression has the value* $72\left(-\frac{1}{3}\right)^3 = -\frac{8}{3}.$

Problem Set Sample Solutions

1. Suppose your class tried to fold an unrolled roll of toilet paper. It was originally 4 in. wide and 30 ft. long. Toilet paper is approximately 0.002 in. thick.

   a. Complete each table and represent the area and thickness using powers of 2.

Number of Folds n	Thickness After n Folds (in.)
0	$0.002 = 0.002 \cdot 2^{0}$
1	$0.004 = 0.002 \cdot 2^{1}$
2	$0.008 = 0.002 \cdot 2^{2}$
3	$0.016 = 0.002 \cdot 2^{3}$
4	$0.032 = 0.002 \cdot 2^{4}$
5	$0.064 = 0.002 \cdot 2^{5}$
6	$0.128 = 0.002 \cdot 2^{6}$

Number of Folds n	Area on Top After n Folds (in^2)
0	$1440 = 1440 \cdot 2^{0}$
1	$720 = 1440 \cdot 2^{-1}$
2	$360 = 1440 \cdot 2^{-2}$
3	$180 = 1440 \cdot 2^{-3}$
4	$90 = 1440 \cdot 2^{-4}$
5	$45 = 1440 \cdot 2^{-5}$
6	$22.5 = 1440 \cdot 2^{-6}$

   b. Create an algebraic function that describes the area in square inches after n folds.

   $A(n) = 1440 \cdot 2^{-n}$, where n is a positive integer.

   c. Create an algebraic function that describes the thickness in inches after n folds.

   $T(n) = 0.002 \cdot 2^{n}$, where n is a positive integer.

2. In the Exit Ticket, we saw the formulas below. The first formula determines the minimum width, W, of a square piece of paper of thickness T needed to fold it in half n times, alternating horizontal and vertical folds. The second formula determines the minimum length, L, of a long rectangular piece of paper of thickness T needed to fold it in half n times, always folding perpendicular to the long side.

$$W = \pi \cdot T \cdot 2^{\frac{3(n-1)}{2}} \qquad L = \frac{\pi T}{6}(2^{n} + 4)(2^{n} - 1)$$

Use the appropriate formula to verify why it is possible to fold a 10 inch by 10 inch sheet of gold foil in half 13 times. Use 0.28 millionths of a meter for the thickness of gold foil.

Given that the thickness of the gold foil is 0.28 millionths of a meter, we have

$$\frac{0.28}{1{,}000{,}000}\,\text{m} \cdot \frac{100\ \text{cm}}{1\ \text{m}} = 0.000028\ \text{cm} \cdot \frac{1\ \text{in}}{2.54\ \text{cm}} = 0.00001102\ \text{in.}$$

Using the formula

$$W = \pi\, T\, 2^{\frac{3(n-1)}{2}}$$

with $n = 13$ and $T = 0.00001102$ in., we get

$$W = \pi(0.00001102)2^{\frac{3(13-1)}{2}}\ \text{in.} \approx 9.1\ \text{in.}$$

3. Use the formula from the Exit Ticket to determine if you can fold an unrolled roll of toilet paper in half more than 10 times. Assume that the thickness of a sheet of toilet paper is approximately 0.002 in. and that one roll is 102 ft. long.

First convert feet to inches. $102 \text{ ft.} = 1224 \text{ in.}$

Then, substitute 0.002 and 10 into the formula for T and n, respectively.

$$L = \frac{\pi(0.002)}{6}(2^{10} + 4)(2^{10} - 1) = 1101.3$$

The roll is just long enough to fold in half 10 times.

4. Apply the properties of exponents to rewrite expressions in the form kx^n, where n is an integer and $x \neq 0$.

a. $(2x^3)(3x^5)(6x)^2$

$2 \cdot 3 \cdot 36x^{3+5+2} = 216x^{10}$

b. $\dfrac{3x^4}{(-6x)^{-2}}$

$3x^4 \cdot 36x^2 = 108x^6$

c. $\dfrac{x^{-3}x^5}{3x^4}$

$\dfrac{1}{3}x^{-3+5-4} = \dfrac{1}{3}x^{-2}$

d. $5(x^3)^{-3}(2x)^{-4}$

$\dfrac{5}{16}x^{-9+(-4)} = \dfrac{5}{16}x^{-13}$

e. $\left(\dfrac{x^2}{4x^{-1}}\right)^{-3}$

$\dfrac{x^{-6}}{4^{-3}x^3} = 64x^{-6-3} = 64x^{-9}$

5. Apply the properties of exponents to verify that each statement is an identity.

a. $\dfrac{2^{n+1}}{3^n} = 2\left(\dfrac{2}{3}\right)^n$ for integer values of n.

$\dfrac{2^{n+1}}{3^n} = \dfrac{2^n2^1}{3^n} = \dfrac{2 \cdot 2^n}{3^n} = 2 \cdot \left(\dfrac{2}{3}\right)^n$

b. $3^{n+1} - 3^n = 2 \cdot 3^n$ for integer values of n.

$3^{n+1} - 3^n = 3^n \cdot 3^1 - 3^n = 3^n(3-1) = 3^n \cdot 2 = 2 \cdot 3^n$

c. $\dfrac{1}{(3^n)^2} \cdot \dfrac{4^n}{3} = \dfrac{1}{3}\left(\dfrac{2}{3}\right)^{2n}$ for integer values of n.

$\dfrac{1}{(3^n)^2} \cdot \dfrac{4^n}{3} = \dfrac{1}{3^{2n}} \cdot \dfrac{(2^2)^n}{3} = \dfrac{1 \cdot 2^{2n}}{3 \cdot 3^{2n}} = \dfrac{1}{3} \cdot \left(\dfrac{2}{3}\right)^{2n}$

MP.3

6. Jonah was trying to rewrite expressions using the properties of exponents and properties of algebra for nonzero values of x. In each problem, he made a mistake. Explain where he made a mistake in each part and provide a correct solution.

Jonah's Incorrect Work

a. $(3x^2)^{-3} = -9x^{-6}$

b. $\frac{2}{3x^5} = 6x^{-5}$

c. $\frac{2x - x^3}{3x} = \frac{2}{3} - x^3$

In part (a), he multiplied 3 by the exponent -3. The correct solution is $3^{-3}x^{-6} = \frac{1}{27}x^{-6}$.

In part (b), he multiplied 2 by 3 when he rewrote x^5. The 3 should remain in the denominator of the expression. The correct solution is $\frac{2}{3}x^{-5}$.

In part (c), he only divided the first term by $3x$, but he should have divided both terms by $3x$. The correct solution is $\frac{2x}{3x} - \frac{x^3}{3x} = \frac{2}{3} - \frac{x^2}{3}$.

7. If $x = 5a^4$, and $a = 2b^3$, express x in terms of b.

By the substitution property, if $x = 5a^4$, and $a = 2b^3$, then $x = 5(2b^3)^4$. Rewriting the right side in an equivalent form gives $x = 80b^{12}$.

8. If $a = 2b^3$, and $b = -\frac{1}{2}c^{-2}$, express a in terms of c.

By the substitution property, if $a = 2b^3$, and $b = -\frac{1}{2}c^{-2}$, then $a = 2\left(-\frac{1}{2}c^{-2}\right)^3$. Rewriting the right side in an equivalent form gives $a = -\frac{1}{4}c^{-6}$.

9. If $x = 3y^4$, and $y = \frac{s}{2x^3}$, show that $s = 54y^{13}$.

Rewrite the equation $y = \frac{s}{2x^3}$ to isolate the variable s.

$$y = \frac{s}{2x^3}$$
$$2x^3y = s$$

By the substitution property, if $s = 2x^3y$, and $x = 3y^4$, then $s = 2(3y^4)^3 \cdot y$. Rewriting the right side in an equivalent form gives $s = 2 \cdot 27y^{12} \cdot y = 54y^{13}$.

10. Do the following without a calculator.

a. Express 8^3 as a power of 2.

$$8^3 = (2^3)^3 = 2^9$$

b. Divide 4^{15} by 2^{10}.

$$\frac{4^{15}}{2^{10}} = \frac{2^{30}}{2^{10}} = 2^{20} \quad \textit{or} \quad \frac{4^{15}}{2^{10}} = \frac{4^{15}}{4^5} = 4^{10}$$

11. Use powers of 2 to help you perform each calculation.

a. $\frac{2^7 \cdot 2^5}{16}$

$\frac{2^7 \cdot 2^5}{16} = \frac{2^7 \cdot 2^5}{2^4} = 2^{7+5-4} = 2^8 = 256$

b. $\frac{512000}{320}$

$\frac{512000}{320} = \frac{512 \cdot 1000}{32 \cdot 10} = \frac{2^9}{2^5} \cdot 100 = 2^4 \cdot 100 = 1600$

12. Write the first five terms of each of the following recursively-defined sequences.

a. $a_{n+1} = 2a_n, a_1 = 3$

$\{3, 6, 12, 24, 48\}$

b. $a_{n+1} = (a_n)^2, a_1 = 3$

$\{3, 9, 81, 6561, 43046721\}$

c. $a_{n+1} = 2(a_n)^2, a_1 = x$, where x is a real number. Write each term in the form kx^n.

$\{x, 2x^2, 8x^4, 128x^8, 32768x^{16}\}$

d. $a_{n+1} = 2(a_n)^{-1}, a_1 = y, (y \neq 0)$. Write each term in the form kx^n.

$\{y, 2y^{-1}, y, 2y^{-1}, y\}$

13. In Module 1, you established the identity
$(1 - r)(1 + r + r^2 + \cdots + r^{n-1}) = 1 - r^n$, where r is a real number and n is a positive integer.
Use this identity to find explicit formulas as specified below.

a. Rewrite the given identity to isolate the sum $1 + r + r^2 + \cdots r^{n-1}$ for $r \neq 1$.

$(1 + r + r^2 + \cdots + r^{n-1}) = \frac{1 - r^n}{1 - r}$

b. Find an explicit formula for $1 + 2 + 2^2 + 2^3 + \cdots + 2^{10}$.

$\frac{1 - 2^{11}}{1 - 2} = 2^{11} - 1$

c. Find an explicit formula for $1 + a + a^2 + a^3 + \cdots + a^{10}$ in terms of powers of a.

$\frac{1 - a^{11}}{1 - a}$

d. Jerry simplified the sum $1 + a + a^2 + a^3 + a^4 + a^5$ by writing $1 + a^{15}$. What did he do wrong?

He assumed that when you add terms with the same base, you also add the exponents. You only add the exponents when you multiply like bases.

e. **Find an explicit formula for $1 + 2a + (2a)^2 + (2a)^3 + \cdots + (2a)^{12}$ in terms of powers of a.**

$$\frac{1-(2a)^{13}}{1-2a}$$

f. **Find an explicit formula for $3 + 3(2a) + 3(2a)^2 + 3(2a)^3 + \ldots + 3(2a)^{12}$ in terms of powers of a.**
Hint: Use part (e).

$$3 \cdot \frac{1-(2a)^{13}}{1-2a}$$

g. **Find an explicit formula for $P + P(1+r) + P(1+r)^2 + P(1+r)^3 + \cdots + P(1+r)^{n-1}$ in terms of powers of $(1+r)$.**

$$P \cdot \frac{1-(1+r)^n}{1-(1+r)} = P \cdot \frac{1-(1+r)^n}{-r}$$

Note to the teacher: Problem 3, part (g) will be important for the financial lessons that occur near the end of this module.