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Exponential and Logarithmic Functions

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[1] Each lesson is ONE day, and ONE day is considered a 45-minute period.

Topics C through E (assessment 1 day, return 1 day, remediation or further applications 4 days)

Algebra II • Module 3

Exponential and Logarithmic Functions

OVERVIEW

In this module, students synthesize and generalize what they have learned about a variety of function families. They extend the domain of exponential functions to the entire real line (**N-RN.A.1**) and then extend their work with these functions to include solving exponential equations with logarithms (**F-LE.A.4**). They explore (with appropriate tools) the effects of transformations on graphs of exponential and logarithmic functions. They notice that the transformations on a graph of a logarithmic function relate to the logarithmic properties (**F-BF.B.3**). Students identify appropriate types of functions to model a situation. They adjust parameters to improve the model, and they compare models by analyzing appropriateness of fit and making judgments about the domain over which a model is a good fit. The description of modeling as, "the process of choosing and using mathematics and statistics to analyze empirical situations, to understand them better, and to make decisions," is at the heart of this module. In particular, through repeated opportunities in working through the modeling cycle (see page 61 of the CCLS), students acquire the insight that the same mathematical or statistical structure can sometimes model seemingly different situations.

This module builds on the work in Algebra I, Modules 3 and 5, where students first modeled situations using exponential functions and considered which type of function would best model a given real world situation. The module also introduces students to the extension standards relating to inverse functions and composition of functions to further enhance student understanding of logarithms.

Topic E is a culminating project spread out over several lessons where students consider applying their knowledge to financial literacy. They plan a budget, consider borrowing money to buy a car and a home, study paying off a credit card balance, and finally, decide how they could accumulate one million dollars.

Focus Standards

Extend the properties of exponents to rational exponents.

N-RN.A.1   Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. *For example, we define* $5^{\frac{1}{3}}$ *to be the cube root of* 5 *because we want* $\left(5^{\frac{1}{3}}\right)^3 = 5^{\left(\frac{1}{3}\right)3}$ *to hold, so* $\left(5^{\frac{1}{3}}\right)^3$ *must equal* 5.

N-RN.A.2[2]   Rewrite expressions involving radicals and rational exponents using the properties of exponents.

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[2] Including expressions where either base or exponent may contain variables.

Reason quantitatively and use units to solve problems.

N-Q.A.2[3]   Define appropriate quantities for the purpose of descriptive modeling.★

Write expressions in equivalent forms to solve problems.

A-SSE.B.3[4]   Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.★

    c.   Use the properties of exponents to transform expressions for exponential functions.

        For example the expression 1.15^t can be rewritten as $\left(1.15^{\frac{1}{12}}\right)^{12t} \approx 1.012^{12t}$ to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.

A-SSE.B.4[5]   Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. *For example, calculate mortgage payments.*★

Create equations that describe numbers or relationships.

A-CED.A.1[6]   Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.*★

Represent and solve equations and inequalities graphically.

A-REI.D.11[7]   Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.★

Understand the concept of a function and use function notation.

F-IF.A.3[8]   Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. *For example, the Fibonacci sequence is defined recursively by $f(0) = f(1) = 1$, $f(n + 1) = f(n) + f(n − 1)$ for $n \geq 1$.*

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[3] This standard will be assessed in Algebra II by ensuring that some modeling tasks (involving Algebra II content or securely held content from previous grades and courses) require the student to create a quantity of interest in the situation being described (i.e., this is not provided in the task). For example, in a situation involving periodic phenomena, the student might autonomously decide that amplitude is a key variable in a situation, and then choose to work with peak amplitude.

[4] Tasks have a real-world context. As described in the standard, there is an interplay between the mathematical structure of the expression and the structure of the situation, such that choosing and producing an equivalent form of the expression reveals something about the situation. In Algebra II, tasks include exponential expressions with rational or real exponents.

[5] This standard includes using the summation notation symbol.

[6] Tasks have a real-world context. In Algebra II, tasks include exponential equations with rational or real exponents, rational functions, and absolute value functions.

[7] In Algebra II, tasks may involve any of the function types mentioned in the standard.

[8] This standard is Supporting Content in Algebra II. This standard should support the Major Content in F-BF.2 for coherence.

Interpret functions that arise in applications in terms of the context.

F-IF.B.4[9]   For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*★

F-IF.B.5   Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function $h(n)$ gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*★

F-IF.B.6   Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.★

Analyze functions using different representations

F-IF.C.7   Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★

   e.   Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

F-IF.C.8[10]   Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

   b.   Use the properties of exponents to interpret expressions for exponential functions. *For example, identify percent rate of change in functions such as $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)^{12t}$, $y = (1.2)^{\frac{t}{10}}$, and classify them as representing exponential growth or decay.*

F-IF.C.9[11]   Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.*

Build a function that models a relationship between two quantities

F-BF.A.1   Write a function that describes a relationship between two quantities.★

   a.   Determine an explicit expression, a recursive process, or steps for calculation from a context.[12]

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[9] Tasks have a real-world context. In Algebra II, tasks may involve polynomial, exponential, logarithmic, and trigonometric functions.

[10] Tasks include knowing and applying $A = Pe^{rt}$ and $A = P\left(1 + \frac{r}{n}\right)^{nt}$.

[11] In Algebra II, tasks may involve polynomial, exponential, logarithmic, and trigonometric functions

[12] Tasks have a real-world context. In Algebra II, tasks may involve linear functions, quadratic functions, and exponential functions.

    b. Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.*[13]

F-BF.A.2 Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.★

Build new functions from existing functions.

F-BF.B.3[14] Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*

F-BF.B.4 Find inverse functions.

    a. Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse. *For example, $f(x) = 2\,x^3$ or $f(x) = (x + 1)/(x – 1)$ for $x \neq 1$.*

Construct and compare linear, quadratic, and exponential models and solve problems.

F-LE.A.2[15] Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).★

F-LE.A.4[16] For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.★

Interpret expressions for functions in terms of the situation they model.

F-LE.B.5[17] Interpret the parameters in a linear or exponential function in terms of a context.★

Foundational Standards

Use properties of rational and irrational numbers.

N-RN.B.3 Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.

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[13] Combining functions also includes composition of functions.
[14] In Algebra II, tasks may involve polynomial, exponential, logarithmic, and trigonometric functions. Tasks may involve recognizing even and odd functions.
[15] In Algebra II, tasks will include solving multi-step problems by constructing linear and exponential functions.
[16] Students learn terminology that logarithm without a base specified is base 10 and that natural logarithm always refers to base e.
[17] Tasks have a real-world context. In Algebra II, tasks include exponential functions with domains not in the integers.

Interpret the structure of expressions.

A-SSE.A.2   Use the structure of an expression to identify ways to rewrite it. *For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.*

Create equations that describe numbers or relationships.

A-CED.A.2   Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.★

A-CED.A.4   Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law $V = IR$ to highlight resistance R.*★

Represent and solve equations and inequalities graphically.

A-REI.D.10   Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).

Understand the concept of a function and use function notation.

F-IF.A.1   Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then $f(x)$ denotes the output of f corresponding to the input x. The graph of f is the graph of the equation $y = f(x)$.

F-IF.A.2   Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.

Construct and compare linear, quadratic, and exponential models and solve problems.

F-LE.A.1   Distinguish between situations that can be modeled with linear functions and with exponential functions.★

      b.   Recognize situations in which one quantity changes at a constant rate per unit interval relative to another.

      c.   Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another.

F-LE.A.3   Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function.★

Focus Standards for Mathematical Practice

MP.1   **Make sense of problems and persevere in solving them.** Students make sense of rational and real number exponents and in doing so are able to apply exponential functions to solve problems involving exponential growth and decay for continuous domains such as time. They explore logarithms numerically and graphically to understand their meaning and how they can be used to solve exponential equations. Students have multiple opportunities to make connections between information presented graphically, numerically, and algebraically and search for similarities between these representations to further understand the underlying mathematical properties of exponents and logarithms. When presented with a wide variety of information related to financial planning, students make sense of the given information and use appropriate formulas to effectively plan for a long-term budget and savings plan.

MP.2   **Reason abstractly and quantitatively.** Students consider appropriate units when exploring the properties of exponents for very large and very small numbers. They reason about quantities when solving a wide variety of problems that can be modeled using logarithms or exponential functions. Students relate the parameters in exponential expressions to the situations they model. They write and solve equations and then interpret their solutions within the context of a problem.

MP.4   **Model with mathematics.** Students use exponential functions to model situations involving exponential growth and decay. They model the number of digits needed to assign identifiers using logarithms. They model exponential growth using a simulation with collected data. The application of exponential functions and logarithms as a means to solve an exponential equation is a focus of several lessons that deal with financial literacy and planning a budget. Here, students must make sense of several different quantities and their relationships as they plan and prioritize for their future financial solvency.

MP.7   **Look for and make use of structure.** Students extend the laws of exponents for integer exponents to rational and real number exponents. They connect how these laws are related to the properties of logarithms and understand how to rearrange an exponential equation into logarithmic form. Students analyze the structure of exponential and logarithmic functions to understand how to sketch graphs and see how the properties relate to transformations of these types of functions. They analyze the structure of expressions to reveal properties such as recognizing when a function models exponential growth versus decay. Students use the structure of equations to understand how to identify an appropriate solution method.

MP.8   **Look for and express regularity in repeated reasoning.** Students discover the properties of logarithms and the meaning of a logarithm by investigating numeric examples. They develop formulas that involve exponentials and logarithms by extending patterns and examining tables and graphs. Students generalize transformations of graphs of logarithmic functions by examining several different cases.

Terminology

New or Recently Introduced Terms

- **Arithmetic Series** (An *arithmetic series* is a series whose terms form an arithmetic sequence.)
- **Geometric Series** (A *geometric series* is a series whose terms form a geometric sequence.)
- **Invertible Function** (Let f be a function whose domain is the set X, and whose image is the set Y. Then f is *invertible* if there exists a function g with domain Y and image X such that f and g satisfy the property:

  For all $x \in X$ and $y \in Y$, $f(x) = y$ if and only if $g(y) = x$.

  The function g is called the *inverse* of f, and is denoted f^{-1}.

  The way to interpret the property is to look at all pairs $(x, y) \in X \times Y$: If the pair (x, y) makes $f(x) = y$ a true equation, then $g(y) = x$ is a true equation. If it makes $f(x) = y$ a false equation, then $g(y) = x$ is false. If that happens for each pair in $X \times Y$, then f and g are invertible and are inverses of each other.)

- **Logarithm** (If three numbers, L, b, and x are related by $x = b^L$, then L is the *logarithm base b of x*, and we write $L = \log_b(x)$. That is, the value of the expression $\log_b(x)$ is the power of b needed to be equivalent to x.

  Valid values of b as a base for a logarithm are $0 < b < 1$ and $b > 1$.)

- **Series** (Let $a_1, a_2, a_3, a_4, \ldots$ be a sequence of numbers. A sum of the form

  $$a_1 + a_2 + a_3 + \cdots + a_n$$

  for some positive integer n is called a *series,* or *finite series,* and is denoted S_n. The a_i's are called the *terms* of the series. The number that the series adds to is called the *sum* of the series.

  Sometimes S_n is called the nth *partial sum*.)

- **e** (Euler's number, e, is an irrational number that is approximately equal to $e \approx 2.7182818284590$.)

- **Σ** (The Greek letter sigma, Σ, is used to represent the sum. There is no rigid way to use Σ to represent a summation, but all notations generally follow the same rules. We will discuss the most common way it is used. Given the sequence $a_1, a_2, a_3, a_4, \ldots$, we can write the sum of the first n terms of the sequence using the expression:

  $$\sum_{k=1}^{n} a_k \ .)$$

Familiar Terms and Symbols[18]

- Compound Interest
- Exponential Decay
- Exponential Expression
- Exponential Growth
- Scientific Notation

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[18] These are terms and symbols students have seen previously.

Suggested Tools and Representations

- Graphing calculator or Desmos online calculator simulation
- Wolfram Alpha Software
- GeoGebra or Geometer's Sketchpad Software
- Excel or other spreadsheet software, such as Calc (part of the OpenOffice suite)

Assessment Summary

Assessment Type	Administered	Format	Standards Addressed
Mid-Module Assessment Task	After Topic B	Constructed response with rubric	N-RN.A.1, N-RN.A.2, N-Q.A.2, A.CED.A.1, F-IF.B.6, F-BF.A.1a, F-LE.A.4
End-of-Module Assessment Task	After Topic E	Constructed response with rubric	A-SSE.B.3c, A-SSE.B.4, A-CED.A.1, A-REI.D.11, F-IF.A.3, F-IF.B.4, F-IF.B.5, F-IF.B.6, F-IF.C.7e, F-IF.C.8b, F-IF.C.9, F-BF.A.1a, F-BF.A.1b, F-BF.A.2, F-BF.B.3, F-BF.B.4a, F-LE.A.2, F-LE.A.4, F-LE.B.5