Name ______________________________ Date ________________

1. Use properties of exponents to explain why it makes sense to define $16^{\frac{1}{4}}$ as $\sqrt[4]{16}$.

2. Use properties of exponents to rewrite each expression as either an integer or as a quotient of integers $\frac{p}{q}$ to show the expression is a rational number.

   a. $\sqrt[4]{2}\,\sqrt[4]{8}$

   b. $\dfrac{\sqrt[3]{54}}{\sqrt[3]{2}}$

   c. $16^{\frac{3}{2}} \cdot \left(\frac{1}{27}\right)^{\frac{2}{3}}$

3. Use properties of exponents to rewrite each expression with only positive, rational exponents. Then find the numerical value of each expression when $x = 9$, $y = 8$, and $z = 16$. In each case, the expression evaluates to a rational number.

a. $\sqrt{\dfrac{xy^2}{(x^3z)^{\frac{1}{2}}}}$

b. $\sqrt[11]{y^2z^4}$

c. $x^{-\frac{3}{2}}y^{\frac{4}{3}}z^{-\frac{3}{4}}$

4. We can use finite approximations of the decimal expansion of $\pi = 3.141519 \ldots$ to find an approximate value of the number 3^{π}.

   a. Fill in the missing exponents in the following sequence of inequalities that represents the recursive process of finding the value of 3^{π}.

$$3^3 < 3^{\pi} < 3^4$$
$$3^{3.1} < 3^{\pi} < 3^{3.2}$$
$$3^{3.14} < 3^{\pi} < 3^{3.15}$$
$$3^{(\quad)} < 3^{\pi} < 3^{(\quad)}$$
$$3^{(\quad\ )} < 3^{\pi} < 3^{(\quad\ )}$$

   b. Explain how this recursive process leads to better and better approximations of the number 3^{π}.

5. A scientist is studying the growth of a population of bacteria. At the beginning of her study, she has 800 bacteria. She notices that the population is quadrupling every hour.

   a. What quantities, including units, need to be identified to further investigate the growth of this bacteria population?

b. The scientist recorded the following information in her notebook, but she forgot to label each row. Label each row to show what quantities, including appropriate units, are represented by the numbers in the table, and then complete the table.

	0	1	2	3	4
	8	32	128		

c. Write an explicit formula for the number of bacteria present after t hours.

d. Another scientist studying the same population notices that the population is doubling every half an hour. Complete the table, and write an explicit formula for the number of bacteria present after x half hours.

Time, t (hours)	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
Time, x (half-hours)	0	1	2	3	4	5	6
Bacteria (hundreds)	8	16	32				

e. Find the time, in hours, when there will be $5{,}120{,}000$ bacteria. Express your answer as a logarithmic expression.

f. A scientist calculated the average rate of change for the bacteria in the first three hours to be 168. Which units should the scientist use when reporting this number?

6. Solve each equation. Express your answer as a logarithm, and then approximate the solution to the nearest thousandth.

   a. $3(10)^{-x} = \frac{1}{9}$

   b. $362\left(10^{\frac{t}{12}}\right) = 500$

   c. $(2)^{3x} = 9$

   d. $300e^{0.4t} = 900$

7. Because atoms and molecules are very small, they are counted in units of *moles*, where $1 \text{ mole} = 6.022 \times 10^{23}$. Concentration of molecules in a liquid is measured in units of moles per liter. The measure of the acidity of a liquid is called the pH of the liquid and is given by the formula

$$\text{pH} = -\log(H),$$

where H is the concentration of hydrogen ions in units of moles per liter.

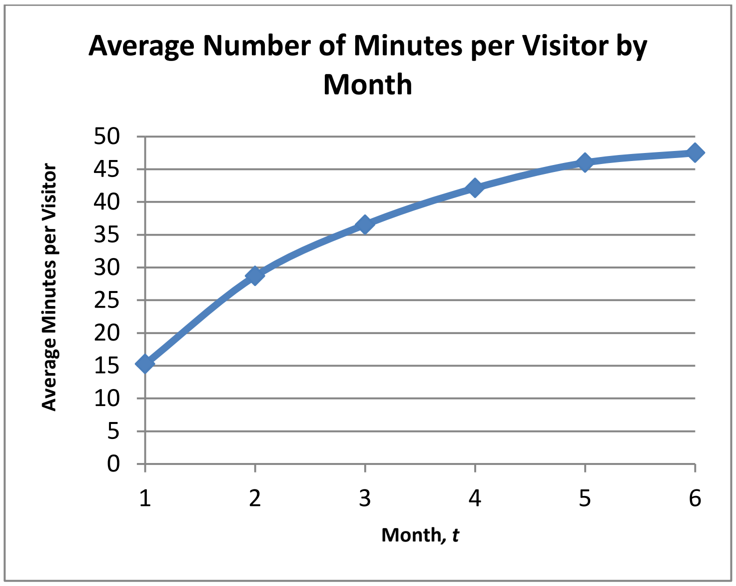
a. Water has a pH value of 7.0. How many hydrogen ions are in one liter of water?

b. If a liquid has a pH value larger than 7.0, does one liter of that liquid contain more or less hydrogen ions than one liter of water? Explain.

c. Suppose that liquid A is more acidic than liquid B, and their pH values differ by 1.2. What is the ratio of the concentration of hydrogen ions in liquid A to the concentration of hydrogen ions in liquid B?

8. A social media site is experiencing rapid growth. The table below shows values of V, the total number of unique visitors in each month, t, for a 6-month period of time. The graph shows the average minutes per visit to the site, M, in each month, t, for the same 6-month period of time.

t, Month	1	2	3	4	5	6
$V(t)$, Number of Unique Visitors	$418{,}000$	$608{,}000$	$1{,}031{,}000$	$1{,}270{,}000$	$2{,}023{,}000$	$3{,}295{,}000$



a. Between which two months did the site experience the most growth in total unique visitors? What is the average rate of change over this time interval?

b. Compute the value of $\frac{V(6)-V(1)}{6-1}$, and explain its meaning in this situation.

c. Between which two months did the average length of a visit change by the least amount? Estimate the average rate of change over this time interval.

d. Estimate the value of $\frac{M(3)-M(2)}{3-2}$ from the graph of M, and explain its meaning in this situation.

e. Based on the patterns they see in the table, the company predicts that the number of unique visitors will double each month after the sixth month. If growth continues at this pace, when will the number of unique visitors reach 1 billion?

A Progression Toward Mastery					
Assessment Task Item		**STEP 1** **Missing or incorrect answer and little evidence of reasoning or application of mathematics to solve the problem.**	**STEP 2** **Missing or incorrect answer but evidence of some reasoning or application of mathematics to solve the problem.**	**STEP 3** **A correct answer with some evidence of reasoning or application of mathematics to solve the problem, <u>or</u> an incorrect answer with substantial evidence of solid reasoning or application of mathematics to solve the problem.**	**STEP 4** **A correct answer supported by substantial evidence of solid reasoning or application of mathematics to solve the problem.**
1	**a** **N-RN.A.1**	Student solution is incorrect or missing. There is little evidence of correct student thinking and little or no written explanation.	Student explanation shows some evidence of reasoning but contains a major omission or limited explanation to support the reasoning.	Student explanation is mathematically correct but communication is limited or may contain minor notation errors.	Student explanation is mathematically correct and clearly stated. Student explanation relies on properties and definitions. Student uses precise and accurate vocabulary and notation.
2	**a–c** **N-RN.A.2**	Student work is inaccurate or missing. Student fails to express any of the expressions correctly as a rational number.	Student correctly rewrites only one of the expressions as a rational number. Student solutions show evidence of major errors or omissions in the work shown.	Student correctly rewrites each expression as a rational number, but the work shown is missing steps or contains minor notational errors. <u>OR</u> Student correctly rewrites two of three parts as rational numbers, and work shown is thorough and precise.	Student correctly rewrites each expression as a rational number and shows sufficient work to support the solution by applying properties and definitions.
3	**a–c** **N-RN.A.2**	Student work is inaccurate or missing, and there is little or no evidence of understanding of how to apply the properties and definitions to rational exponents.	Student rewrites the expressions correctly but only evaluates one expression correctly. <u>OR</u> Student rewrites only one expression correctly but evaluates	Student solutions are largely correct with no more than one or two minor errors. Student solutions show sufficient work to demonstrate evidence of understanding the	Student rewrites each part correctly according to the problem specifications and then evaluates each expression correctly for the given values of the variables.

			the expressions without significant errors for the given values of the variable. Student work indicates difficulty using proper notation.	properties and definitions, and student uses proper notation. <u>OR</u> Student correctly rewrites and evaluates the expressions for all three parts but shows little or no work to support reasoning.	Student solutions show sufficient work to demonstrate evidence of understanding the properties and definitions, and student uses proper notation.
4	**a–b** **F-BF.A.1a** **N-RN.A.1**	Student makes little or no attempt to describe a recursive process to estimate 3^{π}. There is little or no written communication to support reasoning.	Student attempts to explain a recursive process with clear written communication, but solution contains one or more major errors or omissions in thinking.	Student solution is largely correct, but work shown and written communication could be more detailed and precise.	Student solution is correct, and work shown clearly outlines a recursive process for estimating 3^{π}. Student explains thinking in writing clearly and concisely, using proper notation and vocabulary to support work.
5	**a–b** **N-Q.A.2**	Student work is inaccurate, missing, or off-task.	Student identifies one quantity of interest, but the solution fails to address proper units or includes more than one minor error.	Student identifies two quantities of interest, including units, in part (a), and labels the table in part (b) with no more than one minor error or omissions of units.	Student identifies two quantities of interest (number of bacteria and time), including units. Student correctly labels the table, including proper units.
	c–d **F-LE.A.2** **F-BF.A.1a**	Student does not write correct exponential formulas in explicit forms for either function. The work shown has several mathematical errors.	Student writes one of two exponential formulas correctly in explicit form. Solutions contain more than one minor error or show evidence of a major mathematical misconception.	Student writes two explicit exponential formulas that are largely correct with no more than one minor error. Student solutions may not clearly indicate how parameters were determined.	Student correctly writes two explicit exponential formulas to represent the amount of bacteria in each case. Student solution clearly shows evidence of how the parameters of each formula were determined.
	e **F-LE.A.4** **A-CED.A.1**	Student makes little or no attempt to write or solve an equation to answer this problem.	Student writes an incorrect equation, solves it correctly, and expresses the final answer as a logarithm. <u>OR</u> Student equation is correct, but the steps to solve the equation using logarithms show major mathematical errors in student thinking.	Student writes an equation of the form $f(t) = 512{,}000$ and solves it using logarithms. Final solution is not expressed as a logarithm, or the solution contains a minor error. <u>OR</u> Student writes an equation of the form	Student writes an equation of the form $f(t) = 512{,}000$, correctly solves it using logarithms, and expresses the final solution as a logarithm.

				$f(t) = 512{,}000$ and solves it correctly using a graphical or numerical approach.	
	f **N-Q.A.2**	Student makes little or no attempt to answer the question. Work shown is off-task.	Student provides units that agree with the model used in part (c) but explanation is missing.	Student identifies units that do not agree with the model used in part (c). Explanation is otherwise correct.	Student clearly identifies the units and explains the answer. Units agree with the model student created in part (c).
6	**a–d** **F-LE.A.4**	Student fails to correctly solve any equations using logarithms, showing incomplete and incorrect work.	Student correctly solves two equations but shows little or no work and fails to provide the exact or estimated answer. OR Student correctly solves one or two equations using logarithms. Work shown indicates major mathematical errors.	Student correctly solves four equations using logarithms but may not express solutions as both logarithms and estimated to three decimal places. OR Student correctly solves three of four equations. Work is largely correct with only one or two minor errors.	Student correctly solves four equations using logarithms. Work shown uses proper notation, and solutions are expressed exactly as logarithms estimated to three decimal places.
7	**a** **A-CED.A.1**	Student makes little or no attempt to write and solve an equation to answer the question. Work shown contains several significant errors.	Student writes an equation equivalent to $-7 = \log(H)$ but makes significant mathematical errors in solving it. Student does not relate the solution back to the question that was asked.	Student writes and solves an equation equivalent to the equation $-7 = \log(H)$, but the solution contains minor errors, either in solving the equation or answering the question.	Student writes and solves an equation equivalent to the equation $7 = -\log(H)$ and uses the solution to answer the question correctly.
	b **N-Q.A.2**	Student solution is inaccurate and incomplete, showing student had significant difficulty thinking about the quantities in this problem.	Student is not able to correctly interpret the meaning of the pH values when compared to the pH of water. Student explanation gives evidence of partially correct thinking about the quantities in this problem.	Student uses the pH of water to correctly interpret the meaning of pH values more or less than 7.0 in the context of this situation. Student explanation makes sense for the most part but could include more detail or may contain minor vocabulary or notation errors.	Student uses the pH of water to correctly interpret the meaning of pH values more or less than 7.0 in the context of this situation. Student explanation clearly conveys correct thinking and uses proper vocabulary and notation.

	c **A-CED.A.1** **N-Q.A.2**	Student makes little or no attempt to write and solve an equation. Student makes little or no attempt to relate work to the context of the situation.	Student does not write a correct equation but attempts to solve the problem. Student work and explanations show some evidence of correct thinking about the situation but contains some major mathematical errors.	Student correctly writes an equation with a solution that can be used to answer the question. Student attempts to solve the equation. Student work and explanations may contain minor mathematical errors.	Student correctly writes an equation with a solution that can be used to answer the question. Student solves the equation correctly. Student work and explanations clearly show correct thinking, with the solution related back to the situation.
8	**a–d** **F-IF.B.6**	Student is unsuccessful in calculating and interpreting average rate of change from either tables or graphs. Little or no work is shown, and the work that is present contains several mathematical errors.	Student calculates rate of change from tables and graphs correctly but struggles with a correct interpretation. OR Student solutions show major errors in calculating rate of change, but interpretations are largely accurate based on the values they calculated. Student explanations may be limited.	Student calculates and interprets average rate of change from a table or graph in each portion of the problem. Student solutions may contain a few minor errors that do not detract from the overall understanding of the problem.	Student successfully calculates and interprets average rate of change from a table or graph in each portion of the problem. Student explanations and work shown are clear and concise, including proper notation and symbols.
	e **F-LE.A.4**	Student work is off-task, missing, or incorrect. Little or no attempt is made to write either a function or an equation that could be used to solve the problem.	Student fails to create an exponential function that accurately reflects the description in the problem. Student attempts to solve an equation of the form $f(x) = 1{,}000{,}000{,}000$, but the solution, while partially correct, contains some significant mathematical errors.	Student creates an exponential function that accurately reflects the description in the problem. Student writes and solves an equation of the form $f(x) = 1{,}000{,}000{,}000$. Student solution contains minor errors, and student may fail to explicitly interpret the solution in the context of the problem.	Student creates an exponential function that accurately reflects the description in the problem. Student writes and correctly solves an equation of the form $f(x) = 1{,}000{,}000{,}000$, and interprets the solution in the context of the problem.

Name ______________________________ Date ________________

1. Use properties of exponents to explain why it makes sense to define $16^{\frac{1}{4}}$ as $\sqrt[4]{16}$.

We know that

$$\left(16^{\frac{1}{4}}\right)^4 = 16^{\frac{1}{4}} \cdot 16^{\frac{1}{4}} \cdot 16^{\frac{1}{4}} \cdot 16^{\frac{1}{4}} = 16^{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}} = 16^1 = 16,$$

and $2^4 = 16$.

Since there is only one positive real number whose fourth power is 16, and that is $\sqrt[4]{16} = 2$, it makes sense to define $16^{\frac{1}{4}} = \sqrt[4]{16}$.

2. Use properties of exponents to rewrite each expression as either an integer or as a quotient of integers $\frac{p}{q}$ to show the expression is a rational number.

a. $\sqrt[4]{2}\,\sqrt[4]{8}$

$$\sqrt[4]{2}\,\sqrt[4]{8} = 2^{\frac{1}{4}} \cdot 8^{\frac{1}{4}} = 2^{\frac{1}{4}} \cdot (2^3)^{\frac{1}{4}} = 2^{\frac{1}{4}} \cdot 2^{\frac{3}{4}} = 2^{\frac{1}{4}+\frac{3}{4}} = 2^1$$

Thus, $\sqrt[4]{2}\,\sqrt[4]{8} = 2$ is a rational number.

b. $\frac{\sqrt[3]{54}}{\sqrt[3]{2}}$

$$\frac{\sqrt[3]{54}}{\sqrt[3]{2}} = \frac{\left(54^{\frac{1}{3}}\right)}{2^{\frac{1}{3}}} = \frac{(2 \cdot 27)^{\frac{1}{3}}}{2^{\frac{1}{3}}} = \frac{2^{\frac{1}{3}} \cdot 27^{\frac{1}{3}}}{2^{\frac{1}{3}}} = 27^{\frac{1}{3}} = 3$$

Thus, $\frac{\sqrt[3]{54}}{\sqrt[3]{2}} = 3$ is a rational number.

c. $16^{\frac{3}{2}} \cdot \left(\frac{1}{27}\right)^{\frac{2}{3}}$

$$16^{\frac{3}{2}} \cdot \left(\frac{1}{27}\right)^{\frac{2}{3}} = \left(16^{\frac{1}{2}}\right)^3 \cdot \left(\frac{1}{\left(27^{\frac{1}{3}}\right)^2}\right) = 4^3 \cdot \frac{1}{3^2} = \frac{64}{9}$$

Thus, $16^{\frac{3}{2}} \cdot \left(\frac{1}{27}\right)^{\frac{2}{3}} = \frac{64}{9}$ is a rational number.

3. Use properties of exponents to rewrite each expression with only positive, rational exponents. Then find the numerical value of each expression when $x = 9$, $y = 8$, and $z = 16$. In each case, the expression evaluates to a rational number.

   a. $\sqrt{\dfrac{xy^2}{x^3z^{\frac{1}{2}}}}$

$$\sqrt{\frac{xy^2}{x^3z^{\frac{1}{2}}}} = \left(\frac{x \cdot y^2}{x^3z^{\frac{1}{2}}}\right)^{\frac{1}{2}} = \left(\frac{y^2}{x^2\ z^{\frac{1}{2}}}\right)^{\frac{1}{2}} = \frac{y}{x\ z^{\frac{1}{4}}}$$

When $x = 9$, $y = 8$, and $z = 16$, we have

$$\sqrt{\frac{xy^2}{x^3z^{\frac{1}{2}}}} = \frac{8}{9\ 16^{\frac{1}{4}}} = \frac{8}{9 \cdot 2} = \frac{4}{9}.$$

   b. $\sqrt[11]{y^2z^4}$

$$\sqrt[11]{y^2z^4} = (y^2z^4)^{\frac{1}{11}} = y^{\frac{2}{11}}z^{\frac{4}{11}}$$

When $y = 8$ and $z = 16$, we have

$$\sqrt[11]{y^2z^4} = 8^{\frac{2}{11}}16^{\frac{4}{11}} = (2^3)^{\frac{2}{11}} \cdot (2^4)^{\frac{4}{11}} = 2^{\frac{6}{11}} \cdot 2^{\frac{16}{11}} = 2^{\frac{22}{11}} = 2^2 = 4.$$

   c. $x^{-\frac{3}{2}}y^{\frac{4}{3}}z^{-\frac{3}{4}}$

$$x^{-\frac{3}{2}}y^{\frac{4}{3}}z^{-\frac{3}{4}} = \frac{y^{\frac{4}{3}}}{x^{\frac{3}{2}} \cdot z^{\frac{3}{4}}}$$

When $x = 9$, $y = 8$, and $z = 16$, we have

$$x^{-\frac{3}{2}}y^{\frac{4}{3}}z^{-\frac{3}{4}} = \frac{8^{\frac{4}{3}}}{9^{\frac{3}{2}} \cdot 16^{\frac{3}{4}}} = \frac{(2^3)^{\frac{4}{3}}}{(3^2)^{\frac{3}{2}} \cdot (2^4)^{\frac{3}{4}}} = \frac{2^4}{3^3 \cdot 2^3} = \frac{2}{3^3} = \frac{2}{27}.$$

4. We can use finite approximations of the decimal expansion of $\pi = 3.141519\ldots$ to find an approximate value of the number 3^{π}.

   a. Fill in the missing exponents in the following sequence of inequalities.

$$3^3 < 3^{\pi} < 3^4$$
$$3^{3.1} < 3^{\pi} < 3^{3.2}$$
$$3^{3.14} < 3^{\pi} < 3^{3.15}$$
$$3^{(3.141)} < 3^{\pi} < 3^{(3.142)}$$
$$3^{(3.1415)} < 3^{\pi} < 3^{(3.1416)}$$

   b. Explain how this recursive process leads to better and better approximations of the number 3^{π}.

We can get better and better approximations of π by squeezing it between rational over and under estimates that use more and more digits of its decimal expansion.

$$3 < \pi < 4$$
$$3.1 < \pi < 3.2$$
$$3.14 < \pi < 3.15$$
$$\vdots$$

Then, we can estimate 3^{π} by squeezing it between expressions with rational exponents. Because we know how to calculate a number such as $3^{3.14} = 3^3 \cdot 3^{0.14} = 27\left(\sqrt[100]{3^{14}}\right)$, we have a method for calculating the over and under estimates of 3^{π}.

Because $3 < \pi < 4$, we have $3^3 < 3^{\pi} < 3^4$. Thus, $27 < 3^{\pi} < 81$.

Because $3.1 < \pi < 3.2$, we have $3^{3.1} < 3^{\pi} < 3^{3.2}$. Thus, $30.1353 < 3^{\pi} < 33.6347$.

Because $3.14 < \pi < 3.15$, we have $3^{3.14} < 3^{\pi} < 3^{3.15}$. Thus, $31.4891 < 3^{\pi} < 31.8370$.

Continuing this process, we can approximate 3^{π} as closely as we want by starting with more and more digits of π in the exponent.

5. A scientist is studying the growth of a population of bacteria. At the beginning of her study, she has 800 bacteria. She notices that the population is quadrupling every hour.

   a. What quantities, including units, need to be identified to further investigate the growth of this bacteria population?

   We need to have the initial population, P_0, in either units of single bacteria or hundreds of bacteria, the time, t, in hours, and the current population, $P(t)$ at time t, in the same units as the initial population P_0.

   b. The scientist recorded the following information in her notebook, but she forgot to label each row. Label each row to show what quantities, including appropriate units, are represented by the numbers in the table, and then complete the table.

time, t (hours)	0	1	2	3	4
Population, $P(t)$ (hundreds)	8	32	128	*512*	*2048*

   c. Write an explicit formula for the number of bacteria present after t hours.

   After t hours, there are $P(t) = 8(4^t)$ hundred bacteria present.
   (It is also acceptable to model this population by $P(t) = 800(4^t)$ single bacteria.)

   d. Another scientist studying the same population notices that the population is doubling every half an hour. Complete the table, and write an explicit formula for the number of bacteria present after x half hours.

Time, t (hours)	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
Time, x (half-hours)	0	1	2	3	4	5	6
Bacteria (hundreds)	8	16	32	64	128	256	512

   After x half-hours, there are $Q(x) = 8(2^x)$ hundred bacteria present.
   (It is also acceptable to model this population by $Q(x) = 800(2^x)$ single bacteria.)

e. Find the time, in hours, when there will be $5{,}120{,}000$ bacteria. Express your answer as a logarithmic expression.

Students may choose to use the base-2 formula but will need to adjust the value of x, which counts half-hours, to t, which counts full hours, to correctly answer the question. Also, note that 5,120,000 is 51,200 hundred bacteria, so if students modeled the population using single bacteria instead of hundreds, they should solve $800(4^t) = 5{,}120{,}000$. Students may also solve this equation using the base-4 logarithm or base-2 logarithm, giving an equivalent solution that looks a little different..

$$8(4^t) = 51200$$
$$4^t = 6400$$
$$t\log(4) = \log(6400)$$
$$t = \frac{\log(6400)}{\log(4)}$$

f. A scientist calculated the average rate of change for the bacteria in the first three hours to be 168. Which units should the scientist use when reporting this number? Explain how you know.

The average rate of change over the first three hours is given by the formula $\frac{P(3)-P(0)}{3-0}$, which is a quotient of the number of bacteria in hundreds per hour. Thus, the unit should be reported as hundreds of bacteria per hour. Note that if students had modeled the population using single bacteria instead of hundreds, they should answer that the units are in bacteria per hour.

6. Solve each equation. Express your answer as a logarithm, and then approximate the solution to the nearest thousandth.

a. $3(10)^{-x} = \frac{1}{9}$

$$3(10^{-x}) = \frac{1}{9}$$
$$\frac{1}{10^x} = \frac{1}{27}$$
$$10^x = 27$$
$$x = \log(27)$$
$$x \approx 1.431$$

b. $362\left(10^{\frac{t}{12}}\right) = 500$

$$362\left(10^{\frac{t}{12}}\right) = 500$$
$$10^{\frac{t}{12}} = \frac{500}{362}$$
$$\frac{t}{12} = \log\left(\frac{500}{362}\right)$$
$$t \approx 1.683$$

c. $(2)^{3x} = 9$

$$2^{3x} = 9$$
$$\log_2\left(2^{3x}\right) = \log_2(9)$$
$$3x = \log_2(9)$$
$$x = \frac{1}{3}\log_2(9) = \frac{1}{3}\frac{\log(9)}{\log(2)}$$
$$x \approx 1.057$$

d. $300e^{0.4t} = 900$

$$300e^{0.4t} = 900$$
$$e^{0.4t} = 3$$
$$\ln\left(e^{0.4t}\right) = \ln(3)$$
$$0.4t = \ln(3)$$
$$t = 2.5\ln(3)$$
$$t \approx 2.747$$

7. Because atoms and molecules are very small, they are counted in units of *moles*, where $1 \text{ mole} = 6.022 \times 10^{23}$ molecules. Concentration of molecules in a liquid is measured in units of moles per liter. The measure of the acidity of a liquid is called the pH of the liquid and is given by the formula

$$\text{pH} = -\log(H),$$

where H is the concentration of hydrogen ions in units of moles per liter.

a. Water has a pH value of 7.0. How many hydrogen ions are in one liter of water?

$$7 = -\log(H)$$
$$-7 = \log(H)$$
$$H = 10^{-7}$$

Thus, there are approximately 10^{-7} moles of hydrogen in one liter of water. If we multiply this by 6.022×10^{23} ions per mole, we find that there are 6.022×10^{16} hydrogen ions.

b. If a liquid has a pH value larger than 7.0, does one liter of that liquid contain more or fewer hydrogen ions than one liter of water? Explain.

A liquid with a pH value larger than 7.0 will contain fewer hydrogen ions than one liter of water because $H = \frac{1}{10^{pH}}$ and when the pH is larger than 7.0, the value of H will become smaller because the quantities H and 10^{pH} are inversely proportional to one another.

c. Suppose that liquid A is more acidic than liquid B, and their pH values differ by 1.2. What is the ratio of the concentration of hydrogen ions in liquid A to the concentration of hydrogen ions in liquid B?

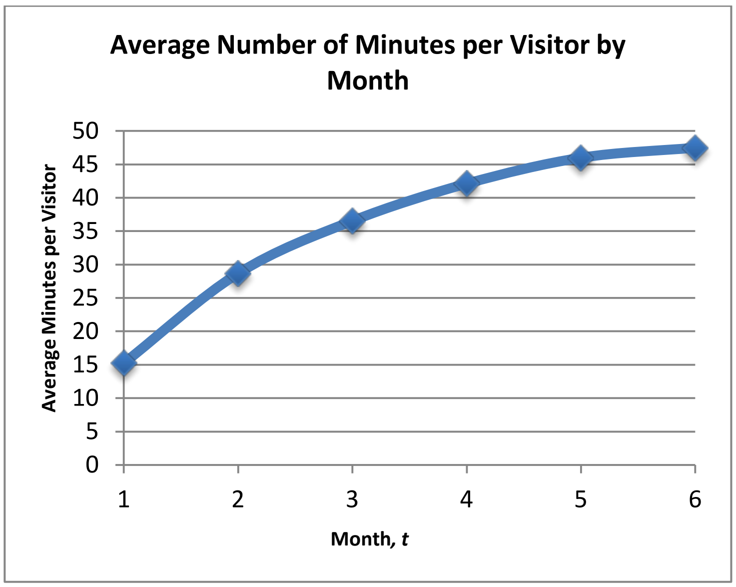
Let H_A be the concentration of hydrogen ions in liquid A, and let H_B be the concentration of hydrogen ions in liquid B. Then, the difference of the pH values is $-\log(H_B) - (-\log(H_A)) = 1.2$. Solve this equation for $\frac{H_A}{H_B}$, the requested ratio.

$$-\log(H_B) + \log(H_A) = 1.2$$
$$\log\left(\frac{H_A}{H_B}\right) = 1.2$$
$$\frac{H_A}{H_B} = 10^{1.2} \approx 15.85$$

Liquid A contains approximately 16 times as many hydrogen ions as liquid B.

8. A social media site is experiencing rapid growth. The table below shows values of V, and the total number of unique visitors in each month, t, for a 6-month period of time. The graph shows the average minutes per visit to the site, M, in each month, t, for the same 6-month period of time.

t, Month	1	2	3	4	5	6
$V(t)$, Number of Unique Visitors	418,000	608,000	1,031,000	1,270,000	2,023,000	3,295,000



a. Between which two months did the site experience the most growth in total unique visitors? What is the average rate of change over this time interval?

$V(2) - V(1) = 190{,}000$ $V(3) - V(2) = 423{,}000$ $V(4) - V(3) = 239{,}000$
$V(5) - V(4) = 753{,}000$ $V(6) - V(5) = 1{,}272{,}000$

The largest growth in the number of visitors occurs between months 5 and 6.

b. Compute the value of $\frac{V(6)-V(1)}{6-1}$, and explain its meaning in this situation.

$$\frac{V(6) - V(1)}{6 - 1} = \frac{3{,}295{,}000 - 418{,}000}{5} = 575{,}400$$

This means that the average monthly growth of visitors to the site between months 1 and 6 is 575,400 visitors per month.

c. Between which two months did the average length of a visit change by the least amount? Estimate the average rate of change over this time interval.

The two neighboring points that have the closest y-values are in months 5 and 6. Estimating values $M(6) \approx 47.5$ and $M(5) \approx 46$ from the graph, we see that the average rate of change over this interval is $\frac{M(6)-M(5)}{6-5} = \frac{47.5-46}{1} = 1.5$ minutes per visitor per month. (Students may read different values from the graph.)

d. Estimate the value of $\frac{M(3)-M(2)}{3-2}$ from the graph of M, and explain its meaning in this situation.

Estimating values $M(3) \approx 37$ and $M(2) \approx 28$ from the graph, we have $\frac{M(3)-M(2)}{3-2} = \frac{37-28}{1} = 9$, meaning that on average, each visit to the website increased by 9 minutes between months 2 and 3.

e. Based on the patterns they see in the table, the company predicts that the number of unique visitors will double each month after the sixth month. If growth continues at this pace, when will the number of unique visitors reach 1 billion?

$$3{,}295{,}000(2)^t = 1{,}000{,}000{,}000$$
$$2^t = 303.49$$
$$\log(2^t) = \log(303.49)$$
$$t\log(2) = \log(303.49)$$
$$t = \frac{\log(303.49)}{\log(2)}$$
$$t \approx 8.25$$

The number of unique visitors will reach 1 billion after 8.25 additional months have passed.