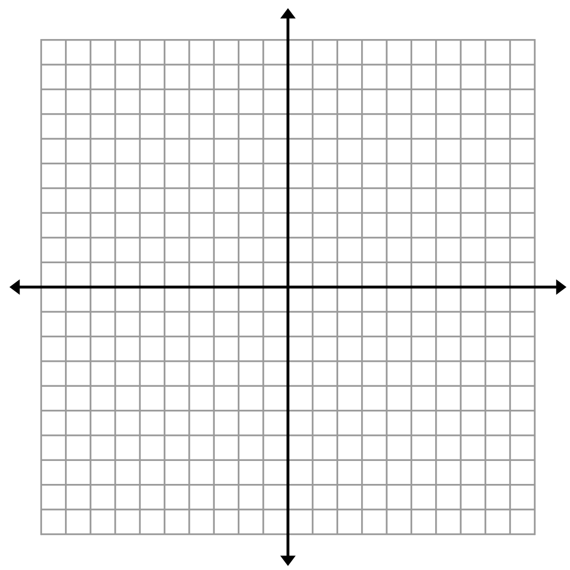
Name ______________________________ Date ____________________

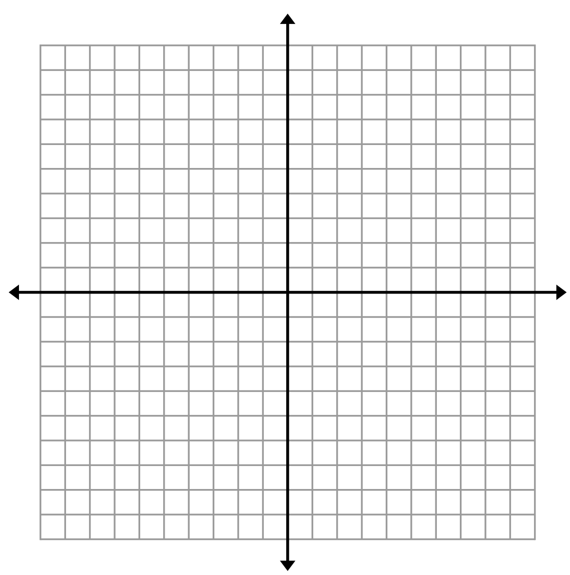
1. For parts (a) to (c),

- Sketch the graph of each pair of functions on the same coordinate axes showing end behavior and intercepts, and
- Describe the graph of g as a series of transformations of the graph of f.

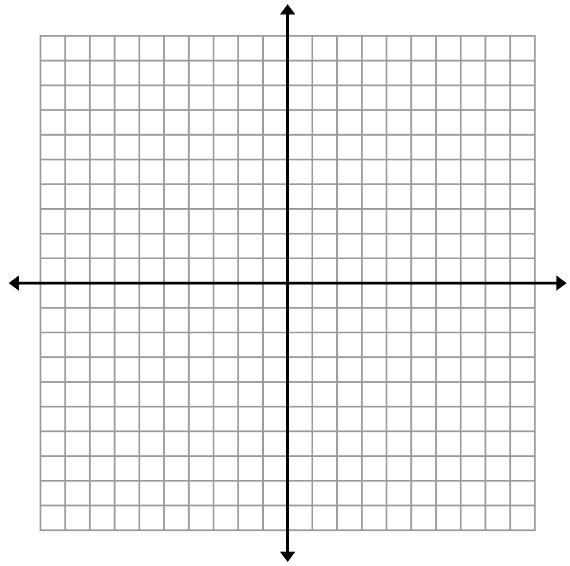
a. $f(x) = 2^x$, and $g(x) = 2^{-x} + 3$



b. $f(x) = 3^x$, and $g(x) = 9^{x-2}$



c. $f(x) = \log_2(x)$, and $g(x) = \log_2(x-1)^2$

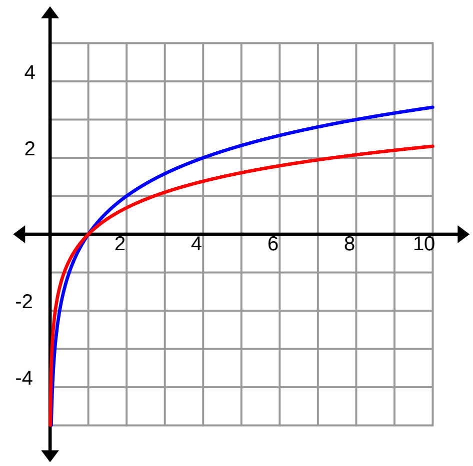


2. Consider the graph of $f(x) = 8^x$. Let $g(x) = f\left(\frac{1}{3}x + \frac{2}{3}\right)$ and $h(x) = 4f\left(\frac{x}{3}\right)$.

a. Describe the graphs of g and h as transformations of the graph of f.

b. Use the properties of exponents to show why the graphs of the functions $g(x) = f\left(\frac{1}{3}x + \frac{2}{3}\right)$ and $h(x) = 4f\left(\frac{x}{3}\right)$ are the same.

3. The graphs of the functions $f(x) = \ln(x)$ and $g(x) = \log_2(x)$ are shown to the right.



   a. Which curve is the graph of f, and which curve is the graph of g? Explain.

   b. Describe the graph of g as a transformation of the graph of f.

   c. By what factor has the graph of f been scaled vertically to produce the graph of g? Explain how you know.

4. Gwyneth is conducting an experiment. She rolls $1{,}000$ dice simultaneously and removes any that have a six showing. She then rerolls all of the dice that remain and again removes any that have a six showing. Gwyneth does this over and over again—rerolling the remaining dice and then removing those that land with a six showing.

   a. Write an exponential function f of the form $f(n) = a \cdot b^{cn}$ for any real number $n \geq 0$ that could be used to model the average number of dice she could expect on the n^{th} roll if she ran her experiment a large number of times.

   b. Gwyneth computed $f(12) = 112.15\ldots$ using the function f. How should she interpret the number $112.15\ldots$ in the context of the experiment?

   c. Explain the meaning of the parameters in your function f in terms of this experiment.

   d. Describe in words the key features of the graph of the function f for $n \geq 0$. Be sure to describe where the function is increasing or decreasing, where it has maximums and minimums (if they exist), and the end behavior.

e. According to the model, on which roll does Gwyneth expect, on average, to find herself with only one die remaining? Write and solve an equation to support your answer to this question.

f. For all of the values in the domain of f, is there any value for which f will predict an average number of 0 dice remaining? Explain why or why not. Be sure to use the domain of the function and the graph to support your reasoning.

Suppose the table below represents the results of one trial of Gwyneth's experiment.

Roll	Number of Dice Left	Roll	Number of Dice Left	Roll	Number of Dice Left
0	1000	10	157	20	26
1	840	11	139	21	22
2	692	12	115	22	15
3	581	13	90	23	13
4	475	14	78	24	10
5	400	15	63	25	6
6	341	16	55	26	2
7	282	17	43	27	1
8	232	18	40	28	0
9	190	19	33		

g. Let g be the function that is defined exactly by the data in the table, i.e., $g(0) = 1000$, $g(1) = 840$, $g(2) = 692$, and so forth, up to $g(28) = 0$. Describe in words how the graph of g looks different from the graph of f. Be sure to use the domain of g and the domain of f to justify your description.

h. Gwyneth runs her experiment hundreds of times, and each time she generates a table like the one above. How are these tables similar to the function f? How are they different?

5. Find the inverse g for each function f.

a. $f(x) = \frac{1}{2}x - 3$

b. $f(x) = \frac{x+3}{x-2}$

c. $f(x) = 2^{3x} + 1$

d. $f(x) = e^{x-3}$

e. $f(x) = \log(2x + 3)$

6. Dani has $\$1{,}000$ in an investment account that earns 3% per year, compounded monthly.

   a. Write a recursive sequence for the amount of money in her account after n months.

   b. Write an explicit formula for the amount of money in the account after n months.

   c. Write an explicit formula for the amount of money in her account after t years.

   d. Boris also has $\$1{,}000$, but in an account that earns 3% per year, compounded yearly. Write an explicit formula for the amount of money in his account after t years.

   e. Boris claims that the equivalent monthly interest rate for his account would be the same as Dani's. Use the expression you wrote in part (d) and the properties of exponents to show why Boris is incorrect.

7. Show that

$$\sum_{k=0}^{n} a \cdot r^k = a\left(\frac{1-r^n}{1-r}\right)$$

where $r \neq 1$.

8. Sami opens an account and deposits $\$100$ into it at the end of each month. The account earns 2% per year compounded monthly. Let $S(n)$ denote the amount of money in her account at the end of n months (just after she makes a deposit). For example, $S(1) = 100$ and $S(2) = 100\left(1 + \frac{0.02}{12}\right) + 100$.

   a. Write a geometric series for the amount of money in the account after 3, 4, and 5 months.

   b. Find a recursive description for $S(n)$.

c. Find an explicit function for $S(n)$, and use it to find $S(12)$.

d. When will Sami have at least $\$5,000$ in her account? Show work to support your answer.

9. Beatrice decides to deposit $\$100$ per month at the end of every month in a bank with an annual interest rate of 5.5% compounded monthly.

a. Write a geometric series to show how much she will accumulate in her account after one year.

b. Use the formula for the sum of a geometric series to calculate how much she will have in the bank after five years if she keeps on investing $\$100$ per month.

10. Nina has just taken out a car loan for $\$12{,}000$. She will pay an annual interest rate of 3% through a series of monthly payments for 60 months, which she pays at the end of each month. The amount of money she has left to pay on the loan at the end of the n^{th} month can be modeled by the function $f(n) = 86248 - 74248(1.0025)^n$ for $0 \le n \le 60$.

At the same time as her first payment (at the end of the first month), Nina placed $\$100$ into a separate investment account that earns 6% per year compounded monthly. She placed $\$100$ into the account at the end of each month thereafter. The amount of money in her savings account at the end of the n^{th} month can be modeled by the function $g(n) = 20000(1.005)^n - 20000$ for $n \ge 0$.

a. Use the functions f and g to write an equation whose solution could be used to determine when Nina will have saved enough money to pay off the remaining balance on her car loan.

b. Use a calculator or computer to graph f and g on the same coordinate plane. Sketch the graphs below, labeling intercepts and indicating end behavior on the sketch. Include the coordinates of any intersection points.

c. How would you interpret the end behavior of each function in the context of this situation?

d. What does the intersection point mean in the context of this situation? Explain how you know.

e. After how many months will Nina have enough money saved to pay off her car loan? Explain how you know.

11. Each function below models the growth of three different trees of different ages over a fixed time interval.

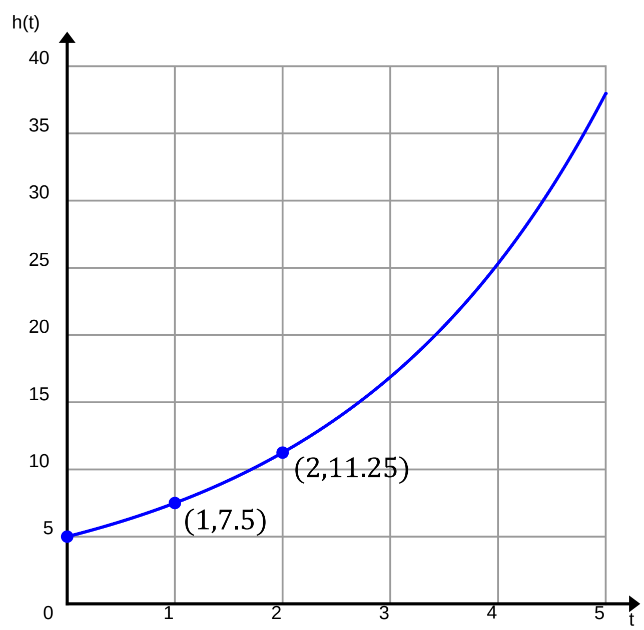
Tree A:

$f(t) = 15(1.69)^{\frac{t}{2}}$, where t is time in years since the tree was 15 feet tall, $f(t)$ is the height of the tree in feet, and $0 \le t \le 4$.

Tree B:

Years since the tree was 5 feet tall, t	Height in feet after t years, $g(t)$
0	5
1	6.3
2	7.6
3	8.9
4	10.2

Tree C: The graph of h is shown where t is years since the tree was 5 feet tall, and $h(t)$ is the height in feet after t years.



a. Classify each function f and g as representing a linear or nonlinear function. Justify your answers.

b. Use the properties of exponents to show that Tree A has a percent rate of change of 30% per year.

c. Which tree, A or C, has the greatest percent rate of change? Justify your answer.

d. Which function has the greatest average rate of change over the interval $[0,4]$, and what does that mean in terms of tree heights?

e. Write formulas for functions g and h, and use them to confirm your answer to part (c).

f. For the exponential models, if the average rate of change of one function over the interval $[0,4]$ is greater than the average rate of change of another function on the same interval, is the percent rate of change also greater? Why or why not?

12. Identify which functions are exponential. For the exponential functions, use the properties of exponents to identify the percent rate of change, and classify them as representing exponential growth or decay.

a. $f(x) = 3(1 - 0.4)^{-x}$

b. $g(x) = \frac{3}{4^x}$

c. $k(x) = 3x^{0.4}$

d. $h(x) = 3^{\frac{x}{4}}$

13. A patient in a hospital needs to maintain a certain amount of a medication in her bloodstream to fight an infection. Suppose the initial dosage is 10 mg, and the patient is given an additional maintenance dosage of 4 mg every hour. Assume that the amount of medication in the bloodstream is reduced by 25% every hour.

    a. Write a function for the amount of the initial dosage that is in the bloodstream after n hours.

    b. Complete the table below to track the amount of medication from the maintenance dosage in the patient's bloodstream for the first five hours.

Hours since initial dose, n	Amount of the medication in the bloodstream from the maintenance dosage at the beginning of each hour
0	0
1	4
2	$4(1+0.75)$
3	
4	
5	

    c. Write a function that models the total amount of medication in the bloodstream after n hours.

d. Use a calculator to graph the function you wrote in part (c). According to the graph, will there ever be more than 16 mg of the medication present in the patient's bloodstream after each dose is administered?

e. Rewrite this function as the difference of two functions (one a constant function and the other an exponential function), and use that difference to justify why the amount of medication in the patient's bloodstream will not exceed 16 mg after each dose is administered.

A Progression Toward Mastery					
Assessment Task Item		**STEP 1** **Missing or incorrect answer and little evidence of reasoning or application of mathematics to solve the problem.**	**STEP 2** **Missing or incorrect answer but evidence of some reasoning or application of mathematics to solve the problem.**	**STEP 3** **A correct answer with some evidence of reasoning or application of mathematics to solve the problem, <u>or</u> an incorrect answer with substantial evidence of solid reasoning or application of mathematics to solve the problem.**	**STEP 4** **A correct answer supported by substantial evidence of solid reasoning or application of mathematics to solve the problem.**
1	**a–c** A-SSE.B.3c F-IF.C.7e F-BF.B.3	Student provides graphs that contain major errors. Labeled intercepts and end behavior are inconsistent with the functions.	Student provides a graph for f that would be correct for g and vice versa. Intercepts and end behavior are correctly shown in at least one graph.	Student provides correct graphs that contain one minor error. Descriptions of transformations taking the graph of f to the graph of g are wrong or incomplete.	Student provides correct graphs that contain one minor error. Descriptions of transformations taking the graph of f to the graph of g are correct.
2	**a** F-BF.B.3	Student draws one or more of the graphs of f, g, and h incorrectly. (While not needed for a correct answer, drawing a graph shows evidence of reasoning.)	Student draws the graphs correctly but does not describe, or gives an incomplete description of, the graphs of g and h as transformations of the graph of f.	Student describes the graphs of g and h as transformations of f, but makes one minor error.	Student describes the graphs of g and h as transformations of f.
	b F-BF.B.3	Student provides little or no evidence of understanding why g and h are equivalent functions.	Student claims that the functions are equivalent because the graphs they drew in part (a) are the same.	Student does the operations to show how the expressions are equal but fails to connect it to the graphs of the functions.	Student performs a sequence of valid operations to show how the expressions are equal, and the student uses the calculation to show that g and h are equivalent functions.

3	**a–c** **F-BF.B.3**	Student incorrectly identifies the blue curve as the graph of g and fails to recognize that the functions are vertical scalings of each other.	Student correctly identifies the graphs (perhaps by noticing $g(2) = 1$) but fails to recognize that the functions are vertical scalings of each other.	Student correctly identifies the graphs and expresses that the graphs are vertical scalings of each other, but fails to find the correct scale factor.	Student correctly identifies the graphs, expresses that the graphs are vertical scalings of each other, and finds the correct scale factor.
4	**a–c** **F-LE.A.2** **F-LE.B.5**	Student does not provide a correct exponential function or function is missing. Student explanations of the meaning of $f(12) = 112.5$ and the meaning of the parameters are missing or contain multiple errors.	Student provides a correct exponential function, but at least one parameter is incorrect given the situation. Explanations contain more than one or two minor errors or inconsistencies.	Student provides a correct exponential function, but explanations of the meaning of $f(12) = 112.5$ or one of the two parameters contain minor errors or are incomplete. <u>OR</u> Student explanations of the point $f(12) = 112.5$ and the meaning of the parameters are correct and consistent with the situation, but one parameter in the function is incorrect.	Student correctly writes an exponential function and provides correct explanations of the meaning of $f(12) = 112.5$ and the meaning of the initial amount (a) and growth rate (b) in a function of the form $f(n) = a \cdot b^n$.
	d **F-IF.B.4**	Student description is incomplete and contains major errors in vocabulary or notation.	Student description is missing two or three essential features and may contain minor errors in vocabulary or notation.	Student description is fairly accurate and complete with no more than one missing component from those listed in Step 4 of the rubric.	Student description is accurate and complete based on the graph of the model written in part (a). Solutions include information about increasing/decreasing intervals, extrema, intercepts, and end behavior. The appropriate domain is also considered in the solution.
	e **F-LE.A.4** **F-BF.B.4a**	Student provides little or no correct work. Equation is not of the form $f(n) = 1$.	Student provides an equation of the form $f(n) = 1$, but the solutions are incomplete or contains major errors. Little or no attempt is made to interpret the irrational solution in terms of a rolled number.	Student provides an equation of the form $f(n) = 1$. Solutions contain no more than one minor error, but student correctly interprets the irrational solution in terms of a rolled number. <u>OR</u> Student writes and solves equation	Student provides an equation of the form $f(n) = 1$. Solutions are correct and solved using logarithms. Student correctly interprets the irrational solution in terms of a rolled number in the context of the situation.

				correctly but does not interpret the solution in the context of the situation.	
	f–g **F-IF.B.5**	Student solutions are incomplete and contain multiple errors relating to interpretation of the domain. Little or no attempt is made to use the graph to support reasoning.	Student struggles to connect the function model to the real world situation and the solutions have one or more major misconceptions or errors, such as not identifying the correct domain of the table. Student does not use the graph to support reasoning.	Student solutions recognize the limitations of the function model to represent this situation perfectly. Explanations are fairly complete and accurate or contain no more than one or two minor errors. Student uses the graph to support reasoning.	Student recognizes that the function will not perfectly model this situation due to differences in the end behavior of the function and the nature of simulation. Student recognizes the model's limitations and compares and contrasts tables and graphs to justify solution.
	h **F-IF.C.9**	Student fails to provide a valid similarity or difference. Little or no explanation to support reasoning is given.	Student provides one valid similarity or one valid difference. Explanation to support reasoning is limited.	Student provides at least one valid similarity and one valid difference and supports the answer with tables, graphs or both. <u>OR</u> Student provides at least one valid similarity and two valid differences with limited explanation to support reasoning.	Student provides at least one valid similarity and two valid differences and supports the answer with tables, graph, or both.
5	**a–e** **F-BF.B.4a**	Student finds the correct inverse for one or zero problems. Work shows a limited understanding of the process of finding the inverse of a function.	Student finds the correct inverse for two parts. Solutions to some parts may contain major mathematical errors or significant problems with using proper notation.	Student finds the correct inverse for at least three of the five parts. Solutions contain no more than one minor error. Work demonstrates that student mostly understands the process and is able to use proper notation.	Student finds the correct inverse for each function, showing sufficient work and using proper notation.
6	**a–e** **F-BF.A.2** **A-SSE.B.3c** **A-CED.A.1** **F-BF.A.1a**	Student solutions are incomplete and inaccurate with major mathematical errors throughout.	Student correctly writes one of the three requested formulas. Solution contains multiple minor errors or one or two major errors.	Student solutions to each part are mostly correct with no more than one minor mathematical error and/or no more than one minor difficulty with the use of proper notation.	Student solutions include correct recursive and explicit formulas and an accurate justification in part (e) using the formulas from parts (c) and (d). Student uses proper notation throughout the solution.

7	**A-SSE.B.4**	Student makes little or no attempt to derive the formula.	Student derivation starts with appropriate identity but is limited and may contain major mathematical errors.	Student derivation is fairly accurate and may contain a minor error in the use of notation or completeness.	Student derivation is accurate, complete, and shows the use of proper notation. Essential steps are shown in the solution.
8	**a–b** **F-IF.A.3** **F-BF.A.2** **F-BF.A.1a**	Student fails to calculate the amount for more than the second month correctly. Work shown is incomplete with major errors or is missing all together.	Student correctly calculates the recursive description or the amounts after two, three, and four months, but not both. There are more than a few mathematical or notation errors.	Student calculations are mostly correct but may contain a minor mathematical or notation error.	Student correctly calculates the amount after two, three, and four months and uses this information to write a correct recursive description for the function. Solution shows the use of proper notation.
	c–d **A-SSE.B.4** **F-LE.A.4** **F-BF.B.4a** **F-BF.A.1a**	Student writes an explicit formula that contains major mathematical errors and is not based on the formula for a geometric series. There is little or no attempt to answer part (d).	Student writes an explicit formula that contains a major mathematical error and the solution to part (d) shows little evidence that student is attempting to solve $S(n) = 5{,}000$.	Student writes a correct explicit formula but struggles to determine successfully when the account will contain $\$5{,}000$ with proper justification and work shown. OR Student writes an explicit formula with a minor error and uses it and the function to determine when the account will contain $\$5{,}000$, giving sufficient justification for the solution.	Student writes a correct explicit formula for the function and successfully uses it to determine when the account will contain $\$5{,}000$. Solution to part (d) is justified by solving the equation algebraically or graphically.
9	**a–b** **A-SSE.B.4** **A-CED.A.1**	Student solutions to both parts are incomplete and contain two or more major mathematical errors.	Student solutions to both parts contain minor mathematical errors. OR Student solution to one part is mostly correct but the other solution contains major errors.	Student solutions to parts (a) and (b) contain no more than one minor error. Solutions may show limited work, but that work shown is correct.	Student writes a correct series in part (a) and uses the formula correctly in part (b) to find the amount after five years. Work is shown to clearly indicate understanding of the application of a geometric series to structured savings accounts.

10	**a–e** **A-REI.D.11** **F-IF.B.4** **F-IF.C.7e** **A-CED.A.1**	Student solutions contain many errors and omissions. Equation in part (a) is missing or incorrect and graphs lack sufficient detail or do not reflect the actual graphs of the equations. Little or no attempt is made to relate the functions to the situation.	Student equates f and g and finds the intersection point of the graphs of f and g. Student struggles to sketch and label key features of the graph, providing incomplete responses and incorrect or missing interpretations of the features of the graphs in terms of the situation.	Student solutions to all five parts are mostly correct and contain no more than one or two minor errors.	Student equates f and g to write the equation, solve it graphically, and correctly interpret the meaning of the x-coordinate of the intersection point of the graphs of f and g. Student accurately sketches, labels, and describes at least three key features of each graph. Student correctly interprets features of the graph in terms of the situation.
11	**a–c** **F-IF.C.9** **F-IF.C.8b**	Student correctly classifies one or no functions. Little or no correct work is shown.	Student correctly classifies two out of three functions as linear or exponential. Percent rate of change identification is missing or incorrect.	Student correctly classifies each function as linear or exponential and calculates the percent rate of change with no more than one minor error. Evidence of using rate of change to help classify functions may be limited.	Student correctly classifies each function as linear or exponential showing evidence of using rate of change to do so. Student correctly identifies a growth factor and interprets it using a percent rate of change for the two exponential functions.
	d **F-IF.B.6**	Student shows little or no correct work. There is little or no attempt to interpret rate of change in terms of the situation.	Student makes a major mathematical error or is only able to calculate the rate of change correctly for one of the three functions. Interpretation of rate of change is limited or incorrect.	Student computes and interprets each rate of change with no more than one minor mathematical error.	Student computes and interprets each rate of change appropriately and uses correct notation and vocabulary in the solution.
	e **F-LE.A.2** **F-IF.B.6**	Student solutions are incorrect or missing.	Student writes either Tree C or Tree B function correctly but has major mathematical errors on the other function. OR Student writes both functions partially correct (e.g., correct slope but not the correct intercept for the linear function).	Student writes a linear function for Tree B and an exponential function for Tree C with no more than one minor mathematical error. Student's solution shows little or no direct evidence of using rate of change to identify the slope.	Student writes a correct linear function for Tree B and a correct exponential function for Tree C by analyzing the information in the respective table and graph. Student uses rate of change to identify the slope of Tree B. The work shown uses proper notation.

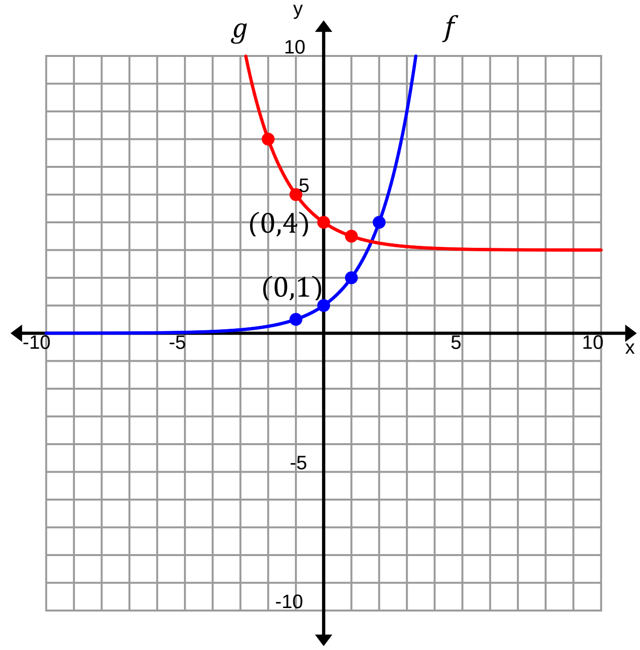
	f **F-IF.C.9**	Student solution is not correct and offers little or no explanation. OR Student solution and explanation are not supported by the given information or are off-task.	Student solution is correct and offers little or no explanation.	Student solution is mostly correct but explanation is limited or contains minor errors.	Student solution is correct and supported using average rate of change based on the functions provided in the problem.
12	**a–d** **F-IF.C.8b**	Student solution is missing or incorrect.	Student identifies two exponential functions correctly including percent rate of change and whether it is growth or decay. Explanation may be missing. OR Student correctly identifies three exponential functions, but the rest of the solution is limited or incorrect.	Student correctly classifies the functions and identifies the percent growth rate in (a), (b), and (d) but explanation is limited or solutions contain minor errors.	Student classifies (a), (b), and (d) as exponential and recognizes that (c) is not exponential. Student rewrites the expression to classify exponential functions as growth or decay and identifies the percent growth rate. Explanations are clear and accurate.
13	**a–b** **A-SSE.B.4**	Student solution is incomplete, inaccurate, or missing.	Student table and/or explicit formula are partially correct with at least one significant error or several minor errors.	Student table and explicit formula are mostly correct, but solution may contain minor mathematical errors.	Student table and explicit formula are correct and clearly show how the maintenance dosage is related to a geometric series.
	c–e **A-SSE.B.4** **F-BF.A.1b**	Student work is incorrect or incomplete with little or no attempt made to accurately graph or correctly rewrite the function.	Student struggles to produce a function that is correct and/or struggles to rewrite the function, which includes one or more significant errors and does not correctly relate the end behavior to the context or to the graph.	Student writes the total amount of the medication as the sum of two functions. Student rewrites the function as the sum of an exponential function and constant function, but student explanation connecting the graph and the situation may be incomplete or contain minor errors.	Student writes the total amount of the medication as the sum of two functions. Student rewrites the function as a sum of an exponential function and a constant function and connects that representation to the graph and to the meaning of the end behavior in the context of the situation.

Name ______________________________ Date ________________

1. For parts (a) to (c),
   - Sketch the graph of each pair of functions on the same coordinate axes showing end behavior and intercepts, and
   - Describe the graph of g as a series of transformations of the graph of f.

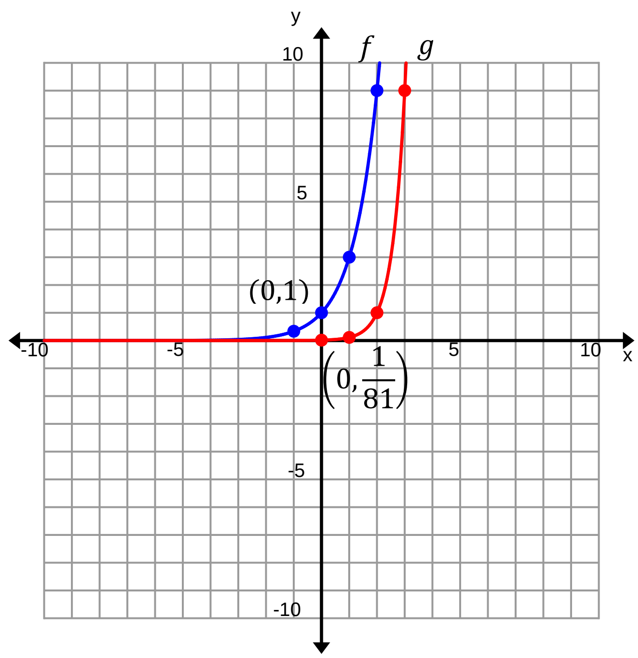
a. $f(x) = 2^x$, and $g(x) = 2^{-x} + 3$

The graph of g is the graph of f reflected across the y-axis and translated vertically 3 units.

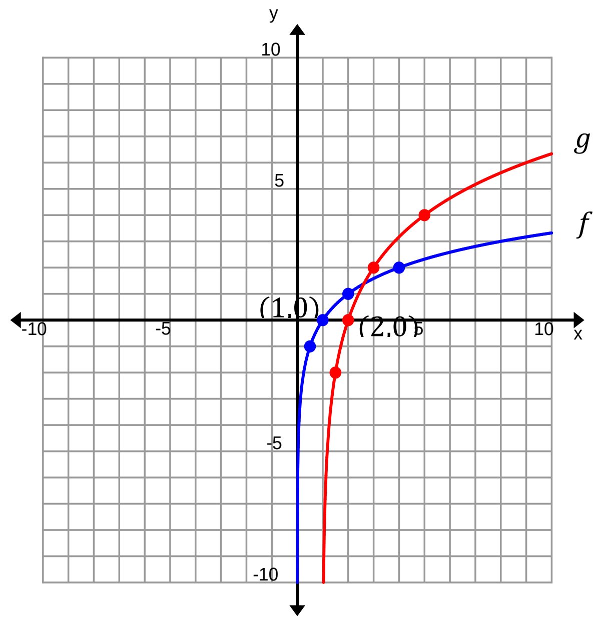


b. $f(x) = 3^x$, and $g(x) = 9^{x-2}$

Answers will vary. For example, because $9^{x-2} = 3^{2(x-2)}$, the graph of g is the graph of f scaled horizontally by a factor of $\frac{1}{2}$ and translated horizontally by 2 units to the right.



c. $f(x) = \log_2(x)$, and $g(x) = \log_2(x-1)^2$



Most likely answer: Because $\log_2(x-1)^2 = 2\log_2(x-1)$, the graph of g is the graph of f scaled vertically by a factor of 2 and translated horizontally by 1 unit to the right.

2. Consider the graph of $f(x) = 8^x$. Let $g(x) = f\left(\frac{1}{3}x + \frac{2}{3}\right)$ and $h(x) = 4f\left(\frac{x}{3}\right)$.

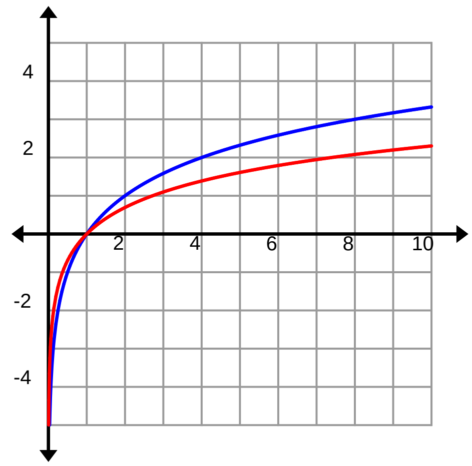
a. Describe the graphs of g and h as transformations of the graph of f.

The graph of g is the graph of f with a horizontal scaling by a factor of 3 and a horizontal translation 2 units to the left. The graph of h is the graph of f scaled vertically by a factor of 4 and horizontally by a factor of 3.

b. Use the properties of exponents to show why the graphs of the functions $g(x) = f\left(\frac{1}{3}x + \frac{2}{3}\right)$ and $h(x) = 4f\left(\frac{x}{3}\right)$ are the same.

Since $g(x) = 8^{\frac{x}{3}+\frac{2}{3}}$, $h(x) = 4 \cdot 8^{\frac{x}{3}}$, $8^{\frac{x}{3}+\frac{2}{3}} = 8^{\frac{x}{3}} \cdot 8^{\frac{2}{3}} = 4 \cdot 8^{\frac{x}{3}}$ for any real number x, and their domains are the same, the functions g and h are equivalent. Therefore, they have identical graphs.

3. The graphs of the functions $f(x) = \ln(x)$ and $g(x) = \log_2(x)$ are shown to the right.



a. Which curve is the graph of f, and which curve is the graph of g? Explain.

The blue curve is the graph of g, and the red curve is the graph of f. The two functions can be compared by converting both of them to the same base. Since $f(x) = \ln(x)$, and $g(x) = \frac{1}{\ln(2)} \cdot \ln(x)$, and $1 < \frac{1}{\ln(2)}$, the graph of g is a vertical stretch of the graph of f; thus, the graph of g is the blue curve.

b. Describe the graph of g as a transformation of the graph of f.

The graph of g is a vertical scaling of the graph of f by a scale factor greater than one.

c. By what factor has the graph of f been scaled vertically to produce the graph of g? Explain how you know.

By the change of base formula, $g(x) = \frac{1}{\ln(2)} \cdot \ln(x)$; thus, the graph of g is a vertical scaling of the graph of f by a scale factor of $\frac{1}{\ln(2)}$.

4. Gwyneth is conducting an experiment. She rolls $1{,}000$ dice simultaneously and removes any that have a six showing. She then rerolls all of the dice that remain and again removes any that have a six showing. Gwyneth does this over and over again—rerolling the remaining dice and then removing those that land with a six showing.

   a. Write an exponential function f of the form $f(n) = a \cdot b^{cn}$ for any real number $n \geq 0$ that could be used to model the average number of dice remaining after the n^{th} roll if she ran her experiment a large number of times.

   $$f(n) = 1000\left(\frac{5}{6}\right)^n$$

   b. Gwyneth computed $f(12) = 112.15 \ldots$ using the function f. How should she interpret the number $112.15 \ldots$ in the context of the experiment?

   The value $f(12) = 112.15$ means that if she was to run her experiment over and over again, the model predicts that the average number of dice left after the 12^{th} roll would be approximately 112.15.

   c. Explain the meaning of the parameters in your function f in terms of this experiment.

   The $1{,}000$ represents the initial amount of dice. The $\frac{5}{6}$ represents the fraction of dice remaining from the previous roll each time she rolls the dice. It represents the fact that there is a 1 in 6 chance of any die landing on a 6, so we would predict that after rolling the dice, about $\frac{5}{6}$ of them would not be removed.

   d. Describe in words the key features of the graph of the function f for $n \geq 0$. Be sure to describe where the function is increasing or decreasing, where it has maximum and minimum (if they exist), and the end behavior.

   This function is decreasing for all $n \geq 0$. The maximum is the starting number of dice, 1000. There is no minimum. The graph decreases at a decreasing rate with $f(x) \to 0$ as $x \to \infty$.

e. According to the model, on which roll does Gwyneth expect, on average, to find herself with only one die remaining? Write and solve an equation to support your answer to this question.

$$1000\left(\frac{5}{6}\right)^n = 1$$

$$\left(\frac{5}{6}\right)^n = \frac{1}{1000}$$

$$\log\left(\frac{5}{6}\right)^n = \log\left(\frac{1}{1000}\right)$$

$$n\log\left(\frac{5}{6}\right) = -3$$

$$n = -\frac{3}{\log\left(\frac{5}{6}\right)}$$

$$n \approx 37.88$$

According to the model, Gwyneth should have 1 die remaining, on average, on the 38th roll.

f. For all of the values in the domain of f, is there any value for which f will predict an average number of 0 dice remaining? Explain why or why not. Be sure to use the domain of the function and the graph to support your reasoning.

The graph of this function shows the end behavior approaching 0 as the number of rolls increases. This function will never predict an average number of 0 dice remaining because a function of the form $f(n) = ab^{cn}$ never takes on the value of zero. Even as the number of trials becomes very large, the value of f will be a positive number.

Suppose the table below represents the results of one trial of Gwyneth's experiment.

Roll	Number of Dice Left	Roll	Number of Dice Left	Roll	Number of Dice Left
0	1000	10	157	20	26
1	840	11	139	21	22
2	692	12	115	22	15
3	581	13	90	23	13
4	475	14	78	24	10
5	400	15	63	25	6
6	341	16	55	26	2
7	282	17	43	27	1
8	232	18	40	28	0
9	190	19	33		

g. Let g be the function that is defined exactly by the data in the table, i.e., $g(0) = 1000$, $g(1) = 840$, $g(2) = 692$, and so forth up to $g(28) = 0$. Describe in words how the graph of g looks different from the graph of f. Be sure to use the domain of g and the domain of f to justify your description.

The domain of g is the set of integers from 0 to 28. The graph of g is discrete, while the graph of f is continuous (a curve). The graph of g has an x-intercept at $(28, 0)$, and the graph of f has no x-intercept. The graphs of both functions decrease, but the graph of f is exponential, which means that the ratio $\frac{f(x+1)}{f(x)} = \frac{5}{6}$ for all x in the domain of f. The function values for g do not have this property.

h. Gwyneth runs her experiment hundreds of times, and each time she generates a table like the one above. How are these tables similar to the function f? How are they different?

Similar: Each table will have the same basic shape as f (i.e., decreasing from 1,000 eventually to 0 with a common ratio between rolls of about $\frac{5}{6}$). Different: (1) The tables are always discrete domains with integer range. (2) Each table will eventually reach 0 after some finite number of rolls, whereas the function f can never take on the value of 0.

5. Find the inverse g for each function f.

a. $f(x) = \frac{1}{2}x - 3$

$$y = \frac{1}{2}x - 3$$
$$x = \frac{1}{2}y - 3$$
$$x + 3 = \frac{1}{2}y$$
$$y = 2x + 6$$
$$g(x) = 2x + 6$$

b. $f(x) = \frac{x+3}{x-2}$

$y = \frac{x+3}{x-2}$

$x = \frac{y+3}{y-2}$

$xy - 2x = y + 3$

$y(x-1) = 2x + 3$

$y = \frac{2x+3}{x-1}$

$g(x) = \frac{2x+3}{x-1}$

c. $f(x) = 2^{3x} + 1$

$y = 2^{3x} + 1$

$x = 2^{3y} + 1$

$x - 1 = 2^{3y}$

$\log_2(x-1) = 3y$

$y = \frac{1}{3}\log_2(x-1)$

$g(x) = \frac{1}{3}\log_2(x-1)$

d. $f(x) = e^{x-3}$

$y = e^{x-3}$

$x = e^{y-3}$

$\ln(x) = y - 3$

$y = \ln(x) + 3$

$g(x) = \ln(x) + 3$

e. $f(x) = \log(2x+3)$

$y = \log(2x+3)$

$x = \log(2y+3)$

$2y + 3 = 10^x$

$y = \frac{1}{2} \cdot 10^x - \frac{3}{2}$

$g(x) = \frac{1}{2} \cdot 10^x - \frac{3}{2}$

6. Dani has $1,000 in an investment account that earns 3% per year, compounded monthly.

a. Write a recursive sequence for the amount of money in her account after n months.

$$a_1 = 1000,\ a_{n+1} = a_n(1.0025)$$

b. Write an explicit formula for the amount of money in the account after n months.

$$a_n = 1000(1.0025)^n$$

c. Write an explicit formula for the amount of money in her account after t years.

$$b_n = 1000(1.0025)^{12t}$$

d. Boris also has $1000, but in an account that earns 3% per year, compounded yearly. Write an explicit formula for the amount of money in his account after t years.

$$V(t) = 1000(1.03)^t$$

e. Boris claims that the equivalent monthly interest rate for his account would be the same as Dani's. Use the expression you wrote in part (d) and the properties of exponents to show why Boris is incorrect.

Boris is incorrect because the formula for the amount of money in the account, based on a monthly rate, is given by $V(t) = 1000(1+i)^{12t}$, where i is the monthly interest rate. The expression from part d, $1000(1.03)^t$, is equivalent to

$1000(1.03)^{\frac{12}{12}t} = 1000(1.03)^{\frac{1}{12}\cdot 12t} = 1000\left((1.03)^{\frac{1}{12}}\right)^{12t}$ by properties of exponents.

Therefore, if the expressions must be equivalent, the quantity given by $1.03^{\frac{1}{12}} \approx 1.00247$ means that his monthly rate is about 0.247%, which is less than 0.25%, given by Dani's formula.

7. Show that

$$\sum_{k=0}^{n} a \cdot r^k = a\left(\frac{1-r^n}{1-r}\right)$$

where $r \neq 1$.

We know that the statement, $1 - r^n = (1 - r)(1 + r + r^2 + r^3 + \ldots + r^{n-1})$ for any real number r such that $r \neq 1$ and positive integers n, is an identity. Dividing both sides of the identity by $1 - r$ gives $\frac{1-r^n}{1-r} = 1 + r^2 + r^3 + \ldots + r^{n-1}$.

Therefore, by factoring the common factor a and substituting, we get the formula,

$$a + ar + ar^2 + \ldots + ar^{n-1} = a(1 + r + r^2 + \ldots + r^{n-1})$$
$$= a\left(\frac{1-r^n}{1-r}\right).$$

8. Sami opens an account and deposits $\$100$ into it at the end of each month. The account earns 2% per year compounded monthly. Let $S(n)$ denote the amount of money in her account at the end of n months (just after she makes a deposit). For example, $S(1) = 100$, and $S(2) = 100\left(1 + \frac{0.02}{12}\right) + 100$.

a. Write a series for the amount of money in the account after 3, 4, and 5 months.

Month	Amount
3	$100\left(1+\frac{0.02}{12}\right)^2 + 100\left(1+\frac{0.02}{12}\right) + 100$
4	$100\left(1+\frac{0.02}{12}\right)^3 + 100\left(1+\frac{0.02}{12}\right)^2 + 100\left(1+\frac{0.02}{12}\right) + 100$
5	$100\left(1+\frac{0.02}{12}\right)^4 + 100\left(1+\frac{0.02}{12}\right)^3 + 100\left(1+\frac{0.02}{12}\right)^2 + 100\left(1+\frac{0.02}{12}\right) + 100$

b. Find a recursive description for $S(n)$.

$S_1 = 100,\ S_{n+1} = S_n\left(1 + \frac{0.02}{12}\right) + 100$

c. Find an explicit formula for $S(n)$, and use it to find $S(12)$.

$$S(n) = \frac{100\left(1-\left(1+\frac{0.02}{12}\right)^n\right)}{1-\left(1+\frac{0.02}{12}\right)}$$

$$S(12) \approx 1211.06$$

d. When will Sami have at least $\$5,000$ in her account? Show work to support your answer.

An algebraic solution is shown, but students could also solve this equation numerically or algebraically.

$$5000 = \frac{100\left(1-\left(1+\frac{0.02}{12}\right)^n\right)}{1-\left(1+\frac{0.02}{12}\right)}$$

$$50\left(-\frac{0.02}{12}\right) = 1-\left(1+\frac{0.02}{12}\right)^n$$

$$\frac{13}{12} = \left(1+\frac{0.02}{12}\right)^n$$

$$\log\left(\frac{13}{12}\right) = \log\left(1+\frac{0.02}{12}\right)^n$$

$$n = \frac{\log\left(\frac{13}{12}\right)}{\log\left(1+\frac{0.02}{12}\right)}$$

9. Beatrice decides to deposit $\$100$ per month at the end of every month in a bank with an annual interest rate of 5.5% compounded monthly.

a. Write a geometric series to show how much she will accumulate in her account after one year.

$$100\left(1+\frac{0.055}{12}\right)^{12}+100\left(1+\frac{0.055}{12}\right)^{11}+\ldots+100\left(1+\frac{0.055}{12}\right)^{1}+100$$

b. Use the formula for the sum of a geometric series to calculate how much she will have in the bank after five years if she keeps on investing $100 per month.

$$100\left(\frac{1-\left(1+\frac{0.055}{12}\right)^{60}}{1-\left(1+\frac{0.055}{12}\right)}\right) \approx 6888.08$$

She will have approximately $6,888 in her account.

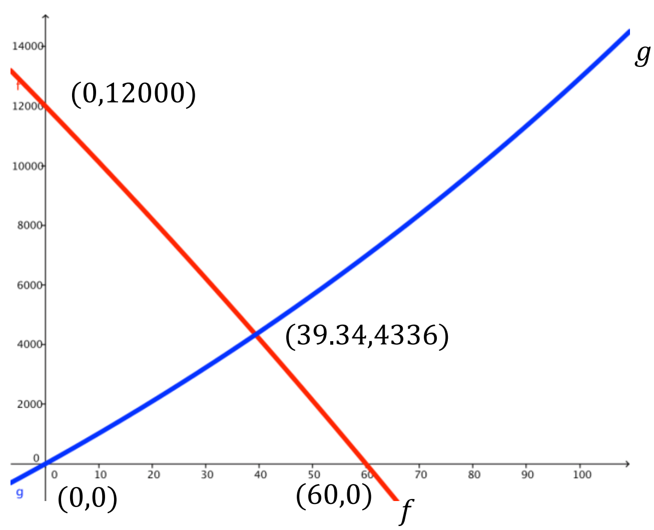
10. Nina has just taken out a car loan for $12,000. She will pay an annual interest rate of 3% through a series of monthly payments for 60 months, which she pays at the end of each month. The amount of money she has left to pay on the loan at the end of the n^{th} month can be modeled by the function $f(n) = 86248 - 74248(1.0025)^n$ for $0 \le n \le 60$.

At the same time as her first payment (at the end of the first month), Nina placed $100 into a separate investment account that earns 6% per year compounded monthly. She placed $100 into the account at the end of each month thereafter. The amount of money in her savings account at the end of the n^{th} month can be modeled by the function $g(n) = 20000(1.005)^n - 20000$ for $n \ge 0$.

a. Use the functions f and g to write an equation whose solution could be used to determine when Nina will have saved enough money to pay off the remaining balance on her car loan.

$$86248 - 74248(1.0025)^n = 20000(1.005)^n - 20000$$

b. Use a calculator or computer to graph f and g on the same coordinate plane. Sketch the graphs below labeling intercepts and indicating end behavior on the sketch. Include the coordinates of any intersection points.



c. How would you interpret the end behavior of each function in the context of this situation?

For g, the end behavior indicates that the amount in the account will continue to grow over time at an exponential rate. For f, the end behavior as $x \to \infty$ does not have any real meaning in this situation since the loan will be paid off after 60 months.

d. What does the intersection point mean in the context of this situation? Explain how you know.

The intersection point describes the time and amount when approximately the amount left on Nina's loan is equal to the amount she has saved. It is only approximate because the actual amounts are compounded/paid/deposited at the end of the month.

e. After how many months will Nina have enough money saved to pay off her car loan? Explain how you know.

After 39.34 months, or at the end of the 40^{th} month, she will have enough money saved to pay off the loan completely. The amount she has at the end of the 40^{th} month in her savings account will be slightly more than \$4,336, and the amount she has left on her loan will be slightly less than \$4,336.

11. Each function below models the growth of three different trees of different ages over a fixed time interval.

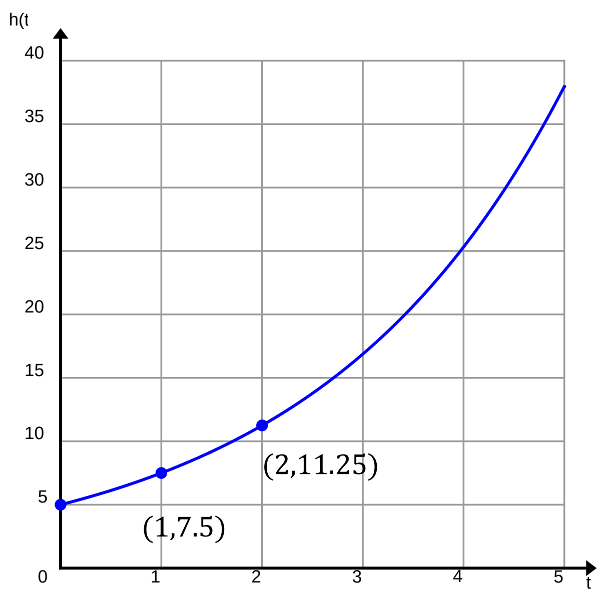
Tree A:

$f(t) = 15(1.69)^{\frac{t}{2}}$, where t is time in years since the tree was 15 feet tall, $f(t)$ is the height of the tree in feet, and $0 \leq t \leq 4$.

Tree B:

Years since the tree was 5 feet tall, t	Height in feet after t years, $g(t)$
0	5
1	6.3
2	7.6
3	8.9
4	10.2

Tree C: The graph of h is shown where t is years since the tree was 5 feet tall, and $h(t)$ is the height in feet after t years.



a. Classify each function f and g as representing a linear or non-linear function. Justify your answers.

The function f is exponential because it is of the form $f(t) = ab^{ct}$, the function g could represent a linear function because its first difference is constant (or it has a constant average rate of change), and the function h is potentially an exponential function because it has a common ratio of 1.5.

b. Use the properties of exponents to show that Tree A has a percent rate of change of 30% per year.

The percent rate of change for f is 30% because $(1.69)^{\frac{t}{2}} = \left((1.69)^{\frac{1}{2}}\right)^{t} = (1.3)^{t} = (1+0.3)^{t}$.

c. Which tree, A or C, has the greatest percent rate of change? Justify your answer?

The percent rate of change for f is 30% by part (b).

The percent rate of change for h is 50% because the common ratio is $\frac{7.5}{5} = \frac{11.25}{7.5} = 1.5$. Tree C has the greatest percent rate of change.

d. Which function has the greatest average rate of change over the interval $[0,4]$, and what does that mean in terms of tree heights?

The average rate of change of f on [0,4] is $\frac{15(1.69)^{\frac{4}{2}} - 15}{4} \approx 7$. The average rate of change of g on the interval [0,4] is $\frac{10.2-5}{4} = 1.3$. The average rate of change of h, by estimating from the graph, is approximately $\frac{25.5-5}{4} = 5.125$. The function with the greatest average rate of change is f, which means that over that four-year period, Tree A grew more feet per year on average than the other two trees.

e. Write formulas for functions g and h, and use them to confirm your answer to part (c).

The functions are given by $g(t) = 5 + 1.3t$ and $h(t) = 5\left(\frac{3}{2}\right)^{t}$. The average rate of change for g can be computed as $\frac{g(4)-g(0)}{4} = \frac{10.2-5}{4} = 1.3$, and the average rate of change for h on the interval [0,4] is $\frac{h(4)-h(0)}{4} = \frac{25.3125-5}{4} = 5.078125$.

f. For the exponential models, if the average rate of change of one function over the interval $[0,4]$ is greater than the average rate of change of another function on the same interval, is the percent rate of change also greater? Why or why not?

No. The average rate of change of f is greater than h over the interval of $[0,4]$, but the percent rate of change of h is greater. The average rate of change is the rate of change over a specific interval, which varies with the interval chosen. The percent rate of change of an exponential function is the percent increase or decrease between the value of the function at x and the value of the function at $x+1$ for any real number x; the percent rate of change is constant for an exponential function.

12. Identify which functions are exponential. For the exponential functions, use the properties of exponents to identify the percent rate of change, and classify them as representing exponential growth or decay.

a. $f(x) = 3(1-0.4)^{-x}$

Since $(1-0.4)^{-x} = ((0.6)^{-1})^x = \left(\left(\frac{6}{10}\right)^{-1}\right)^x = \left(\frac{10}{6}\right)^x \approx (1.666)^x$, the percent rate of change for f is approximately 66%, which represents an exponential growth.

b. $g(x) = \frac{3}{4^x}$

Since $\frac{3}{4^x} = 3\left(\frac{1}{4}\right)^x = 3(0.25)^x = 3(1-0.75)^x$, the percent rate of change for g is -75%, which represents an exponential decay.

c. $k(x) = 3x^{0.4}$

Since $3x^{0.4} = 3x^{\frac{4}{10}} = 3\sqrt[10]{x^4}$, this is not an exponential function (it does not have a common ratio, for example).

d. $h(x) = 3^{\frac{x}{4}}$

$3^{\frac{x}{4}} = \left(3^{\frac{1}{4}}\right)^x \approx 1.316^x$. The percent growth is approximately 31.6%, which represents an exponential growth.

13. A patient in a hospital needs to maintain a certain amount of a medication in her bloodstream to fight infection. Suppose the initial dosage is 10 mg, and the patient is given an additional maintenance dosage of 4 mg every hour. Assume that the amount of medication in the bloodstream is reduced by 25% every hour.

a. Write a function for the amount of the initial dosage that is in the bloodstream after n hours.

$$b(t) = 10(0.75)^n$$

b. Complete the table below to track the amount of medication from the maintenance dosage in the patient's bloodstream for the first five hours.

Hours since initial dose, n	Amount of the medication in the bloodstream from the maintenance dosage at the beginning of each hour
0	0
1	4
2	$4(1 + 0.75)$
3	$4(1 + 0.75 + 0.75^2)$
4	$4(1 + 0.75 + 0.75^2 + 0.75^3)$
5	$4(1 + 0.75 + 0.75^2 + \dots + 0.75^4)$

c. Write a function that models the total amount of medication in the bloodstream after n hours.

$$d(n) = 10(0.75)^n + 4\left(\frac{1 - 0.75^n}{1 - 0.75}\right)$$

d. Use a calculator to graph the function you wrote in part (c). According to the graph, will there ever be more than 16 mg of the medication present in the patient's bloodstream after each dose is administered?

According to the graph of the function, the amount of the medication approaches 16 mg as $n \to \infty$.

e. Rewrite this function as the difference of two functions (one a constant function and the other an exponential function), and use that difference to justify why the amount of medication in the patient's bloodstream will not exceed 16 mg after each dose is administered.

Rewriting this function gives $d(n) = 16 - 6(0.75)^n$. Thus, the function is the difference between a constant function and an exponential function that is approaching a value of 0 as n increases. The amount of the medication will always be 16 reduced by an ever-decreasing quantity.