Lesson 17: Trigonometric Identity Proofs

Classwork

Opening Exercise

We have seen that $\sin(\alpha + \beta) \neq \sin(\alpha) + \sin(\beta)$. So, what is $\sin(\alpha + \beta)$? Begin by completing the following table:

$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$	$\mathbf{sin}(\boldsymbol{\alpha})$	$\mathbf{sin}(\boldsymbol{\beta})$	$\mathbf{sin}(\boldsymbol{\alpha}+\boldsymbol{\beta})$	$\mathbf{sin}(\boldsymbol{\alpha})\,\mathbf{cos}(\boldsymbol{\beta})$	$\mathbf{sin}(\boldsymbol{\alpha})\,\mathbf{sin}(\boldsymbol{\beta})$	$\mathbf{cos}(\boldsymbol{\alpha})\,\mathbf{cos}(\boldsymbol{\beta})$	$\mathbf{cos}(\boldsymbol{\alpha})\,\mathbf{sin}(\boldsymbol{\beta})$
$\frac{\pi}{6}$	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{1}{2}$	$\mathbf{\frac{\sqrt{3}}{2}}$	$\frac{\sqrt{3}}{4}$	$\frac{1}{2}$		
$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\mathbf{1}$			$\frac{\sqrt{3}}{4}$	
$\frac{\pi}{4}$	$\frac{\pi}{6}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$\mathbf{\frac{1+\sqrt{3}}{2\sqrt{2}}}$	$\frac{\sqrt{3}}{2\sqrt{2}}$			
$\frac{\pi}{4}$	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\mathbf{1}$		$\frac{1}{2}$	$\frac{1}{2}$	
$\frac{\pi}{3}$	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$\mathbf{\frac{\sqrt{3}}{2}}$				$\frac{\sqrt{3}}{4}$
$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\mathbf{\frac{1+\sqrt{3}}{2\sqrt{2}}}$		$\frac{\sqrt{3}}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$	

Use the following table to formulate a conjecture for $\cos(\alpha + \beta)$:

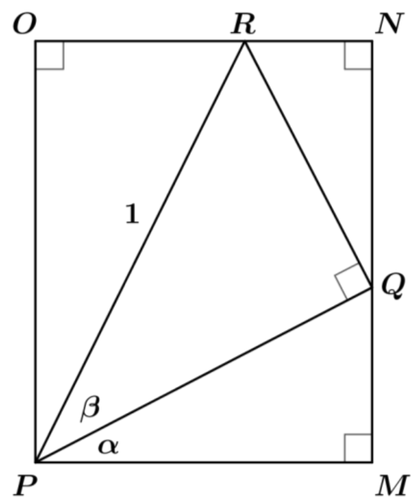
$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$	$\mathbf{cos}(\boldsymbol{\alpha})$	$\mathbf{cos}(\boldsymbol{\beta})$	$\mathbf{cos}(\boldsymbol{\alpha}+\boldsymbol{\beta})$	$\mathbf{sin}(\boldsymbol{\alpha})\,\mathbf{cos}(\boldsymbol{\beta})$	$\mathbf{sin}(\boldsymbol{\alpha})\,\mathbf{sin}(\boldsymbol{\beta})$	$\mathbf{cos}(\boldsymbol{\alpha})\,\mathbf{cos}(\boldsymbol{\beta})$	$\mathbf{cos}(\boldsymbol{\alpha})\,\mathbf{sin}(\boldsymbol{\beta})$
$\frac{\pi}{6}$	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{1}{2}$	$\mathbf{\frac{1}{2}}$	$\frac{\sqrt{3}}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{\sqrt{3}}{4}$
$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\mathbf{0}$	$\frac{1}{4}$	$\frac{\sqrt{3}}{4}$	$\frac{\sqrt{3}}{4}$	$\frac{3}{4}$
$\frac{\pi}{4}$	$\frac{\pi}{6}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	$\mathbf{\frac{\sqrt{3}-1}{2\sqrt{2}}}$	$\frac{\sqrt{3}}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$	$\frac{\sqrt{3}}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$
$\frac{\pi}{4}$	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\mathbf{0}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{\pi}{3}$	$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\mathbf{-\frac{1}{2}}$	$\frac{\sqrt{3}}{4}$	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{\sqrt{3}}{4}$
$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\mathbf{\frac{1-\sqrt{3}}{2\sqrt{2}}}$	$\frac{\sqrt{3}}{2\sqrt{2}}$	$\frac{\sqrt{3}}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$

Examples 1–2: Formulas for $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$

1. One conjecture is that the formula for the sine of the sum of two numbers is $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$. The proof can be a little long, but it is fairly straightforward. We will prove only the case when the two numbers are positive, and their sum is less than $\frac{\pi}{2}$.

   a. Let α and β be positive real numbers such that $0 < \alpha + \beta < \frac{\pi}{2}$.

   b. Construct rectangle $MNOP$ such that $PR = 1, m\angle PQR = 90°$, $m\angle RPQ = \beta$, and $m\angle QPM = \alpha$. See the figure at the right.



   c. Fill in the blanks in terms of α and β:

      i. $m\angle RPO =$ _____________.

      ii. $m\angle PRO =$ _______________.

      iii. Therefore, $\sin(\alpha + \beta) = PO$.

      iv. $RQ = \sin($_____$)$.

      v. $PQ = \cos($_____$)$.

   d. Let's label the angle and length measurements as shown.

   e. Use this new figure to fill in the blanks in terms of α and β:

      i. Why does $\sin(\alpha) = \frac{MQ}{\cos(\beta)}$?

      ii. Therefore, $MQ =$ _________.

      iii. $m\angle RQN =$ ____________.

   f. Now consider ΔRQN. Since $cos(\alpha) = \frac{QN}{sin(\beta)}$,

      $QN =$ _________.

   g. Label these lengths and angle measurements in the figure.

h. Since $MNOP$ is a rectangle, $OP = MQ + QN$.

i. Thus, $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$.

Note that we have only proven the formula for the sine of the sum of two real numbers α and β in the case where $0 < \alpha + \beta < \frac{\pi}{2}$. A proof for all real numbers α and β breaks down into cases that are proven similarly to the case we have just seen. Although we are omitting the full proof, this formula holds for all real numbers α and β.

Thus, for any real numbers $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$,

$$\mathbf{\sin(\boldsymbol{\alpha} + \boldsymbol{\beta}) = \sin(\boldsymbol{\alpha})\cos(\boldsymbol{\beta}) + \cos(\boldsymbol{\alpha})\sin(\boldsymbol{\beta})}.$$

2. Now let's prove our other conjecture, which is that the formula for the cosine of the sum of two numbers is
$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta).$$
Again, we will prove only the case when the two numbers are positive, and their sum is less than $\frac{\pi}{2}$. This time, we will use the sine addition formula and identities from previous lessons instead of working through a geometric proof.

   Fill in the blanks in terms of α and β:

   Let α and β be any real numbers. Then,

$$\begin{aligned}
\cos(\alpha + \beta) &= \sin\left(\frac{\pi}{2} - (\underline{\qquad\qquad})\right) \\
&= \sin\big((\underline{\qquad\qquad}) - \beta\big) \\
&= \sin\big((\underline{\qquad\qquad}) + (-\beta)\big) \\
&= \sin(\underline{\qquad\qquad})\cos(-\beta) + \cos(\underline{\qquad\qquad})\sin(-\beta) \\
&= \cos(\alpha)\cos(-\beta) + \sin(\alpha)\sin(-\beta) \\
&= \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta).
\end{aligned}$$

Thus, for all real numbers $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$,

$$\mathbf{\cos(\boldsymbol{\alpha} + \boldsymbol{\beta}) = \cos(\boldsymbol{\alpha})\cos(\boldsymbol{\beta}) - \sin(\boldsymbol{\alpha})\sin(\boldsymbol{\beta})}.$$

Exercises 1–2: Formulas for $\sin(\alpha - \beta)$ and $\cos(\alpha - \beta)$

1. Rewrite the expression $\sin(\alpha - \beta)$ as follows: $\sin\big(\alpha + (-\beta)\big)$. Use the rewritten form to find a formula for the sine of the difference of two angles, recalling that the sine is an odd function.

2. Now use the same idea to find a formula for the cosine of the difference of two angles. Recall that the cosine is an even function.

Thus, for all real numbers α and β,

$$\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta), \text{ and}$$

$$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta).$$

Exercises 3–5

3. Derive a formula for $\tan(\alpha+\beta)$ in terms of $\tan(\alpha)$ and $\tan(\beta)$ for $\frac{2n+1}{2}\pi < \theta < \frac{2n+3}{2}\pi$, for any integer n.

   Hint: Use the addition formulas for sine and cosine.

4. Derive a formula for $\sin(2u)$ in terms of $\sin(u)$ and $\cos(u)$ for all real numbers u.

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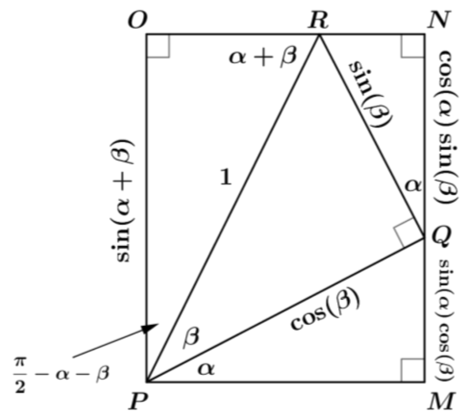
Problem Set

1. Prove the formula
$$\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta) \text{ for } 0<\alpha+\beta<\frac{\pi}{2}$$
using the rectangle $MNOP$ in the figure at the right and calculating PM, RN, and RO in terms of α and β.



2. Derive a formula for $\tan(2u)$ for $u \neq \frac{\pi}{4}+\frac{k\pi}{2}$ and $u \neq \frac{\pi}{2}+k\pi$, for all integers k.

3. Prove that $\cos(2u)=2\cos^2(u)-1$ is true for any real number u.

4. Prove that $\frac{1}{\cos(x)}-\cos(x)=\sin(x)\cdot\tan(x)$ is true for $x \neq \frac{\pi}{2}+k\pi$, for all integers k.

5. Write as a single term: $\cos\left(\frac{\pi}{4}+\theta\right)+\cos\left(\frac{\pi}{4}-\theta\right)$.

6. Write as a single term: $\sin(25°)\cos(10°)-\cos(25°)\sin(10°)$.

7. Write as a single term: $\cos(2x)\cos(x)+\sin(2x)\sin(x)$.

8. Write as a single term: $\frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{\cos(\alpha)\cos(\beta)}$, where $\cos(\alpha)\neq 0$ and $\cos(\beta)\neq 0$.

9. Prove that for all values of θ, $\cos\left(\frac{3\pi}{2}+\theta\right)=\sin(\theta)$.

10. Prove that for all values of θ, $\cos(\pi-\theta)=-\cos(\theta)$.