Lesson 17: Trigonometric Identity Proofs

Classwork

Opening Exercise

We have seen that . So, what is ? Begin by completing the following table:

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Use the following table to formulate a conjecture for :

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Examples 1–2: Formulas for and

1. One conjecture is that the formula for the sine of the sum of two numbers is . The proof can be a little long, but it is fairly straightforward. We will prove only the case when the two numbers are positive, and their sum is less than .
	1. Let and be positive real numbers such that .
	2. Construct rectangle such that , and See the figure at the right.
	3. Fill in the blanks in terms of and
		1. \_\_\_\_\_\_\_\_\_\_\_\_\_.
		2. \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_.
		3. Therefore, .
		4. .
		5. .
	4. Let’s label the angle and length measurements as shown.
	5. Use this new figure to fill in the blanks in terms of and
		1. Why does ?
		2. Therefore,\_\_\_\_\_\_\_\_\_.
		3. \_\_\_\_\_\_\_\_\_\_\_\_.
	6. Now consider Since

\_\_\_\_\_\_\_\_\_.

* 1. Label these lengths and angle measurements in the figure.
	2. Since is a rectangle,
	3. Thus,.

Note that we have only proven the formula for the sine of the sum of two real numbers and in the case where . A proof for all real numbers and breaks down into cases that are proven similarly to the case we have just seen. Although we are omitting the full proof, this formula holds for all real numbers and .

**Thus, for any real numbers and**

**.**

1. Now let’s prove our other conjecture, which is that the formula for the cosine of the sum of two numbers is

Again, we will prove only the case when the two numbers are positive, and their sum is less than . This time, we will use the sine addition formula and identities from previous lessons instead of working through a geometric proof.

Fill in the blanks in terms of and :

Let and be any real numbers. Then,

**Thus, for all real numbers and**

Exercises 1–2: Formulas for and

1. Rewrite the expression as follows: . Use the rewritten form to find a formula for the sine of the difference of two angles, recalling that the sine is an odd function.
2. Now use the same idea to find a formula for the cosine of the difference of two angles. Recall that the cosine is an even function.

**Thus, for all real numbers and**

Exercises 3–5

1. Derive a formula for in terms of and ) for for any integer .

Hint: Use the addition formulas for sine and cosine.

1. Derive a formula for in terms of and for all real numbers.
2. Derive a formula for in terms of and for all real numbers.

Problem Set

1. Prove the formula

 for

using the rectangle in the figure at the right and calculating and in terms of and .

1. Derive a formula for for and , for all integers .
2. Prove that is true for any real number .
3. Prove that is true for , for all integers .
4. Write as a single term: .
5. Write as a single term:
6. Write as a single term:
7. Write as a single term: , where and
8. Prove that for all values of , .
9. Prove that for all values of ,