Lesson 16: Proving Trigonometric Identities

Student Outcomes

- Students prove simple identities involving the sine function, cosine function, and secant function.
- Students recognize features of proofs of identities.

Lesson Notes

Students find that in some circumstances, they can start with a false statement and logically arrive at a true statement; so, students should avoid beginning a proof with the statement to be proven. Instead, they should work on transforming one side of the equation into the other. In this lesson, they will prove several simple identities.

Classwork

Opening Exercise (10 minutes)

Have students work in pairs on the Opening Exercise.

MP.3

Opening Exercise

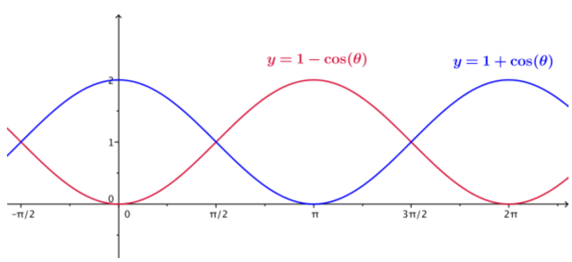
Which of these statements is a trigonometric identity? Provide evidence to support your claim.

Statement 1: $\sin^2(\theta) = 1 - \cos^2(\theta)$ **for θ any real number.**

Statement 2: $1 - \cos(\theta) = 1 - \cos(\theta)$ **for θ any real number.**

Statement 3: $1 - \cos(\theta) = 1 + \cos(\theta)$ **for θ any real number.**

Statement 1 is found by subtracting $\cos^2(\theta)$ from each side of the Pythagorean identity. Since the Pythagorean identity is true for all real numbers, Statement 1 is also an identity. The functions on either side of the equal sign in Statement 2 are identical, so they are also equivalent. The graphs of the functions on either side of the equal sign in Statement 3 intersect, as shown in the figure below, but the functions are not equivalent. Thus, Statements 1 and 2 are valid identities, but Statement 3 is false.



Using Statements 1 and 2, create a third identity, Statement 4, whose left side is $\frac{\sin^2(\theta)}{1-\cos(\theta)}$.

Students are likely to divide the equation in Statement 1 by the equation in Statement 2 to get something like the following statement:

Statement 4: $\frac{\sin^2(\theta)}{1-\cos(\theta)} = 1 + \cos(\theta)$ ***for all*** θ ***for which the functions on both sides are defined.***

For which values of θ is this statement valid?

The left side of the equation in Statement 4 is not defined when $\cos(\theta) = 1$***, which happens when*** θ ***is a multiple of*** 2π***. The right side of the equation is defined for all*** θ***. Thus, the equation is true wherever both sides are defined, which is for all*** θ ***that are not multiples of*** 2π***.***

Scaffolding:

- Ask students struggling to see that these are identities to substitute particular values for θ in to the left and the right sides of these equations separately to verify that they are true equations.
- Demonstrate how to determine Statement 4 for students having trouble seeing it.
- To challenge students, ask them to generate another identity using Statements 1 and 2 and explain for which values of θ it is valid.

MP.3

Discuss in pairs what it might mean to "prove" an identity. What might it take to prove, for example, that the following statement is an identity?

$\frac{\sin^2(\theta)}{1-\cos(\theta)} = 1 + \cos(\theta)$ **where** $\theta \neq 2\pi k$**, for all integers** k**.**

Students might come up with various suggestions. They might say, for example, that $\frac{\sin^2(\theta)}{1-\cos(\theta)} = 1 + \cos(\theta)$***, for*** $\theta \neq 2\pi k$ ***for all integers k is an identity if*** $\frac{\sin^2(\theta)}{1-\cos(\theta)}$ ***and*** $1+\cos(\theta)$ ***define the same functions or if they take the same values for the variable*** θ***. The functions on the two sides should have the same domain, but that alone is not enough to make the equation an identity.***

Scaffolding:

- Student pairs may need to first discuss what it means to prove anything. Circulate to assist those having trouble with the question and to find those who might present their answer.

To prove an identity, you have to use logical steps to show that one side of the equation in the identity can be transformed into the other side of the equation using already established identities such as the Pythagorean identity or the properties of operation (commutative, associative, and distributive properties). It is not correct to start with what you want to prove and work on both sides of the equation at the same time, as the following exercise shows.

Exercise (8 minutes)

Students should work on this exercise in groups. Its purpose is to show that if we start with the goal of a proof, we can end up "proving" a statement that is false. Part of Standard MP.3 involves distinguishing correct reasoning from flawed reasoning, and if there is a flaw in the argument, explaining what the flaw is. In this exercise, students see how a line of reasoning can go wrong.

Begin by asking students to take out their calculators and quickly graph the equations $y = \sin(x) + \cos(x)$ and $y = -\sqrt{1 + 2\sin(x)\cos(x)}$ to determine whether $\sin(\theta) + \cos(\theta) = -\sqrt{1 + 2\sin(\theta)\cos(\theta)}$ for all θ for which both functions are defined is a valid identity. Students should see from the graphs that the functions are not equivalent.

Scaffolding:

If students have trouble seeing the problem here, you might have them consider the following similar argument:

- [a] $1 = (-1)$, so squaring each side, we get
  [b] $1 = 1$, which is an identity.
- Therefore, squaring each side of a false statement can yield an identity. That does not make the original statement true.

Exercise

Take out your calculators and quickly graph the equations $y = \sin(x) + \cos(x)$ and $y = -\sqrt{1 + 2\sin(x)\cos(x)}$ to determine whether $\sin(\theta) + \cos(\theta) = -\sqrt{1 + 2\sin(\theta)\cos(\theta)}$ for all θ for which both functions are defined is a valid identity. You should see from the graphs that the functions are not equivalent.

Suppose that Charles did not think to graph the equations to see if the given statement was a valid identity, so he set about proving the identity using algebra and a previous identity. His argument is shown below.

First, [1] $\sin(\theta) + \cos(\theta) = -\sqrt{1 + 2\sin(\theta)\cos(\theta)}$ **for θ any real number.**

Now, using the multiplication property of equality, square both sides, which gives

[2] $\sin^2(\theta) + 2\sin(\theta)\cos(\theta) + \cos^2(\theta) = 1 + 2\sin(\theta)\cos(\theta)$ **for θ any real number.**

Using the subtraction property of equality, subtract $2\sin(\theta)\cos(\theta)$ from each side, which gives

[3] $\sin^2(\theta) + \cos^2(\theta) = 1$ **for θ any real number.**

Statement [3] is the Pythagorean identity. So, replace $\sin^2(\theta) + \cos^2(\theta)$ by 1 to get

[4] $1 = 1$, **which is definitely true.**

Therefore, the original statement must be true.

Does this mean that the student has proven that statement [1] is an identity? Discuss with your group whether it is a valid proof. If you decide it is not a valid proof, then discuss with your group how and where his argument went wrong.

No, statement [1] is not an identity; in fact, it is not true as we showed above by graphing the functions on the two sides of the equation. The sequence of statements is not a proof because it starts with a false statement in statement [1]. Squaring both sides of the equation is an irreversible step that alters the solutions to the equation. When squaring both sides of an equation, we have assumed that the equality exists, and that amounts to assuming what one is trying to prove. A better approach to prove an identity is valid would be to take one side of the equation in the proposed identity and work on it until one gets the other side.

The logic used by Charles is essentially, "If Statement [1] is true, then Statement [1] is true," which does not establish that Statement [1] is true. Make sure that students understand that all statements in a proof, particularly the first step of a proof, must be known to be true and must follow logically from the preceding statements in order for the proof to be valid.

Example 1 (10 minutes): Two Proofs of Our New Identity

Example 1: Two Proofs of Our New Identity

Work through these two different ways to approach proving the identity $\frac{\sin^2(\theta)}{1-\cos(\theta)} = 1 + \cos(\theta)$ where $\theta \neq 2\pi k$, for integers k. The proofs make use of some of the following properties of equality and real numbers. Here a, b, and c stand for arbitrary real numbers.

Reflexive property of equality	$a = a$
Symmetric property of equality	If $a = b$, then $b = a$.
Transitive property of equality	If $a = b$ and $b = c$, then $a = c$.
Addition property of equality	If $a = b$, then $a + c = b + c$.
Subtraction property of equality	If $a = b$, then $a - c = b - c$.
Multiplication property of equality	If $a = b$, then $a \times c = b \times c$.
Division property of equality	If $a = b$ and $c \neq 0$, then $a \div c = b \div c$.
Substitution property of equality	If $a = b$, then b may be substituted for a in any expression containing a.
Associative properties	$(a + b) + c = a + (b + c)$ and $a(bc) = (ab)c$.
Commutative properties	$a + b = b + a$ and $ab = ba$.
Distributive property	$a(b + c) = ab + ac$ and $(a + b)c = ac + bc$.

Fill in the missing parts of the proofs below.

A. We start with Statement 1 from the opening activity and divide both sides by the same expression, $1 - \cos(\theta)$. This step will introduce division by zero when $1 - \cos(\theta) = 0$ and will change the set of values of θ for which the identity is valid.

PROOF.

Step	Left Side of Equation		Equivalent Right Side	Domain	Reason
1	$\sin^2(\theta) + \cos^2(\theta)$	$=$	1	θ any real number	Pythagorean identity
2	$\sin^2(\theta)$	$=$	$1 - \cos^2(\theta)$	θ any real number	*Subtraction property of equality*
3	$\sin^2(\theta)$	$=$	$(1 - \cos(\theta))(1 + \cos(\theta))$	θ any real number	*Distributive property*
4	$\frac{\sin^2(\theta)}{1 - \cos(\theta)}$	$=$	$\frac{(1 - \cos(\theta))(1 + \cos(\theta))}{1 - \cos(\theta)}$	*$\theta \neq 2\pi k$ for all integers k*	*Division property of equality*
5	$\frac{\sin^2(\theta)}{1 - \cos(\theta)}$	$=$	$1 + \cos(\theta)$	$\theta \neq 2\pi k$ for all integers k	Substitution property of equality using $\frac{1 - \cos(\theta)}{1 - \cos(\theta)} = 1$

Let θ be a real number so that $\theta \neq 2\pi k$, for all integers k. Since $\sin^2(\theta) = 1 - \cos^2(\theta)$, we can divide both sides by $1 - \cos(\theta)$ when $\cos(\theta) \neq 1$ to give $\frac{\sin^2(\theta)}{1-\cos(\theta)} = \frac{1-\cos^2(\theta)}{1-\cos(\theta)}$. Factoring the numerator of the right side, we have $\frac{\sin^2(\theta)}{1-\cos(\theta)} = \frac{(1-\cos(\theta))(1+\cos(\theta))}{1-\cos(\theta)}$; thus, $\frac{\sin^2(\theta)}{1-\cos(\theta)} = 1 + \cos(\theta)$ where $\theta \neq 2\pi k$, for all integers k.

B. **Or, we can start with the more complicated side of the identity we want to prove and use algebra and prior trigonometric definitions and identities to transform it to the other side. In this case, the more complicated expression is $\frac{\sin^2(\theta)}{1-\cos(\theta)}$.**

PROOF.

Step	Left Side of Equation		Equivalent Right Side	Domain	Reason
1	$\frac{\sin^2(\theta)}{1-\cos(\theta)}$	$=$	$\frac{1-\cos^2(\theta)}{1-\cos(\theta)}$	$\theta \neq 2\pi k$, for all integers k	Substitution property of equality using $\sin^2(\theta) = 1-\cos^2(\theta)$
2	$\frac{\sin^2(\theta)}{1-\cos(\theta)}$	$=$	$\frac{(1-\cos(\theta))(1+\cos(\theta))}{1-\cos(\theta)}$	*$\theta \neq 2\pi k$, for all integers k*	Distributive property
3	$\frac{\sin^2(\theta)}{1-\cos(\theta)}$	$=$	$1+\cos(\theta)$	*$\theta \neq 2\pi k$, for all integers k*	*Substitution property of equality using $\frac{1-\cos(\theta)}{1-\cos(\theta)} = 1$*

Let θ be a real number so that $\theta \neq 2\pi k$, for all integers k. We begin with the expression $\frac{\sin^2(\theta)}{1-\cos(\theta)}$. According to the Pythagorean identity, $\sin^2(\theta) = 1-\cos^2(\theta)$. Substituting this into our expression, we have $\frac{\sin^2(\theta)}{1-\cos(\theta)} = \frac{1-\cos^2(\theta)}{1-\cos(\theta)}$. Factoring the numerator gives $\frac{\sin^2(\theta)}{1-\cos(\theta)} = \frac{(1-\cos(\theta))(1+\cos(\theta))}{1-\cos(\theta)}$. Thus, $\frac{\sin^2(\theta)}{1-\cos(\theta)} = 1+\cos(\theta)$ where $\theta \neq 2\pi k$, for all integers k.

Exercises 1–2 (12 minutes)

Students should work on these exercises individually and then share their results either in a group or with the whole class. Before beginning to prove an identity, students might want to take some scratch paper and work out the main ideas of the proof, taking into account the values for which the functions on either side of the equation are not defined. Then, they can restrict the values of x or θ at the beginning of the proof and not have to worry about it at every step.

Scaffolding:

- While students work independently on these exercises, work with a small group that would benefit from more teacher modeling in a small group setting.

Exercises 1–2

Prove that the following are trigonometric identities, beginning with the side of the equation that seems to be more complicated and starting the proof by restricting x to values where the identity is valid. Make sure that the complete identity statement is included at the end of the proof.

1. $\tan(x) = \frac{\sec(x)}{\csc(x)}$ for real numbers $x \neq \frac{\pi}{2} + \pi k$, for all integers k.

The more complicated side of the equation is $\frac{\sec(x)}{\csc(x)}$, so we begin with it. First, we eliminate values of x that are not in the domain.

PROOF. Let x be a real number so that $x \neq \frac{\pi}{2} + \pi k$, for all integers k.

Applying the definitions of the secant and cosecant functions, we have

$$\frac{\sec(x)}{\csc(x)} = \frac{\frac{1}{\cos(x)}}{\frac{1}{\sin(x)}}.$$

Simplifying the complex fraction gives

$$\begin{aligned}\frac{\sec(x)}{\csc(x)} &= \frac{1}{\cos(x)} \cdot \frac{\sin(x)}{1} \\ &= \frac{\sin(x)}{\cos(x)} \\ &= \tan(x).\end{aligned}$$

Thus, $\tan(x) = \frac{\sec(x)}{\csc(x)}$ for $x \neq \frac{\pi}{2} + \pi k$, for all integers k.

2. $\cot(x) + \tan(x) = \sec(x)\csc(x)$ for all real $x \neq \frac{\pi}{2}n$ for integer n.

The sides seem equally complicated, but the left side has two terms, so we begin with it. In general, functions composed of multiple terms (or a product of multiple terms) can be seen as more complicated than functions having a single term. However, a valid proof can be written starting on either side of the equation.

PROOF. Let x be a real number so that $x \neq \frac{\pi}{2}k$, for all integers k. Then, we express $\cot(x) + \tan(x)$ in terms of $\sin(x)$ and $\cos(x)$ and find a common denominator.

$$\begin{aligned}\cot(x) + \tan(x) &= \frac{\cos(x)}{\sin(x)} + \frac{\sin(x)}{\cos(x)} \\ &= \frac{\cos^2(x)}{\sin(x)\cos(x)} + \frac{\sin^2(x)}{\sin(x)\cos(x)}.\end{aligned}$$

Adding and applying the Pythagorean identity and then converting to the secant and cotangent functions gives

$$\begin{aligned}\cot(x) + \tan(x) &= \frac{1}{\sin(x)\cos(x)} \\ &= \frac{1}{\sin(x)} \cdot \frac{1}{\cos(x)} \\ &= \csc(x)\sec(x).\end{aligned}$$

Therefore, $\cot(x) + \tan(x) = \sec(x)\csc(x)$, where $x \neq \frac{\pi}{2}n$ for all integers n.

Closing (1 minute)

Ask students to explain how to prove a trigonometric identity, either in writing, to a partner, or as a class. Some key points they should mention are listed below.

- Start with stating the values of the variable—usually x or θ—for which the identity is valid.
- Work from a fact that is known to be true from either a prior identity or algebraic fact.
- A general plan is to start with the more complicated side of the equation you are trying to establish and transform it using a series of steps that can each be justified by prior facts and rules of algebra. The goal is to create a sequence of equations that are logically equivalent and that end with the desired equation for your identity.
- Note that we cannot start with the equation we want because that is assuming what we are trying to prove. "If A is true, then A is true" does not logically establish that A is true.

Exit Ticket (4 minutes)

Name ______________________________ Date ______________

Lesson 16: Proving Trigonometric Identities

Exit Ticket

Prove the following identity:

$\tan(\theta)\sin(\theta) + \cos(\theta) = \sec(\theta)$ for real numbers θ, where $\theta \neq \frac{\pi}{2} + \pi k$, for all integers k.

Exit Ticket Sample Solutions

Prove the following identity:

$\tan(\theta)\sin(\theta) + \cos(\theta) = \sec(\theta)$ **for real numbers θ, where $\theta \neq \frac{\pi}{2} + \pi k$, for all integers k.**

Begin with the more complicated side. Find a common denominator, use the Pythagorean identity, and then convert the fraction to its reciprocal.

PROOF: Let θ be any real number so that $\theta \neq \frac{\pi}{2} + \pi k$, for all integers k. Then,

$$\begin{aligned}\tan(\theta)\sin(\theta) + \cos(\theta) &= \frac{\sin(\theta)}{\cos(\theta)}\sin(\theta) + \cos(\theta) \\ &= \frac{\sin^2(\theta)}{\cos(\theta)} + \frac{\cos^2(\theta)}{\cos(\theta)} \\ &= \frac{1}{\cos(\theta)} \\ &= \sec(\theta).\end{aligned}$$

Therefore, $\tan(\theta)\sin(\theta) + \cos(\theta) = \sec(\theta)$, where $\theta \neq \frac{\pi}{2} + \pi k$, for all integers k.

Problem Set Sample Solutions

The first problem is designed to reinforce that the sine function is not linear. The formulas for $\sin(x + y)$ and $\cos(x + y)$ will be introduced in the next lesson.

1. **Does $\sin(x + y)$ equal $\sin(x) + \sin(y)$ for all real numbers x and y?**

   a. **Find each of the following: $\sin\left(\frac{\pi}{2}\right)$, $\sin\left(\frac{\pi}{4}\right)$, $\sin\left(\frac{3\pi}{4}\right)$.**

   $\sin\left(\frac{\pi}{2}\right) = 1$, $\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$, *and* $\sin\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}$

   b. **Are $\sin\left(\frac{\pi}{2} + \frac{\pi}{4}\right)$ and $\sin\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{4}\right)$ equal?**

   No, because $\sin\left(\frac{\pi}{2} + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$, *and* $\sin\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{4}\right) = 1 + \frac{\sqrt{2}}{2}$.

   c. **Are there any values of x and y for which $\sin(x + y) = \sin(x) + \sin(y)$?**

   Yes. If either x or y is zero, or if both x and y are multiples of π, this is a true statement. In many other cases it is not true, so it is not true in general.

2. Use $\tan(x) = \frac{\sin(x)}{\cos(x)}$ and identities involving the sine and cosine functions to establish the following identities for the tangent function. Identify the values of x where the equation is an identity.

a. $\tan(\pi - x) = \tan(x)$

Let x be a real number so that $x \neq \frac{\pi}{2} + \pi k$, for any integer k. Then,

$$\tan(\pi - x) = \frac{\sin(\pi - x)}{\cos(\pi - x)} = \frac{\sin(x)}{-\cos(x)}$$
$$= -\frac{\sin(x)}{\cos(x)} = -\tan(x).$$

Thus, $\tan(\pi - x) = -\tan(x)$, where $x \neq \frac{\pi}{2} + k\pi$, for all integers k.

b. $\tan(x + \pi) = \tan(x)$

Let x be a real number so that $x \neq \frac{\pi}{2} + \pi k$, for any integer k. Then,

$$\tan(x + \pi) = \frac{\sin(x + \pi)}{\cos(x + \pi)} = \frac{-\sin(x)}{-\cos(x)}$$
$$= \frac{\sin(x)}{\cos(x)} = \tan(x).$$

Thus, $\tan(x + \pi) = \tan(x)$, where $x \neq \frac{\pi}{2} + k\pi$, for all integers k.

c. $\tan(2\pi - x) = -\tan(x)$

Let x be a real number so that $x \neq \frac{\pi}{2} + \pi k$, for any integer k. Then,

$$\tan(2\pi - x) = \frac{\sin(2\pi - x)}{\cos(2\pi - x)} = \frac{-\sin(x)}{\cos(x)}$$
$$= -\frac{\sin(x)}{\cos(x)} = -\tan(x).$$

Thus, $\tan(2\pi - x) = -\tan(x)$, where $x \neq \frac{\pi}{2} + k\pi$, for all integers k.

d. $\tan(-x) = -\tan(x)$

Let x be a real number so that $x \neq \frac{\pi}{2} + \pi k$, for any integer k. Then,

$$\tan(-x) = \frac{\sin(-x)}{\cos(-x)} = \frac{-\sin(x)}{\cos(x)}$$
$$= -\frac{\sin(x)}{\cos(x)} = -\tan(x).$$

Thus, $\tan(-x) = -\tan(x)$, where $x \neq \frac{\pi}{2} + k\pi$, for all integers k.

3. **Rewrite each of the following expressions as a single term. Identify the values of theta for which the original expression and your expression are equal:**

a. $\cot(\theta)\sec(\theta)\sin(\theta)$

$$\cot(\theta)\sec(\theta)\sin(\theta) = \frac{\cos(\theta)}{\sin(\theta)}\cdot\frac{1}{\cos(\theta)}\cdot\frac{\sin(\theta)}{1}$$
$$= 1.$$

The expressions are equal where $x \neq \frac{\pi}{2}k$, for all integers k.

b. $\left(\frac{1}{1-\sin(x)}\right)\left(\frac{1}{1+\sin(x)}\right)$

$$\left(\frac{1}{1-\sin(x)}\right)\left(\frac{1}{1+\sin(x)}\right) = \frac{1}{1-\sin^2(x)}$$
$$= \frac{1}{\cos^2(x)}$$
$$= \sec^2(x).$$

The expressions are equal where $x \neq \frac{\pi}{2} + k\pi$, for all integers k.

c. $\frac{1}{\cos^2(x)} - \frac{1}{\cot^2(x)}$

$$\frac{1}{\cos^2(x)} - \frac{1}{\cot^2(x)} = \frac{1}{\cos^2(x)} - \frac{\sin^2(x)}{\cos^2(x)}$$
$$= \frac{1-\sin^2(x)}{\cos^2(x)}$$
$$= 1.$$

The expressions are equal where $x \neq \frac{\pi}{2}k$, for all integers k.

d. $\frac{(\tan(x)-\sin(x))(1+\cos(x))}{\sin^3(x)}$

$$\frac{(\tan(x)-\sin(x))(1+\cos(x))}{\sin^3(x)} = \frac{\left(\frac{\sin(x)}{\cos(x)} - \sin(x)\right)(1+\cos(x))}{\sin^3(x)}$$
$$= \frac{\sin(x)\left(\frac{1-\cos(x)}{\cos(x)}\right)(1+\cos(x))}{\sin^3(x)}$$
$$= \frac{\left(\frac{1-\cos^2(x)}{\cos(x)}\right)}{\sin^2(x)}$$
$$= \frac{1}{\cos(x)}$$
$$= \sec(x).$$

The expressions are equal where $\theta \neq \frac{\pi}{2} + k\pi$, for all integers k.

4. **Prove that for any two real numbers a and b,**

$$\sin^2(a) - \sin^2(b) + \cos^2(a)\sin^2(b) - \sin^2(a)\cos^2(b) = 0.$$

PROOF. Let a and b be any real numbers. Then,

$$\begin{aligned}\sin^2(a) - \sin^2(b) + \cos^2(a)\sin^2(b) - \sin^2(a)\cos^2(b) &= \sin^2(a)(1-\cos^2(b)) - \sin^2(b)(1-\cos^2(a))\\ &= \sin^2(a)\sin^2(b) - \sin^2(b)\sin^2(a)\\ &= 0\end{aligned}$$

Therefore, for all real numbers a and b, $\sin^2(a) - \sin^2(b) + \cos^2(a)\sin^2(b) - \sin^2(a)\cos^2(b) = 0$.

5. **Prove that the following statements are identities for all values of θ for which both sides are defined, and describe that set.**

   a. $\cot(\theta)\sec(\theta) = \csc(\theta)$

   PROOF. Let θ be a real number so that $\theta \neq \frac{\pi}{2}k$, for all integers k. Then,

$$\begin{aligned}\cot(\theta)\sec(\theta) &= \frac{\cos(\theta)}{\sin(\theta)}\cdot\frac{1}{\cos(\theta)}\\ &= \frac{1}{\sin(\theta)}\\ &= \csc(\theta).\end{aligned}$$

   Therefore, if $\theta \neq \frac{\pi}{2}k$, then $\cot(\theta)\sec(\theta) = \csc(\theta)$.

   b. $(\csc(\theta) + \cot(\theta))(1 - \cos(\theta)) = \sin(\theta)$

   PROOF. Let θ be a real number so that $\theta \neq \frac{\pi}{2}k$, for all integers k. Then,

$$\begin{aligned}(\csc(\theta) + \cot(\theta))(1 - \cos(\theta)) &= \frac{1+\cos(\theta)}{\sin(\theta)}(1-\cos(\theta))\\ &= \frac{1-\cos^2(\theta)}{\sin(\theta)}\\ &= \frac{\sin^2(\theta)}{\sin(\theta)}\\ &= \sin(\theta).\end{aligned}$$

   Therefore, if $\theta \neq \frac{\pi}{2}k$, then $(\csc(\theta) + \cot(\theta))(1 - \cos(\theta)) = \sin(\theta)$.

   c. $\tan^2(\theta) - \sin^2(\theta) = \tan^2(\theta)\sin^2(\theta)$

   PROOF. Let θ be a real number so that $\theta \neq \frac{\pi}{2}k$, for all integers k. Then,

$$\begin{aligned}\tan^2(\theta) - \sin^2(\theta) &= \frac{\sin^2(\theta)}{\cos^2(\theta)} - \sin^2(\theta)\\ &= \frac{\sin^2(\theta)(1-\cos^2(\theta))}{\cos^2(\theta)}\\ &= \frac{\sin^2(\theta)\sin^2(\theta)}{\cos^2(\theta)}\\ &= \tan^2(\theta)\sin^2(\theta).\end{aligned}$$

   Therefore, if $\theta \neq \frac{\pi}{2}k$, then $\tan^2(\theta) - \sin^2(\theta) = \tan^2(\theta)\sin^2(\theta)$.

d. $\frac{4+\tan^2(x)-\sec^2(x)}{\csc^2(x)} = 3\sin^2(x)$

PROOF. Let x be a real number so that $x \neq \frac{\pi}{2}k$, for all integers k. Then,

$$\begin{aligned}\frac{4+\tan^2(x)-\sec^2(x)}{\csc^2(x)} &= \frac{4+\frac{\sin^2(x)}{\cos^2(x)}-\frac{1}{\cos^2(x)}}{\frac{1}{\sin^2(x)}}\\ &= \frac{\frac{4\cos^2(x)+\sin^2(x)-1}{\cos^2(x)}}{\frac{1}{\sin^2(x)}}\\ &= \frac{\sin^2(x)(4\cos^2(x)-\cos^2(x))}{\cos^2(x)}\\ &= 3\sin^2(x).\end{aligned}$$

Therefore, if $x \neq \frac{\pi}{2}k$, then $\frac{4+\tan^2(x)-\sec^2(x)}{\csc^2(x)} = 3\sin^2(x)$.

e. $\frac{(1+\sin(\theta))^2+\cos^2(\theta)}{1+\sin(\theta)} = 2$

PROOF. Let θ be a real number so that $\theta \neq -\frac{\pi}{2}+2k\pi$, for all integers k. Then,

$$\begin{aligned}\frac{(1+\sin(\theta))^2+\cos^2(\theta)}{1+\sin(\theta)} &= \frac{1+2\sin(\theta)+\sin^2(\theta)+\cos^2(\theta)}{1+\sin(\theta)}\\ &= \frac{2+2\sin(\theta)}{1+\sin(\theta)}\\ &= 2.\end{aligned}$$

Therefore, if $\theta \neq -\frac{\pi}{2}+2k\pi$ for all integers k, then $\frac{(1+\sin(\theta))^2+\cos^2(\theta)}{1+\sin(\theta)} = 2$.

6. Prove that the value of the following expression does not depend on the value of y:

$$\cot(y)\frac{\tan(x)+\tan(y)}{\cot(x)+\cot(y)}.$$

If $y \neq \frac{\pi}{2}+k\pi$ for all integers k, then

$$\begin{aligned}\cot(y)\frac{\tan(x)+\tan(y)}{\cot(x)+\cot(y)} &= \frac{\cos(y)}{\sin(y)}\cdot\frac{\frac{\sin(x)}{\cos(x)}+\frac{\sin(y)}{\cos(y)}}{\frac{\cos(x)}{\sin(x)}+\frac{\cos(y)}{\sin(y)}}\\ &= \frac{\cos(y)}{\sin(y)}\cdot\frac{\frac{\sin(x)\cos(y)+\cos(x)\sin(y)}{\cos(x)\cos(y)}}{\frac{\cos(x)\sin(y)+\sin(x)\cos(y)}{\sin(x)\sin(y)}}\\ &= \frac{\cos(y)}{\sin(y)}\cdot\frac{\sin(x)\sin(y)}{\cos(x)\cos(y)}\\ &= \frac{\sin(x)}{\cos(x)}\\ &= \tan(x).\end{aligned}$$

Therefore, $\cot(y)\frac{\tan(x)+\tan(y)}{\cot(x)+\cot(y)} = \tan(x)$ for all values of x and y for which both sides of the equation are defined. Thus, the expression does not depend on the value of y.