Lesson 5: Extending the Domain of Sine and Cosine to All Real Numbers

Classwork

Opening Exercises

* 1. Suppose that a group of $360$ coworkers pool their money, buying a single lottery ticket every day with the understanding that if any ticket was a winning ticket, the group would split the winnings evenly, and they would donate any left over money to the local high school. Using this strategy, the group won $\$1,000$. How much money was donated to the school?
	2. What if the winning ticket was worth $\$250,000$? Using the same plan as in part (a), how much money would be donated to the school?
	3. What if the winning ticket was worth $\$540,000$? Using the same plan as in part (a), how much money would be donated to the school?

Exercises 1–5

1. Find $cos(405°)$ and $sin\left(405°\right).$ Identify the measure of the reference angle.
2. Find $cos(840°)$ and $sin\left(840°\right).$ Identify the measure of the reference angle.
3. Find $cos(1680°)$ and $sin\left(1680°\right).$ Identify the measure of the reference angle.
4. Find $cos(2115°)$ and $sin\left(2115°\right).$ Identify the measure of the reference angle.
5. Find $cos(720030°)$ and $sin\left(720030°\right).$ Identify the measure of the reference angle.

Exercises 6–10

1. Find $cos(-30°)$ and $sin\left(-30°\right).$ Identify the measure of the reference angle.
2. Find $cos(-135°)$ and $sin\left(-135°\right).$ Identify the measure of the reference angle.
3. Find $cos(-1320°)$ and $sin\left(-1320°\right).$ Identify the measure of the reference angle.
4. Find $cos(-2205°)$ and $sin\left(-2205°\right).$ Identify the measure of the reference angle.
5. Find $cos(-2835°)$ and $sin\left(-2835°\right).$ Identify the measure of the reference angle.

Discussion

**Case 1:** What about the values of the sine and cosine function of other amounts of rotation that produce a terminal ray along the positive $x$-axis, such as $1080°$?

Our definition of a reference angle is the angle formed by the terminal ray and the $x$-axis, but our terminal ray lies along the $x$-axis so the terminal ray and the $x$-axis form a zero angle.

How would we assign values to $cos(1080°)$ and $\sin((1080°))$?

What if we rotated around $24000°$, which is $400$ turns? What are $cos⁡(24000°)$ and $\sin(\left(24000°\right))$?

State a generalization of these results:

If $θ=n∙360°$, for some integer $n,$ then $\cos(\left(θ\right)=\\_\\_\\_\\_\\_)$, and $\sin(\left(θ\right))=\\_\\_\\_\\_\\_\\_$.

**Case 2:** What about the values of the sine and cosine function of other amounts of rotation that produce a terminal ray along the negative $x$-axis, such as $540°$?

How would we assign values to $cos(540°)$ and $\sin((540°))$?

What are the values of $cos⁡(900°)$ and $\sin((900°))$? How do you know?

State a generalization of these results:

If $θ=n∙360°+180°$, for some integer $n,$ then $\cos(\left(θ\right)=\\_\\_\\_\\_\\_)$, and $\sin(\left(θ\right))=\\_\\_\\_\\_\\_\\_$.

**Case 3:** What about the values of the sine and cosine function for rotations that are $90°$ more than a number of full turns, such as $-630°$?How would we assign values to $cos\left(-630°\right),$ and $\sin((-630°))?$

Can we generalize to any rotation that produces a terminal ray along the positive $y$-axis?

State a generalization of these results:

*If* $θ=n∙360°+90°$*, for some integer* $n,$ *then* $\cos(\left(θ\right)=\\_\\_\\_\\_\\_)$*, and* $\sin(\left(θ\right))=\\_\\_\\_\\_\\_\\_$*.*

**Case 4:** What about the values of the sine and cosine function for rotations whose terminal ray lies along the negative $y$-axis, such as $-810°$?

****How would we assign values to $cos(-810°)$ and $\sin((-810°))$?

Can we generalize to any rotation that produces a terminal ray along the negative $y$-axis?

State a generalization of these results:

*If* $θ=n∙360°+270°$*, for some integer* $n,$ *then* $\cos(\left(θ\right)=\\_\\_\\_\\_\\_)$*, and* $\sin(\left(θ\right))=\\_\\_\\_\\_\\_\\_$*.*

Discussion

Let $θ$ be any real number. In the Cartesian plane, rotate the initial ray by $θ$ degrees about the origin. Intersect the resulting terminal ray with the unit circle to get a point $(x\_{θ},y\_{θ})$ in the coordinate plane. The value of $sin⁡(θ)$ is $y\_{θ},$ and the value of $\cos(\left(θ\right))$ is $x\_{θ}$.

Lesson Summary

In this lesson we formalized the definition of the sine and cosine functions of a number of degrees of rotation, $θ$. We rotate the initial ray made from the positive $x$-axis through $θ$ degrees, going counterclockwise if $θ>0$ and clockwise if $θ<0$. The point $P$ is defined by the intersection of the terminal ray and the unit circle.

* The value of $cos\left(θ\right)$ is the $x$-coordinate of $P$.
* The value of $sin\left(θ\right)$ is the $y$-coordinate of $P$.
* The sine and cosine functions have domain of all real numbers and range $[-1,1]$.

Problem Set

1. Fill in the chart; write the quadrant where the terminal ray is located after rotation by $θ, $the measures of the reference angles, and the values of the sine and cosine functions for the indicated rotation numbers.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Number of degrees of rotation,$$θ$$ | Quadrant | Measure of ReferenceAngle | $$cos(θ)$$ | $$sin(θ)$$ |
| $$690$$ |  |  |  |  |
| $$810$$ |  |  |  |  |
| $$1560$$ |  |  |  |  |
| $$1440$$ |  |  |  |  |
| $$855$$ |  |  |  |  |
| $$-330$$ |  |  |  |  |
| $$-4500$$ |  |  |  |  |
| $$-510$$ |  |  |  |  |
| $$-135$$ |  |  |  |  |
| $$-1170$$ |  |  |  |  |

1. Using geometry, Jennifer correctly calculated that $\sin(15°=)\frac{1}{2}\sqrt{2-\sqrt{3}}$ . Based on this information, fill in the chart:

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Number of degrees of rotation,$$θ$$ | Quadrant | Measure of Reference Angle | $$\cos((θ))$$ | $$\sin((θ))$$ |
| $$525$$ |  |  |  |  |
| $$705$$ |  |  |  |  |
| $$915$$ |  |  |  |  |
| $$-15$$ |  |  |  |  |
| $$-165$$ |  |  |  |  |
| $$-705$$ |  |  |  |  |

1. Suppose $θ$ represents a quantity in degrees, and that $\sin((θ)=0.5)$. List the first six possible positive values that $θ$ can take.
2. Suppose $θ$ represents a quantity in degrees, and that $\sin((θ°)=-0.5)$. List six possible negative values that $θ$ can take.
3. Suppose $θ$ represents a quantity in degrees. Is it possible that $cos\left(θ°\right)=\frac{1}{2}$ and $sin\left(θ°\right)=\frac{1}{2}$?
4. Jane says that since the reference angle for a rotation through $-765°$ has measure $45°$, then $cos\left(-765°\right)=cos\left(45°\right),$ and $sin\left(-765°\right)=sin(45°)$. Explain why she is or is not correct.
5. Doug says that since the reference angle for a rotation through $765°$ has measure $45°$, then $cos\left(765°\right)=cos\left(45°\right),$ and $sin\left(765°\right)=sin(45°)$. Explain why he is or is not correct.