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## Trigonometric Functions

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<sup>1</sup> Each lesson is ONE day, and ONE day is considered a 45-minute period.

## Algebra II • Module 2

## Trigonometric Functions

## OVERVIEW

Module 2 builds on students' previous work with units (**N-Q.A.1**) and with functions (**F-IF.A.1**, **F-IF.A.2**, **F-IF.B.4**, **F-IF.C.7e**, **F-BF.A.1**, **F-BF.B.3**) from Algebra I, and with trigonometric ratios and circles (**G-SRT.C.6**, **G-SRT.C.7**, **G-SRT.C.8**) from high school Geometry. Included in Topic A is preparation for extension standard **F-TF.A.3**. Extension standard **F-TF.C.9** is also discussed in Topic B as preparation for the Pre-Calculus and Advanced Topics course.

Topic A starts by asking students to graph the height of a Ferris wheel as a function of time and uses that study to help define the sine, cosine, and tangent functions as functions from all (or most) real numbers to the real numbers. A precise definition of sine and cosine (as well as tangent and the co-functions) is developed using transformational geometry from high school Geometry. This precision leads to a discussion of a mathematically *natural* unit of measurement for angle measures, a radian, and students begin to build fluency with values of sine, cosine, and tangent at  $\frac{\pi}{6}$ ,  $\frac{\pi}{4}$ ,  $\frac{\pi}{3}$ ,  $\frac{\pi}{2}$ ,  $\pi$ , etc. The topic concludes with students graphing the sine and cosine functions and noticing various aspects of the graph, which they write down as simple trigonometric identities.

In Topic B, students make sense of periodic phenomena as they model them with trigonometric functions. They identify the periodicity, midline, and amplitude from graphs of data and use them to construct sinusoidal functions that model situations from both the biological and physical sciences. They extend the concept of polynomial identities to trigonometric identities and prove simple trigonometric identities such as the Pythagorean identity; these identities are then used to solve problems.

The Mid-Module Assessment follows Topic A. The End-of-Module Assessment follows Topic B.

## Focus Standards

## Analyze functions using different representations.

- F-IF.C.7** Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.\*
- e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

**Extend the domain of trigonometric functions using the unit circle.**

- F-TF.A.1** Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.
- F-TF.A.2** Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

**Model periodic phenomena with trigonometric functions.**

- F-TF.B.5** Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.\*

**Prove and apply trigonometric identities.**

- F-TF.C.8** Prove the Pythagorean identity  $\sin^2(\theta) + \cos^2(\theta) = 1$  and use it to find  $\sin(\theta)$ ,  $\cos(\theta)$ , or  $\tan(\theta)$  given  $\sin(\theta)$ ,  $\cos(\theta)$ , or  $\tan(\theta)$  and the quadrant of the angle.

**Summarize, represent, and interpret data on two categorical and quantitative variables.**

- S-ID.B.6a** Represent data on two quantitative variables on a scatter plot, and describe how the variables are related.
- Fit a function to the data; use functions fitted to data to solve problems in the context of the data. *Use given functions or choose a function suggested by the context. Emphasize linear, quadratic, and exponential models.*

## Extension Standards

**Extend the domain of trigonometric functions using the unit circle.**

- F-TF.A.3** (+) Use special triangles to determine geometrically the values of sine, cosine, tangent for  $\pi/3$ ,  $\pi/4$  and  $\pi/6$  and use the unit circle to express the values of sine, cosine, and tangent for  $\pi - x$ ,  $\pi + x$ , and  $2\pi - x$  in terms of their values for  $x$ , where  $x$  is any real number.

**Prove and apply trigonometric identities.**

- F-TF.C.9** (+) Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems.

## Foundational Standards

**Reason quantitatively and use units to solve problems.**

- N-Q.A.2** Define appropriate quantities for the purpose of descriptive modeling.

**Understand the concept of a function and use function notation.**

- F-IF.A.1** Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If  $f$  is a function and  $x$  is an element of its domain, then  $f(x)$  denotes the output of  $f$  corresponding to the input  $x$ . The graph of  $f$  is the graph of the equation  $y = f(x)$ .
- F-IF.A.2** Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.

**Build a function that models a relationship between two quantities.**

- F-BF.A.1** Write a function that describes a relationship between two quantities.\*
- Determine an explicit expression, a recursive process, or steps for calculation from a context.
  - Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.*
  - (+) Compose functions. *For example, if  $T(y)$  is the temperature in the atmosphere as a function of height, and  $h(t)$  is the height of a weather balloon as a function of time, then  $T(h(t))$  is the temperature at the location of the weather balloon as a function of time.*

**Build new functions from existing functions.**

- F-BF.B.3** Identify the effect on the graph of replacing  $f(x)$  by  $f(x) + k$ ,  $k f(x)$ ,  $f(kx)$ , and  $f(x + k)$  for specific values of  $k$  (both positive and negative); find the value of  $k$  given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*

**Define trigonometric ratios and solve problems involving right triangles.**

- G-SRT.C.6** Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.
- G-SRT.C.7** Explain and use the relationship between the sine and cosine of complementary angles.
- G-SRT.C.8** Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.\*

## Focus Standards for Mathematical Practice

- MP.1 Make sense of problems and persevere in solving them.** Students look for entry points into studying the “height” of the sun above the ground, first by realizing that no such quantity exists, then by surmising that the notion can still be profitably analyzed in terms of trigonometric ratios. They use this and other concrete situations to extend concepts of trigonometry studied in previous years, which were initially limited to angles between zero and ninety-degrees, to the full range of inputs; they also solve challenges about circular motion.
- MP.2 Reason abstractly and quantitatively.** Students extend the study of trigonometry to the domain of all (or almost all) real inputs. By focusing only on the linear components of circular motion (the vertical or the horizontal displacement of a point in orbit), students develop the means to analyze periodic phenomena. Students also extend a classic proof of the Pythagorean theorem to discover trigonometric addition formulas.
- MP.3 Construct viable arguments and critique the reasoning of others.** The vertical and horizontal displacements of a Ferris wheel passenger car are both periodic. Students conjecture how these functions are related to the trigonometric ratios they studied in geometry, making plausible arguments by modeling the Ferris wheel with a circle in the coordinate plane. Also, students construct valid arguments to extend trigonometric identities to the full range of inputs.
- MP.4 Model with mathematics.** The main modeling activity of this module is to analyze the vertical and horizontal displacement of a passenger car of a Ferris wheel. As they make assumptions and simplify the situation, they discover the need for sine and cosine functions to model the periodic motion using sinusoidal functions. Students then model a large number of other periodic phenomena by fitting sinusoidal functions to data given about tides, sound waves, and daylight hours; they then solve problems using those functions in the context of that data.
- MP.7 Look for and make use of structure.** Students recognize the periodic nature of a phenomenon and look for suitable values of midline, amplitude, and frequency for it. The periodicity and properties of cyclical motion shown in graphs helps students to recognize different trigonometric identities, and structure in standard proofs (of the Pythagorean theorem, for example) provides the means to extend familiar trigonometric results to a wider range of input values.
- MP.8 Look for and express regularity in repeated reasoning.** In repeatedly graphing different sinusoidal functions, students identify how parameters within the function give information about the amplitude, midline, and frequency of the function. They express this regularity in terms of a general formula for sinusoidal functions and use the formula to quickly write functions that model periodic data.

## Terminology

### New or Recently Introduced Terms

- **Radian** (A *radian* angle is the angle subtended by an arc of a circle that is equal in length to the radius of the circle. A *radian* (1 rad) is a unit of rotational measure given by a rotation by a radian angle.)
- **Periodic Function** (A function  $f$  whose domain is a subset of the real numbers is said to be *periodic* with period  $P > 0$  if the domain of  $f$  contains  $x + P$  whenever it contains  $x$ , and if  $f(x + P) = f(x)$  for all real numbers  $x$  in its domain.)
- **Sine** (Let  $\theta$  be any real number. In the Cartesian plane, rotate the initial ray by  $\theta$  radians about the origin. Intersect the resulting terminal ray with the unit circle to get a point  $(x_\theta, y_\theta)$ . The value of  $\sin(\theta)$  is  $y_\theta$ .)
- **Cosine** (Let  $\theta$  be any real number. In the Cartesian plane, rotate the initial ray by  $\theta$  radians about the origin. Intersect the resulting terminal ray with the unit circle to get a point  $(x_\theta, y_\theta)$ . The value of  $\cos(\theta)$  is  $x_\theta$ .)
- **Sinusoidal Function** (A periodic function is *sinusoidal* if it can be written in the form  $f(x) = A \sin(\omega(x - h)) + k$  for real numbers  $A$ ,  $\omega$ ,  $h$ , and  $k$ . In this form,
  - $|A|$  is called the *amplitude* of the function,
  - $\frac{2\pi}{|\omega|}$  is the *period* of the function,
  - $\frac{|\omega|}{2\pi}$  is the *frequency* of the function,
  - $h$  is called the *phase shift*, and
  - the graph of  $y = k$  is called the *midline*.

Furthermore, we can see that the graph of the *sinusoidal function*  $f$  is obtained by first vertically scaling the graph of the sine function by  $A$ , then horizontally scaling the resulting graph by  $\frac{1}{\omega}$ , and, finally, by horizontally and vertically translating the resulting graph by  $h$  and  $k$  units, respectively.)

- **Period** (The *period*  $P$  is the distance between two consecutive maximal points, or two consecutive minimal points, on the graph of the sinusoidal function.)
- **Amplitude** (The *amplitude* is the distance between a maximal point of the graph of the sinusoidal function and the midline.)
- **Frequency** (The *frequency* of a periodic function is the unit rate of the constant rate defined by the number of cycles per unit length.)
- **Midline** (The *midline* is the horizontal line that is halfway between the maximal line and the minimal line.)
- **Trigonometric Identity** (A *trigonometric identity* is a statement that two trigonometric functions are equivalent.)
- **Tangent** (Let  $\theta$  be any real number such that  $\theta \neq \frac{\pi}{2} + k\pi$  for all integers  $k$ . In the Cartesian plane, rotate the initial ray by  $\theta$  radians about the origin. Intersect the resulting terminal ray with the unit circle to get a point  $(x_\theta, y_\theta)$ . The value of  $\tan(\theta)$  is  $\frac{y_\theta}{x_\theta}$ .)

- **Secant** (Let  $\theta$  be any real number such that  $\theta \neq \frac{\pi}{2} + k\pi$  for all integers  $k$ . In the Cartesian plane, rotate the initial ray by  $\theta$  radians about the origin. Intersect the resulting terminal ray with the unit circle to get a point  $(x_\theta, y_\theta)$ . The value of  $\sec(\theta)$  is  $\frac{1}{x_\theta}$ .)
- **Cosecant** (Let  $\theta$  be any real number such that  $\theta \neq k\pi$  for all integers  $k$ . In the Cartesian plane, rotate the initial ray by  $\theta$  radians about the origin. Intersect the resulting terminal ray with the unit circle to get a point  $(x_\theta, y_\theta)$ . The value of  $\csc(\theta)$  is  $\frac{1}{y_\theta}$ .)
- **Cotangent** (Let  $\theta$  be any real number such that  $\theta \neq k\pi$  for all integers  $k$ . In the Cartesian plane, rotate the initial ray by  $\theta$  radians about the origin. Intersect the resulting terminal ray with the unit circle to get a point  $(x_\theta, y_\theta)$ . The value of  $\cot(\theta)$  is  $\frac{x_\theta}{y_\theta}$ .)

### Familiar Terms and Symbols<sup>2</sup>

- Degree
- Sine, Cosine, Tangent
- Circle
- Rotation
- Identity
- Asymptote
- Even and Odd Functions

### Suggested Tools and Representations

- Graphing Calculator
- Wolfram Alpha Software
- Geometer's Sketchpad Software

### Assessment Summary

Assessment Type	Administered	Format	Standards Addressed
Mid-Module Assessment Task	After Topic A	Constructed response with rubric	F-IF.C.7e, F-TF.A.1, F-TF.A.2
End-of-Module Assessment Task	After Topic B	Constructed response with rubric	F-IF.C.7e, F-TF.B.5, F-TF.C.8, S-ID.B.6a

<sup>2</sup> These are terms and symbols students have seen previously.