Lesson 40: Obstacles Resolved—A Surprising Result

Student Outcomes

- Students understand the Fundamental Theorem of Algebra; that all polynomial expressions factor into linear terms in the realm of complex numbers. Consequences, in particular, for quadratic and cubic equations are understood.

Lesson Notes

There is no real consensus in the literature about what exactly constitutes the Fundamental Theorem of Algebra; it is stated differently in different texts. The two-part theorem stated in this lesson encapsulates the main ideas of the theorem and its corollaries while remaining accessible to students. The first part of what is stated here as the Fundamental Theorem of Algebra is the one that we are not mathematically equipped to prove or justify at this level; this part states that every polynomial equation has at least one solution in the complex numbers and will need to be accepted without proof. The consequence of this first part is what we find really interesting–that every polynomial expression factors into the same number of linear factors as its degree. Justification for this second part of the Fundamental Theorem of Algebra is accessible for students as long as we can accept the first part without needing proof. Since every polynomial of degree $n \geq 1$ will factor into n linear factors, we know that any polynomial function of degree n will have n zeros (including repeated zeros).

Classwork

Opening Exercise (5 minutes)

At the beginning of the lesson, we focus on the most familiar of polynomial expressions, the quadratic equations. Ensure that students understand the link provided by the Remainder Theorem between solutions of polynomial equations and factors of the associated polynomial expression. Allow students time to work on the Opening Exercise, and then debrief.

Opening Exercise

Write each of the following quadratic expressions as a product of linear factors. Verify that the factored form is equivalent.

1. $x^2 + 12x + 27$ $(x+3)(x+9)$
2. $x^2 - 16$ $(x+4)(x-4)$
3. $x^2 + 16$ $(x+4i)(x-4i)$
4. $x^2 + 4x + 5$ $(x+2+i)(x+2-i)$

Discussion (7 minutes)

Remind students about the Remainder Theorem, studied earlier in the module. The Remainder Theorem states that if P is a polynomial function, and $P(a) = 0$ for some value of a, then $x - a$ is a factor of P. The Remainder Theorem plays an important role in the development of this lesson, linking the solutions of a polynomial equation to the factors of the associated polynomial expression.

- With a partner, describe any patterns you see in the Opening Exercise.
- Can every quadratic polynomial be written in terms of linear factors? If so, how many linear factors?
    - *Yes, two.*
- How do you know?
    - *Every quadratic equation has two solutions that can be found using the quadratic formula. These solutions of the equation lead to linear factors of the quadratic polynomial.*
- What types of solutions can a quadratic equation have? What does this mean about the graph of the corresponding function?
    - *The equation has either two real solutions, one real solution, or two complex solutions. These situations correspond to the graph having two x-intercepts, one x-intercept, or no x-intercepts.*

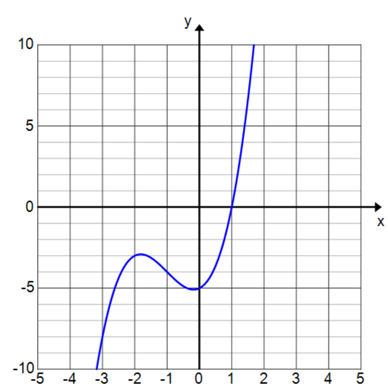
Be sure that students realize that real numbers are also complex numbers; if a is real, then we can write it as $a + 0i$.

Example 1 (8 minutes)

The purpose of this example is to help students move fluently between the concepts of x-intercepts of the graph of a polynomial equation $y = P(x)$, the solutions of the polynomial equation $P(x) = 0$, and the factors in the factored form of the associated polynomial P. Talk the students through parts (a)–(e), and then allow them time to work alone or in pairs on part (f) before completing the discussion.

Example 1

Consider the polynomial $P(x) = x^3 + 3x^2 + x - 5$ whose graph is shown to the right.



a. Looking at the graph, how do we know that there is only one real solution?

The graph has only one x-intercept.

b. Is it possible for a cubic polynomial function to have no zeros?

No. Since the opposite ends of the graph of a cubic function go in opposite directions, the graph must cross the x-axis at some point. Since the graph must have an x-intercept, the function must have a zero.

c. From the graph, what appears to be one solution to the equation $x^3 + 3x^2 + x - 5 = 0$?

The only real solution appears to be 1.

d. How can we verify that this is a solution?

Evaluate the function at 1; that is, verify that $P(1) = 0$.

$$P(1) = (1)^3 + 3(1)^2 + 1 - 5 = 1 + 2 + 1 - 5 = 0$$

e. According to the Remainder Theorem, what is one factor of the cubic expression $x^3 + 3x^2 + x - 5$?

$(x - 1)$

f. Factor out the expression you found in part (e) from $x^3 + 3x^2 + x - 5$.

Using polynomial division, we see that $x^3 + 3x^2 + x - 5 = (x - 1)(x^2 + 4x + 5)$.

g. What are all of the solutions to $x^3 + 3x^2 + x - 5 = 0$?

The quadratic equation $x^2 + 4x + 5 = 0$ has solutions $2 - i$ and $2 + i$ by the quadratic formula, so the original equation has solutions 1, $2 - i$, and $2 + i$.

h. Write the expression $x^3 + 3x^2 + x - 5$ in terms of linear factors.

The factored form of the cubic expression is $x^3 + 3x^2 + x - 5 = (x - 1)(x - (2 - i))(x - (2 + i))$.

Scaffolding:

- For students who are struggling with part (g), point out that the remaining quadratic polynomial is the same as one of the problems from the Opening Exercise.
- As an extension, ask students to create a polynomial equation that has three real and two complex solutions.

- We established earlier in the lesson that all quadratic expressions can be written in terms of two linear factors. How many factors did our cubic expression have?
  - *Three.*
- Is it true that every cubic expression can be factored into three linear factors?
  - *Yes, because a cubic equation will always have at least one real solution that corresponds to a linear factor of the expression. What is left over will be a quadratic expression, which can be written in terms of two linear factors.*

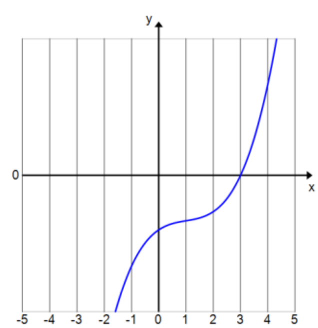
If students don't seem ready to answer the last question or are unsure of the answer, let them work through Exercise 1 and then re-address it.

Exercises 1–2 (6 minutes)

Give students time to work through the two exercises and then lead the discussion that follows.

Exercises 1–2

Write each polynomial in terms of linear factors. The graph of $y = x^3 - 3x^2 + 4x - 12$ is provided for Exercise 2.



1. $f(x) = x^3 + 5x$

   $f(x) = x(x + i\sqrt{5})(x - i\sqrt{5})$

2. $g(x) = x^3 - 3x^2 + 4x - 12$

   $g(x) = (x - 3)(x + 2i)(x - 2i)$

Discussion (3 minutes)

- Do your results from Exercises 1 and 2 agree with our conclusions from Example 1?
  - *Yes, each cubic function could be written as a product of three linear factors.*

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- Make a conjecture about what might happen if we factored a degree 4 polynomial. What about a degree 5 polynomial? Explain your reasoning.
  - *A degree 4 polynomial should have 4 linear factors. Based on the previous examples, it seems that a polynomial has as many linear factors as its degree. Similarly, a degree 5 polynomial should be able to be written as a product of 5 linear factors.*
- Our major conclusion in this lesson is a two-part theorem known as the Fundamental Theorem of Algebra (FTA). Part 1 of the Fundamental Theorem of Algebra says that every polynomial equation has at least one solution in the complex numbers. Does that agree with our experience?
  - *Yes.*

> *Scaffolding:*
> To meet a variety of student needs, ask students to:
> - Restate the FTA in their own words (either in writing or verbally).
> - Illustrate the FTA with an example. For instance, show that the FTA is true for some polynomial function.
> - Apply the FTA to some examples. For instance, how many linear factors would be in the factored form of $x^5 - 3x + 1$?

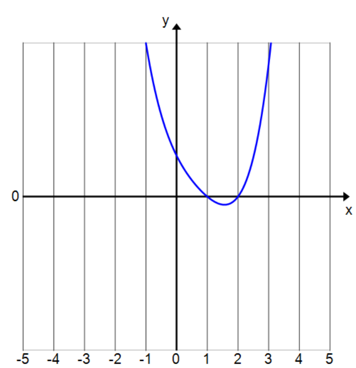
- Think about how we factor a polynomial expression P: We find one solution a to $P(x) = 0$, then we factor out the term $(x - a)$. We are left with a new polynomial expression of one degree lower than P, so we look for another solution, and repeat until we have factored everything into linear parts.
- Consider the polynomial $P(x) = x^4 - 3x^3 + 6x^2 - 12x + 8$ in the next example.

Example 2 (8 minutes)

While we do not have the mathematical tools or experience needed to prove the Fundamental Theorem of Algebra (either part), this example illustrates how the logic of the second part of the FTA works. Lead students through this example, allowing time for factoring and discussion at each step.

Example 2

Consider the polynomial function $P(x) = x^4 - 3x^3 + 6x^2 - 12x + 8$, whose corresponding graph $y = x^4 - 3x^3 + 6x^2 - 12x + 8$ is shown to the right. How many zeros does P have?



a. **Part 1 of the Fundamental Theorem of Algebra says that this equation will have at least one solution in the complex numbers. How does this align with what we can see in the graph to the right?**

Since the graph has 2 x-intercepts, there appear to be 2 zeros to the function. We were guaranteed one zero, but we know there are at least two.

b. **Identify one zero from the graph.**

One zero is 1. (The other is 2.)

c. **Use polynomial division to factor out one linear term from the expression $x^4 - 3x^3 + 6x^2 - 12x + 8$.**

$x^4 - 3x^3 + 6x^2 - 12x + 8 = (x - 1)(x^3 - 2x^2 + 4x - 8)$

d. **Now we have a cubic polynomial to factor. We know by part 1 of the Fundamental Theorem of Algebra that a polynomial function will have at least one real zero. What is that zero in this case?**

The original polynomial function had real zeros at 1 and 2, so the cubic function $P(x) = x^3 - 2x^2 + 4x - 8$ has a zero at 2.

e. **Use polynomial division to factor out another linear term of $x^4 - 3x^3 + 6x^2 - 12x + 8$.**

$x^4 - 3x^3 + 6x^2 - 12x + 8 = (x-1)(x^3 - 2x^2 + 4x - 8) = (x-1)(x-2)(x^2+4)$

f. **Are we done? Can we factor this polynomial any further?**

At this point, we can see that $x^2 + 4 = (x+2i)(x-2i)$, so

$$x^4 - 3x^3 + 6x^2 - 12x + 8 = (x-1)(x-2)(x+2i)(x-2i).$$

g. **Now that the polynomial is in factored form, we can quickly see how many solutions there are to the original equation $x^4 - 3x^3 + 6x^2 - 12x + 8 = 0$.**

If $x^4 - 3x^3 + 6x^2 - 12x + 8 = 0$, then $(x-1)(x-2)(x+2i)(x-2i) = 0$, so the solutions are 1, 2, $2i$ and $-2i$. So, the polynomial P has 4 zeros; 2 are real numbers, and 2 are complex numbers.

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h. **What if we had started with a polynomial function of degree 8?**

We would find the first zero, and factor out a linear term, leaving a polynomial of degree 7. We would then find another zero, factor out a linear term, leaving a polynomial of degree 6. We would repeat this process until we had a quadratic polynomial remaining; then, we would factor that with the help of the quadratic formula. We would have 8 linear factors at the end of the process that correspond to the 8 zeros of the original function.

The logic we just followed leads to part 2 of the Fundamental Theorem of Algebra, which is the result that we have already conjectured: A polynomial of degree $N \geq 1$ will factor into N linear factors with complex coefficients. Collectively, these two results are often just referred to as the Fundamental Theorem of Algebra. Although we have only worked with polynomials with real coefficients, the FTA applies to polynomial functions with real coefficients, such as $P(x) = x^3 + 2x^2 - 4$ as well as to polynomial functions with non-real coefficients, such as $P(x) = x^3 + 3ix^2 + 4 - 2i$. We have not attempted to justify the first part, but the students should be able to justify the second part of the theorem.

Fundamental Theorem of Algebra

1. Every polynomial function of degree $N \geq 1$ with real or complex coefficients has at least one real or complex zero.
2. Every polynomial of degree $N \geq 1$ with real or complex coefficients factors into N linear terms with real or complex coefficients.

- Why is the Fundamental Theorem of Algebra so "fundamental" to mathematics?
  - *The Fundamental Theorem says that the complex number system contains every zero of every polynomial function. We do not need to look anywhere else to find zeros to these types of functions.*

- Notice that the Fundamental Theorem just tells us that the factorization of the polynomial exists; it does not help us actually find it. If we had been given a polynomial function that did not have any real zeros, it would have been very hard to start the factorization process.

Closing (3 minutes)

- With a partner, summarize the key points of this lesson.
- What does the Fundamental Theorem of Algebra guarantee?
  - *A polynomial of degree $N \geq 1$ will factor into N linear factors, and the associated function will have N zeros, some of which may be repeated.*
- Why is this important?
  - *The Fundamental Theorem of Algebra ensures that there are as many zeros as we'd expect for a polynomial function, and that factoring will always (in theory) work to find solutions to polynomial equations.*
- Illustrate the Fundamental Theorem of Algebra with an example.

Lesson Summary

Every polynomial function of degree n, for $n \geq 1$, has n roots over the complex numbers, counted with multiplicity. Therefore, such polynomials can always be factored into n linear factors, and the obstacles to factoring we saw before have all disappeared in the larger context of allowing solutions to be complex numbers.

The Fundamental Theorem of Algebra:

1. **If P is a polynomial function of degree $n \geq 1$, with real or complex coefficients, then there exists at least one number r (real or complex) such that $P(r) = 0$.**
2. **If P is a polynomial function of degree $n \geq 1$, given by $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with real or complex coefficients a_i, then P has exactly n zeros $r_1, r_2, \ldots, r_n$ (not all necessarily distinct), such that $P(x) = a(x - r_1)(x - r_2)\cdots(x - r_n)$.**

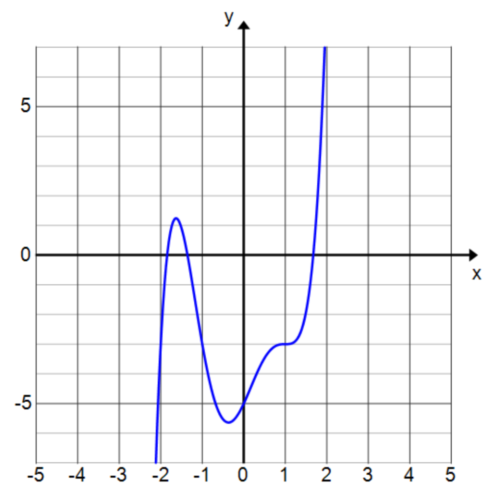
Exit Ticket (5 minutes)

Name ______________________________ Date ______________

Lesson 40: Obstacles Resolved—A Surprising Result

Exit Ticket

Consider the degree 5 polynomial function $P(x) = x^5 - 4x^3 + 2x^2 + 3x - 5$, whose graph is shown below. You do not need to factor this polynomial to answer the questions below.



1. How many linear factors is P guaranteed to have? Explain.

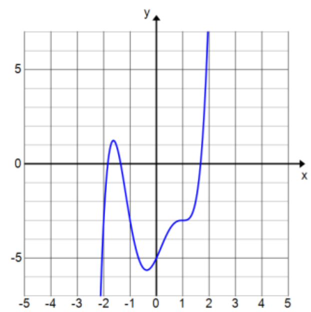
2. How many zeros does P have? Explain.

3. How many real zeros does P have? Explain.

4. How many complex zeros does P have? Explain.

Exit Ticket Sample Solutions

Consider the degree 5 polynomial function $P(x) = x^5 - 4x^3 + 2x^2 + 3x - 5$ whose graph is shown below. You do not need to factor this polynomial to answer the questions below.



1. How many linear factors is P guaranteed to have? Explain.

   The polynomial expression must have 5 linear factors. The Fundamental Theorem of Algebra guarantees that a polynomial function can be written in terms of linear factors and must have the same number of linear factors as its degree.

2. How many zeros does P have? Explain.

   Since P can be written in terms of 5 linear factors, the equation P must have 5 zeros (counted with multiplicity).

3. How many real zeros does P have? Explain.

   The graph crosses the x-axis 3 times, which means that three of the zeros are real numbers.

4. How many complex zeros does P have? Explain.

   Since P must have 5 total zeros and only 3 of them are real, there must be 2 complex zeros.

Problem Set Sample Solutions

1. Write each quadratic function below in terms of linear factors.

   a. $f(x) = x^2 - 25$

      $f(x) = (x+5)(x-5)$

   b. $f(x) = x^2 + 25$

      $f(x) = (x+5i)(x-5i)$

   c. $f(x) = 4x^2 + 25$

      $f(x) = (2x+5i)(2x-5i)$

   d. $f(x) = x^2 - 2x + 1$

      $f(x) = (x-1)(x-1)$

   e. $f(x) = x^2 - 2x + 4$

      $f(x) = (x - 1 + i\sqrt{3})(x - 1 - i\sqrt{3})$

2. Consider the polynomial function $P(x) = (x^2+4)(x^2+1)(2x+3)(3x-4)$.

   a. Express P in terms of linear factors.

      $P(x) = (x+2i)(x-2i)(x+i)(x-i)(2x+3)(3x-4)$

   b. Fill in the blanks of the following sentence.

      The polynomial P has degree ________ and can, therefore, be written in terms of ________ linear factors. The function P has ________ zeros. There are _______ real zeros and ________ complex zeros. The graph of $y = P(x)$ has ________ x-intercepts.

      The polynomial P has degree $\underline{6}$ and can, therefore, be written in terms of $\underline{6}$ linear factors. The function P has $\underline{6}$ solutions. There are $\underline{2}$ real zeros and $\underline{4}$ complex zeros. The graph of $y = P(x)$ has $\underline{2}$ x-intercepts.

3. Express each cubic function below in terms of linear factors.

   a. $f(x) = x^3 - 6x^2 - 27x$

   $f(x) = x(x-9)(x+3)$

   b. $f(x) = x^3 - 16x^2$

   $f(x) = x^2(x-16)$

   c. $f(x) = x^3 + 16x$

   $f(x) = x(x+4i)(x-4i)$

4. For each cubic function below, one of the zeros is given. Express each cubic function in terms of linear factors.

   a. $f(x) = 2x^3 - 9x^2 - 53x - 24$; $f(8) = 0$

   $f(x) = (x-8)(2x+1)(x+3)$

   b. $f(x) = x^3 + x^2 + 6x + 6$; $f(-1) = 0$

   $f(x) = (x+1)(x+i\sqrt{6})(x-i\sqrt{6})$

5. Determine if each statement is always true or sometimes false. If it is sometimes false, explain why it is not always true.

   a. A degree 2 polynomial function will have two linear factors.

   Always true.

   b. The graph of a degree 2 polynomial function will have two x-intercepts.

   False. It is possible for the solutions to a degree 2 polynomial to be complex, in which case the graph would not cross the x-axis. It is also possible for the graph to have only one x-intercept if the vertex lies on the x-axis.

   c. The graph of a degree 3 polynomial function might not cross the x-axis.

   False. A degree 3 polynomial must cross the x-axis at least one time.

   d. A polynomial function of degree n can be written in terms of n linear factors.

   Always true.

6. Consider the polynomial function $f(x) = x^6 - 9x^3 + 8$.

   a. How many linear factors does $x^6 - 9x^3 + 8$ have? Explain.

   Since the degree is 6, the polynomial must have 6 linear factors.

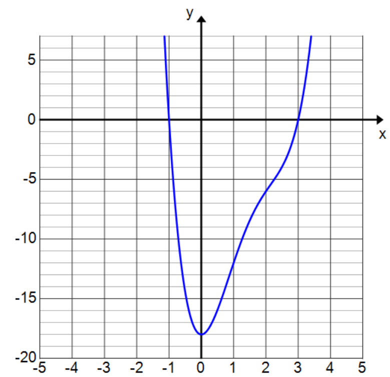
   b. How is this information useful for finding the zeros of f?

   We know that the function has 6 zeros since there are 6 linear factors. Each factor corresponds to a zero of the function.

   c. Find the zeros of f. (Hint: Let $Q = x^3$. Rewrite the equation in terms of Q to factor.)

   $1, 2, -1+i\sqrt{3}, -1-i\sqrt{3}, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}$

7. Consider the polynomial function $P(x) = x^4 - 6x^3 + 11x^2 - 18$.



   a. Use the graph to find the real zeros of P

      The real zeros are -1 and 3.

   b. Confirm that the zeros are correct by evaluating the function P at those values.

      $P(-1) = 0$ *and* $P(3) = 0$

   c. Express P in terms of linear factors.

      $P(x) = (x+1)(x-3)\left(x-\left(2+i\sqrt{2}\right)\right)\left(x-\left(2-i\sqrt{2}\right)\right)$

   d. Find all zeros of P.

      $-1, 3, 2-i\sqrt{2}, 2+i\sqrt{2}$

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8. Penny says that the equation $x^3 - 8 = 0$ has only one solution, $x = 2$. Use the Fundamental Theorem of Algebra to explain to her why she is incorrect.

   Because $x^3 - 8$ is a degree 3 polynomial, the Fundamental Theorem of Algebra guarantees that $x^3 - 8$ can be written as the product of three linear factors; therefore, the corresponding equation has 3 solutions. One of the 3 solutions is 2. We know that 2 cannot be the only solution because $(x-2)(x-2)(x-2) \neq x^3 - 8$.

9. Roger says that the equation $x^2 - 12x + 36 = 0$ has only one solution, 6. Regina says Roger is wrong and that the Fundamental Theorem of Algebra guarantees that a quadratic equation must have two solutions. Who is correct and why?

   Roger is correct. While the Fundamental Theorem of Algebra guarantees that a quadratic polynomial can be written in terms of two linear factors, the factors are not necessarily distinct. We know that $x^2 - 12x + 36 = (x-6)(x-6)$, so the equation $x^2 - 12x + 36 = 0$ has only one solution, which is 6.