Lesson 39: Factoring Extended to the Complex Realm

Student Outcomes

- Students solve quadratic equations with real coefficients that have complex solutions. Students extend polynomial identities to the complex numbers.
- Students note the difference between solutions to the equation and the x-intercepts of the graph of said equation.

Lesson Notes

This lesson extends the factoring concepts and techniques covered in Topic B of this module to the complex number system and specifically addresses **N-CN.C.7**. Students will learn how to solve and express the solutions to any quadratic equation. Students observe that complex solutions to polynomial equations with real coefficients occur in conjugate pairs, and that only real solutions to polynomial equations are also the x-intercepts of the graph of the related polynomial function. In essence, this is the transition lesson to the next lesson on factoring all polynomials into linear factors.

Classwork

Opening (1 minute)

Since your introduction to the complex number system, you have hopefully recognized its theme of sharing arithmetic and algebraic properties with the real numbers. Today, we extend factoring polynomial expressions and finding solutions to polynomial equations to the complex realm.

Opening Exercise (8 minutes)

Have students individually complete this opening exercise. Students will eventually identify the expressions in this exercise as cases of the identity $(x + ai)(x - ai) = x^2 + a^2$. But, for now, allow them the experience of the algebra and to confirm for themselves that the imaginary terms combine to 0 in each example, resulting in polynomials in standard form with real coefficients. Invite students to the board to write their solutions, and let the class have the first opportunity to correct any mistakes, should it be necessary.

Opening Exercise

Rewrite each expression as a polynomial in standard form.

a. $(x+i)(x-i)$

$$\begin{aligned}(x+i)(x-i) &= x^2 + ix - ix - i^2\\ &= x^2 - i^2\\ &= x^2 - (-1)\\ &= x^2 + 1\end{aligned}$$

b. $(x+5i)(x-5i)$

$$\begin{aligned}(x+5i)(x-5i) &= x^2+5ix-5ix-25i^2\\ &= x^2-25i^2\\ &= x^2-25(-1)\\ &= x^2+25\end{aligned}$$

c. $(x-(2+i))(x-(2-i))$

$$\begin{aligned}\big(x-(2+i)\big)\big(x-(2-i)\big) &= x^2-(2+i)x-(2-i)x+[(2+i)(2-i)]\\ &= x^2-2x-ix-2x+ix+[4-i^2]\\ &= x^2-4x+[4—(-1)]\\ &= x^2-4x+5\end{aligned}$$

Discussion (5 minutes)

Here we begin a dialogue that discusses patterns and regularity observed in the Opening Exercise. As you pose each question, give students time to discuss them with a partner or in their small groups. To encourage students to be accountable for responding to questions during discussion, you can have them write answers on personal whiteboards, show a thumbs-up when they have an idea, whisper their idea to a partner before asking for a response with the whole group, or show their agreement or disagreement to a question by showing a thumbs up/down.

MP.7

- Do you observe any patterns among parts (a)–(c) in the opening exercise?
  - *After each expression is expanded and like terms are collected, we have quadratic polynomials with real coefficients. The imaginary terms were opposites and combined to* 0.
- How could you generalize the patterns into a rule (or identity)?
  - *Parts (a) and (b) are instances of the identity*
    $(x+ai)(x-ai)=x^2+a^2$.
- What about part (c)? Do you notice an instance of the same identity?
  - *Yes.*
    $\big(x-(2+i)\big)\big(x-(2-i)\big)=\big((x-2)-i\big)\big((x-2)+i\big)$
- Where have we seen a similar identity to $(x+ai)(x-ai)=x^2+a^2$?
  - *Recall the polynomial identity* $(x+a)(x-a)=x^2-a^2$ *from Lesson 6.*
- Recall the quick mental arithmetic we learned in Lesson 7. Can you compute $(3+2i)(3-2i)$ really quickly?
  - *Yes.* $(3+2i)(3-2i)=3^2+2^2=9+4=13$
- How about $(9+4i)(9-4i)$?
  - $(9+4i)(9-4i)=9^2+4^2=81+16=97$

Exercises 1–2 (5 minutes)

Students understand that the expansion of $(x+ai)(x-ai)$ is a polynomial with real coefficients; the imaginary terms disappear when working through the algebra. Now, students are expected to understand this process in reverse; in other words, they factor polynomials with real coefficients but complex factors.

Exercises 1–4

Completely factor the following polynomial expressions.

1. $x^2 + 9$

$$x^2 + 9 = (x + 3i)(x - 3i)$$

2. $x^2 + 5$

$$x^2 + 5 = (x + i\sqrt{5})(x - i\sqrt{5})$$

Discussion (6 minutes)

This discussion is the introduction to conjugate pairs in the context of complex numbers. Relate this idea back to the idea of conjugate pairs for radical expressions from Lesson 29.

- In Lesson 29, we saw that the conjugate of an expression such as $x + \sqrt{5}$ is the expression $x - \sqrt{5}$, and if we multiply a radical expression by its conjugate, the result is a rational expression—the radical part disappears.
$$(x + \sqrt{5})(x - \sqrt{5}) = x^2 - x\sqrt{5} + x\sqrt{5} - 5 = x^2 - 5$$
- Analogously, complex numbers $a + bi$ have conjugates. The conjugate of $a + bi$ is $a - bi$. Then, we see that
$$(a + bi)(a - bi) = a^2 - abi + abi - b^2i^2 = a^2 + b^2.$$
- We have observed, for real values of x and a, that the expression $(x + ai)(x - ai)$ is a real number. The factors $(x + ai)$ and $(x - ai)$ form a conjugate pair.

Scaffolding:

- Show conjugate pairs graphically by graphing (as in the previous lessons) a parabola with 0, 1, and 2 solutions and cubic curves with a various number of solutions. Let student determine visually what is possible.
- Additionally, consider having students complete a Frayer diagram for the term "conjugate."
- As an extension, ask students to generate conjugate pairs.

Quadratic expressions with real coefficients, as we have seen, can be decomposed into real factors or non-real complex factors. However, non-real factors must be members of a conjugate pair; hence, a quadratic expression with real coefficients cannot have exactly one complex factor.

- Similarly, quadratic equations can have real or non-real complex solutions. If there are complex solutions, they will be conjugates of each other.
- Can a polynomial equation with real number coefficients have just one complex solution?
  - *No. If there is a complex solution, then the conjugate is also a solution. Complex solutions come in pairs.*
- Now, can a polynomial equation have real and non-real solutions?
  - *Yes, as long as all non-real complex solutions occur in conjugate pairs.*
  - *For example, the polynomial equation $(x^2 + 1)(x^2 - 1) = 0$ has two real solutions, 1 and -1, and two complex solutions. The complex solutions, i and $-i$, form a conjugate pair.*
- If you know that $2i$ is a solution to the polynomial equation $P(x) = 0$, can you tell me another solution?
  - *Complex solutions come in conjugate pairs, so if $2i$ is a solution to the equation, then its conjugate, $-2i$, is also a solution.*

At this point, have students write down or discuss with their neighbors what they have learned so far. The teacher should walk around the room and check for understanding.

Exercise 3 (6 minutes)

Students should work in groups of 2–4 on this exercise. Invite students to the board to present their solutions.

Scaffolding:

- Consider having groups work Exercise 3 with different polynomials:

$4x^3 - x$
$x^3 - 4x^2 + 29x$
$x^3 - 6x^2 + 25x$
$x^4 - 1$
$x^4 + 3x^2 - 4$
$x^4 - 4x^3 + 5x^2$
$x^4 - 4x^3 + 13x^2$
$x^4 - 2x^3 - 10x^2$
$x^4 + 13x^2 + 36$

3. Consider the polynomial $P(x) = x^4 - 3x^2 - 4$.

a. What are the solutions to $x^4 - 3x^2 - 4 = 0$?

$$x^4 - 3x^2 - 4 = 0$$
$$(x^2)^2 - 3x^2 - 4 = 0$$
$$(x^2 + 1)(x^2 - 4) = 0$$
$$(x + i)(x - i)(x + 2)(x - 2) = 0$$

The solutions are $-i$, i, -2, and 2.

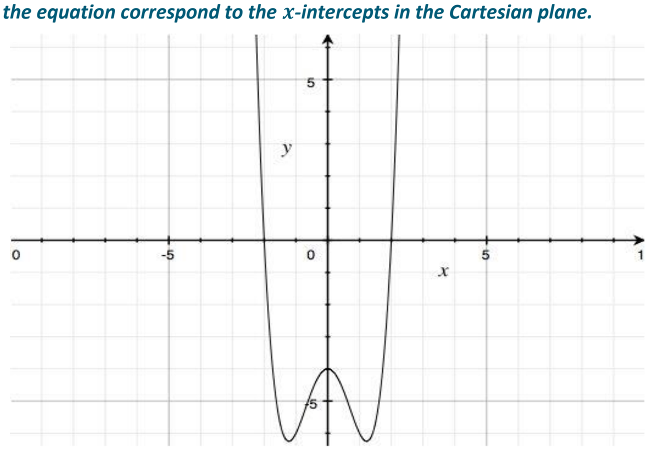
b. How many x-intercepts does the graph of the equation $y = x^4 - 3x^2 - 4$ have? What are the coordinates of the x-intercepts?

The graph of $y = x^4 - 3x^2 - 4$ has two x-intercepts: $(-2, 0)$ and $(2, 0)$.

MP.3

c. Are solutions to the polynomial equation $P(x) = 0$ the same as the x-intercepts of the graph of $y = P(x)$? Justify your reasoning.

No. Only the real solutions to the equation are x-intercepts of the graph. By comparing the graph of the polynomial in part (b) to the equation's solutions from part (c), you can see that only the real number solutions to the equation correspond to the x-intercepts in the Cartesian plane.



Exercise 4 (5 minutes)

Transition students to the next exercise by announcing that we now want to reverse our thinking. In the previous problem, we solved an equation to find the solutions. Now, pose the question: Can we construct an equation if we know its solutions? You may want to remind students that when a polynomial equation is written in factored form $(x - r_1)(x - r_2) \dots (x - r_n) = 0$, the solutions to the equation are $r_1, r_2, \dots, r_n$. Students will apply what they learned in the previous exercise to create a polynomial equation given its solutions. The problems scaffold from easier to more difficult. Students are encouraged to rewrite the factored form to show the polynomial in standard form for additional practice with complex numbers, but you may choose to have them leave the polynomial in factored form if time is a concern. Have students work with a partner on this exercise.

4. Write a polynomial P with the lowest possible degree that has the given solutions. Explain how you generated each answer.

a. $-2, 3, -4i, 4i$

The polynomial P has two real zeroes and two complex zeros. Since the two complex zeros are members of a conjugate pair, P may have as few as four total factors. Therefore, P has degree at least 4.

$$\begin{aligned} P(x) &= (x+2)(x-3)(x+4i)(x-4i) \\ &= (x^2-x-6)(x^2-16i^2) \\ &= (x^2-x-6)(x^2+16) \\ &= x^4-x^3-6x^2+16x^2-16x-96 \\ &= x^4-x^3+10x^2-16x-96 \end{aligned}$$

b. $-1, 3i$

The polynomial P has one real zero and two complex zeros because complex zeros come in pairs. Since $3i$ and $-3i$ form a conjugate pair, P has at least three total factors. Therefore, P has degree at least 3.

$$\begin{aligned} P(x) &= (x+1)(x-3i)(x+3i) \\ &= (x+1)(x^2-9i^2) \\ &= (x+1)(x^2+9) \\ &= x^3+x^2+9x+9 \end{aligned}$$

c. $0, 2, 1+i, 1-i$

Since $1+i$ and $1-i$ are complex conjugates, P is at least a 4th degree polynomial.

$$\begin{aligned} P(x) &= x(x-2)\big(x-(1+i)\big)\big(x-(1-i)\big) \\ &= x(x-2)[(x-1)-i][(x-1)+i] \\ &= x(x-2)[(x-1)^2-i^2] \\ &= x(x-2)[(x^2-2x+1)+1 \\ &= x(x-2)(x^2-2x+2) \\ &= x(x^3-2x^2+2x-2x^2+4x-4) \\ &= x(x^3-4x^2+6x-4) \\ &= x^4-4x^3+6x^2-4x \end{aligned}$$

d. $\sqrt{2}, -\sqrt{2}, 3, 1+2i$

Since$1+2i$ is a complex solution to $P(x)=0$, its conjugate, $1-2i$, must also be a complex solution. Thus, P is at least a fifth-degree polynomial.

$$\begin{aligned} P(x) &= \big(x-\sqrt{2}\big)\big(x+\sqrt{2}\big)(x-3)\big(x-(1+2i)\big)\big(x-(1-2i)\big) \\ &= (x^2-2)(x-3)[(x-1)-2i][(x-1)+2i] \\ &= (x^2-2)(x-3)[(x-1)^2-4i^2] \\ &= (x^2-2)(x-3)[(x^2-2x+1)+4] \\ &= (x^2-2)(x-3)(x^2-2x+5) \\ &= (x^3-3x^2-2x+6)(x^2-2x+5) \\ &= x^5-5x^4+9x^3-5x^2-22x+30 \end{aligned}$$

e. $2i, 3 - i$

The complex conjugates of $2i$ and $3 - i$ are $-2i$ and $3 + i$, respectively. So, P is at least a fourth-degree polynomial.

$$\begin{aligned} P(x) &= (x - 2i)(x + 2i)\big(x - (3 - i)\big)\big(x - (3 + i)\big) \\ &= (x^2 - 4i^2)[(x - 3) + i][(x - 3) - i] \\ &= (x^2 + 4)[(x - 3)^2 - i^2] \\ &= (x^2 + 4)[(x^2 - 6x + 9) + 1] \\ &= (x^2 + 4)(x^2 - 6x + 10) \\ &= x^4 - 6x^3 + 14x^2 - 24x + 40 \end{aligned}$$

Closing (3 minutes)

Have students break into small groups to discuss what they learned today. Today's lesson is summarized in the box below.

Lesson Summary

- **Polynomial equations with real coefficients can have real or complex solutions or they can have both.**
- **Complex solutions to polynomial equations with real coefficients are always members of a conjugate pair.**
- **Real solutions to polynomial equations are the same as x-intercepts of the associated graph, but complex solutions are not.**

Exit Ticket (6 minutes)

In this Exit Ticket, students solve quadratic equations with real and complex solutions.

Name ____ Date ____________________

Lesson 39: Factoring Extended to the Complex Realm

Exit Ticket

1. Solve the quadratic equation $x^2 + 9 = 0$. What are the x-intercepts of the graph of the function $f(x) = x^2 + 9$?

2. Find the solutions to $2x^5 - 5x^3 - 3x = 0$. What are the x-intercepts of the graph of the function $f(x) = 2x^5 - 5x^3 - 3x$?

Exit Ticket Sample Solutions

1. Solve the quadratic equation $x^2 + 9 = 0$. What are the x-intercepts of the graph of the function $f(x) = x^2 + 9$?

$$x^2 + 9 = 0$$
$$x^2 = -9$$
$$x = \sqrt{-9} \text{ or } x = -\sqrt{-9}$$
$$x = 3\sqrt{-1} \text{ or } x = -3\sqrt{-1}$$
$$x = 3i \text{ or } x = -3i$$

The x-intercepts of the graph of the function $f(x) = x^2 + 9$ are any real solutions to the equation $x^2 + 9 = 0$. However, since both solutions to $x^2 + 9 = 0$ are not real, the function $f(x) = x^2 + 9$ does not have any x-intercepts.

2. Find the solutions to $2x^5 - 5x^3 - 3x = 0$. What are the x-intercepts of the graph of the function $f(x) = 2x^5 - 5x^3 - 3x$?

$$(2x^4 - 5x^2 - 3) = 0$$
$$x(x^2 - 3)(2x^2 + 1) = 0$$
$$x(x + \sqrt{3})(x - \sqrt{3})(2x^2 + 1) = 0$$
$$x(x + \sqrt{3})(x - \sqrt{3})\left(x + \frac{i\sqrt{2}}{2}\right)\left(x - \frac{i\sqrt{2}}{2}\right) = 0$$

Thus, $x = 0$, $x = -\sqrt{3}$, $x = \sqrt{3}$, $x = -\frac{i\sqrt{2}}{2}$, *or* $x = \frac{i\sqrt{2}}{2}$.

The solutions are $0, \sqrt{3}, -\sqrt{3}, \frac{i\sqrt{2}}{2}$, *and* $-\frac{i\sqrt{2}}{2}$.

The x-intercepts of the graph of the function $f(x) = 2x^5 - 5x^3 - 3x$ are the real solutions to the equation $2x^5 - 5x^3 - 3x = 0$, so the x-intercepts are 0, $\sqrt{3}$, and $-\sqrt{3}$.

Problem Set Sample Solutions

1. Rewrite each expression in standard form.

a. $(x + 3i)(x - 3i)$

$x^2 + 3^2 = x^2 + 9$

b. $(x - a + bi)(x - (a + bi))$

$$\begin{aligned}(x - a + bi)(x - (a + bi)) &= ((x - a) + bi)((x - a) - bi)\\ &= (x - a)^2 + b^2\\ &= x^2 - 2ax + a^2 + b^2\end{aligned}$$

c. $(x + 2i)(x - i)(x + i)(x - 2i)$

$$\begin{aligned}(x + 2i)(x - 2i)(x + i)(x - i) &= (x^2 + 2^2)(x^2 + 1^2)\\ &= (x^2 + 4)(x^2 + 1)\\ &= x^4 + 5x^2 + 4\end{aligned}$$

d. $(x+i)^2 \cdot (x-i)^2$

$$\begin{aligned}(x+i)(x-i)\cdot(x+i)(x-i) &= (x^2+1)(x^2+1)\\ &= x^4+2x^2+1\end{aligned}$$

2. **Suppose in Problem 1 that you had no access to paper, writing utensils, or technology. How do you know that the expressions in parts (a)–(d) are polynomials with real coefficients?**

In part (a), the identity $(x+ai)(x-ai)=x^2+a^2$ can be applied. Since the number a is real, the resulting polynomial will have real coefficients. The remaining three expressions can all be rearranged to take advantage of the conjugate pairs identity. In parts (c) and (d), regrouping terms will produce products of polynomial expressions with real coefficients, which will again have real coefficients.

3. **Write a polynomial equation of degree 4 in standard form that has the solutions $i, -i, 1, -1$.**

The first step is writing the equation in factored form:

$$(x+i)(x-i)(x+1)(x-1)=0.$$

Then, use the commutative property to rearrange terms and apply the difference of squares formula:

$$\begin{aligned}(x+i)(x-i)(x+1)(x-1) &= (x^2+1)(x^2-1)\\ &= x^4-1.\end{aligned}$$

So, the standard form of the equation is

$$x^4-1=0.$$

4. **Explain the difference between x-intercepts and solutions to an equation. Give an example of a polynomial with real coefficients that has twice as many solutions as x-intercepts. Write it in standard form.**

The x-intercepts are the real solutions to a polynomial equation with real coefficients. The solutions to an equation can be real or not real. The previous problem is an example of a polynomial with twice as many solutions than x-intercepts. Or, we could consider the equation $x^4-6x^3+13x^2-12x+4=0$, which has zeros of multiplicity 2 at both 1 and 2.

5. **Find the solutions to $x^4-5x^2-36=0$ and the x-intercepts of the graph of $y=x^4-5x^2-36$.**

$$(x^2+4)(x^2-9)=0$$
$$(x+2i)(x-2i)(x+3)(x-3)=0$$

Since the solutions are $2i$, $-2i$, 3, and -3, and only real solutions to the equation are x-intercepts of the graph, the x-intercepts are 3 and -3.

6. **Find the solutions to $2x^4-24x^2+40=0$ and the x-intercepts of the graph of $y=2x^4-24x^2+40$.**

$$2(x^4-12x^2+20)=0$$
$$2(x^2-10)(x^2-2)=0$$

Since all of the solutions $\sqrt{10}$, $-\sqrt{10}$, $\sqrt{2}$ and $-\sqrt{2}$ are real numbers, the x-intercepts of the graph are $\sqrt{10}$, $-\sqrt{10}$, $\sqrt{2}$ and $-\sqrt{2}$.

7. **Find the solutions to $x^4-64=0$ and the x-intercepts of the graph of $y=x^4-64$.**

$$(x^2+8)(x^2-8)=0$$
$$(x+\sqrt{8}i)(x-\sqrt{8}i)(x+\sqrt{8})(x-\sqrt{8})=0$$

The x-intercepts are $2\sqrt{2}$ and $2-\sqrt{2}$.

8. Use the fact that $x^4 + 64 = (x^2 - 4x + 8)(x^2 + 4x + 8)$ to explain how you know that the graph of $y = x^4 + 64$ has no x-intercepts. You need not find the solutions.

The x-intercepts of $y = x^4 + 64$ are solutions to $(x^2 - 4x + 8)(x^2 + 4x + 8) = 0$. Both $x^2 - 4x + 8 = 0$ and $x^2 + 4x + 8 = 0$ have negative discriminant values of -16, so the equations $x^2 - 4x + 8 = 0$ and $x^2 + 4x + 8 = 0$ have no real solutions. Thus, the equation $x^4 + 64 = 0$ has no real solutions, and the graph of $y = x^4 + 64$ has no x-intercepts.

Since $x^4 + 64 = 0$ has no real solutions, the graph of $y = x^4 + 64$ has no x-intercepts.