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Lesson 37: A Surprising Boost from Geometry

Student Outcomes

* Students define a complex number in the form , where and are real numbers and the imaginary unit satisfies . Students geometrically identify as a multiplicand effecting a counterclockwise rotation of the real number line. Students locate points corresponding to complex numbers in the complex plane.
* Students understand complex numbers as a superset of the real numbers; i.e., a complex number is real when . Students learn that complex numbers share many similar properties of the real numbers: associative, commutative, distributive, addition/subtraction, multiplication, etc.

Lesson Notes

Students first receive an introduction to the imaginary unit and develop an algebraic and geometric understanding of the complex numbers (**N-CN.A.1**). The lesson then underscores that complex numbers also satisfy the properties of operations that real numbers do (**N-CN.A.2**). Finally, students perform exercises to reinforce their understanding of and facility with complex numbers in an algebraic arena. This lesson ties into the work in the next lesson, which involves complex solutions to quadratic equations (**N-CN.C.7**).

Complex numbers are neither *imaginary*, as in make believe, nor *complex*, as in complicated. Students first encounter them when they classify equations such as as having no real number solutions. At that point, we do not introduce the possibility that a solution exists within a superset of the real numbers called the complex numbers. At the end of this module, we briefly introduce the idea that every polynomial of degree has values for which where is a whole number and is a real or complex number. Further, in preparation for students’ work in Precalculus, we state (but do not expect students to know) that can be written as the product of linear factors, a result known as the Fundamental Theorem of Algebra. The usefulness of complex numbers as solutions to polynomial equations comes with a cost: While real numbers can be ordered (put in order from smallest to greatest), complex numbers cannot be compared; for example, the complex number is not larger or smaller than . However, this is a small price to pay. Students will begin to see just how important complex numbers are to geometry and computer science in Modules 1 and 2 in Precalculus. In college level science and engineering courses, complex numbers are used in conjunction with differential equations to model circular motion and periodic phenomena in two dimensions.

Classwork

Opening (1 minute)

We introduce a geometric context for complex numbers by demonstrating the analogous relationship between rotations in the plane and multiplication. The intention is for students to develop a deep understanding of through geometry.

* Today, we define a new number system that allows us to identify solutions to some equations that have no real number solutions. The complex numbers, as you will see, in fact share many properties with the real numbers with which you are familiar. We will be taking a geometric approach to introducing complex numbers.

Opening Exercise (5 minutes)

Have students work alone on this motivating Opening Exercise. This exercise provides the context and invites the necessity for defining an alternative number system, namely the complex numbers. Go over parts (a), (b), and (c) with the class; then, suggest that part (d) may be solvable using an alternative number system. Have students table this thought while beginning a geometrically-oriented discussion.

*Scaffolding:*

* There were times in the past when people would have said that an equation such as also had no solution.

**Opening Exercise**

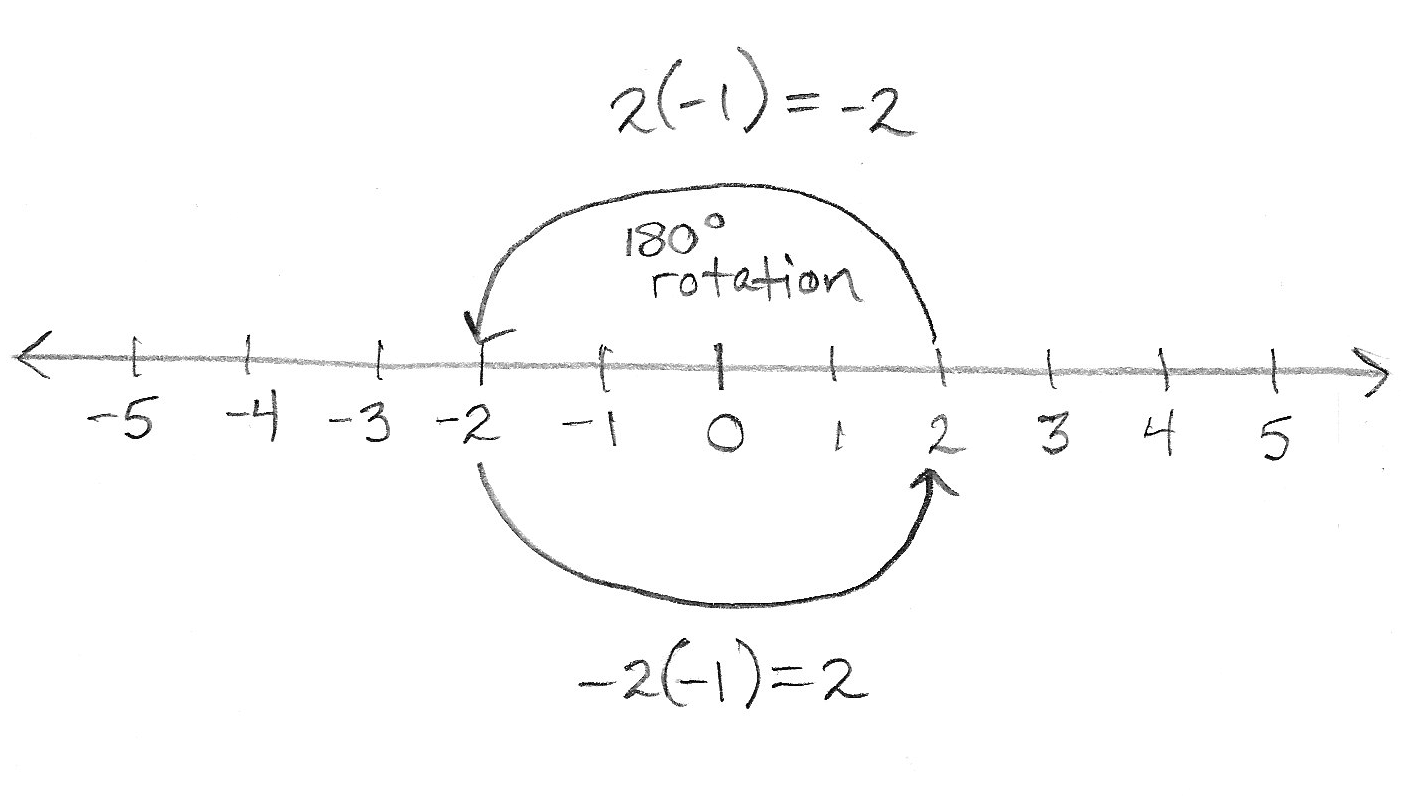
**Solve each equation for .**

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|  |  |
|  | **No real solution** |

Discussion (20 minutes)

Before beginning, allow students to prepare graph paper for drawing images as the discussion unfolds. At the close of this discussion, have students work with partners to summarize at least one thing they learned; then, provide time for some teacher-guided note-taking to capture the definition of the imaginary unit and its connection to geometric rotation.

Recall that multiplying by rotates the number line in the plane by about the point .

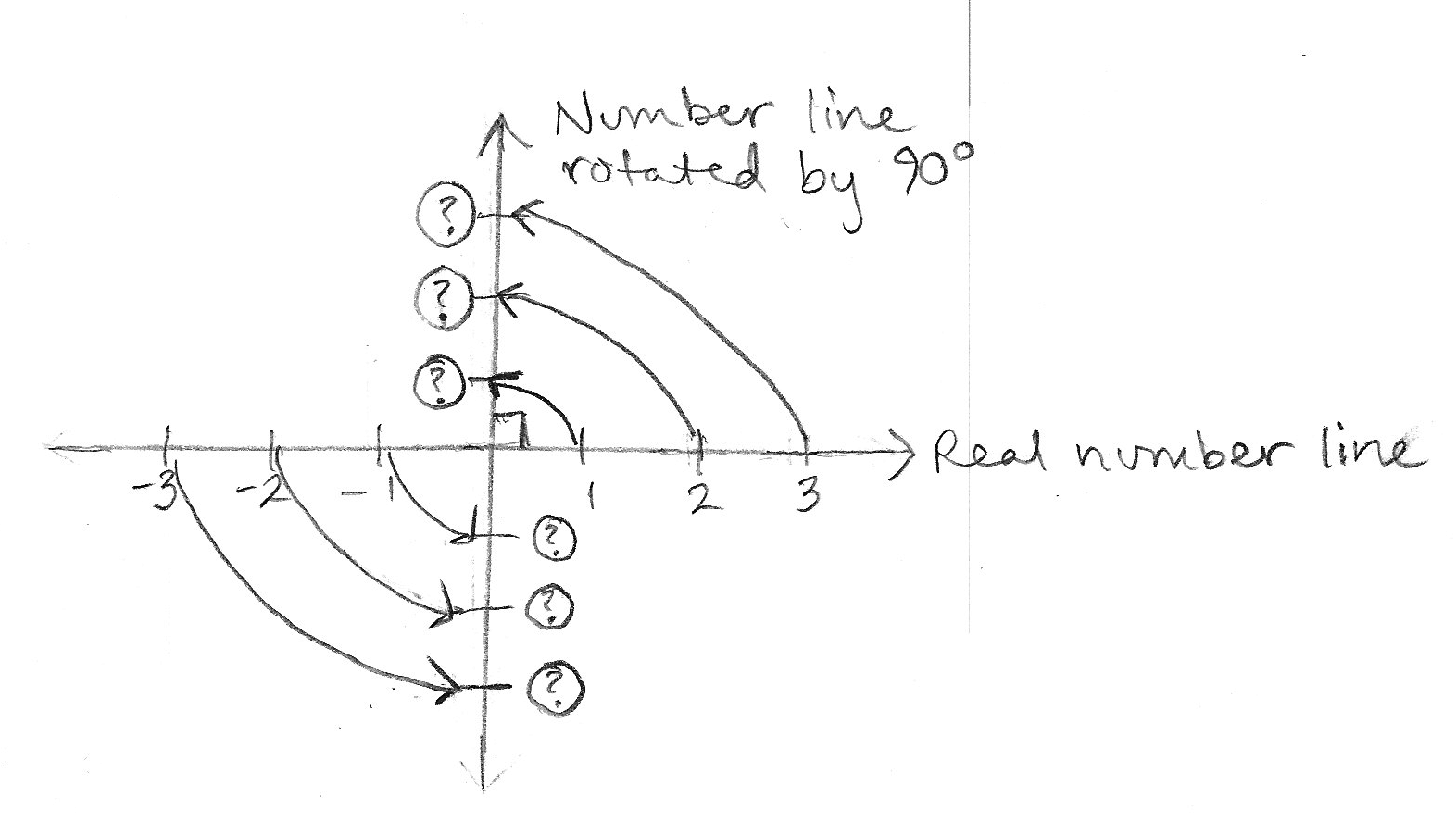


*Scaffolding:*

* You can demonstrate the rotation concept by drawing the number line carefully on a piece of white paper, drawing an identical number line on a transparency, putting a pin at zero, and rotating the transparency to show that the number line is rotating. For example, go from to . This, of course, is the same as multiplying by .

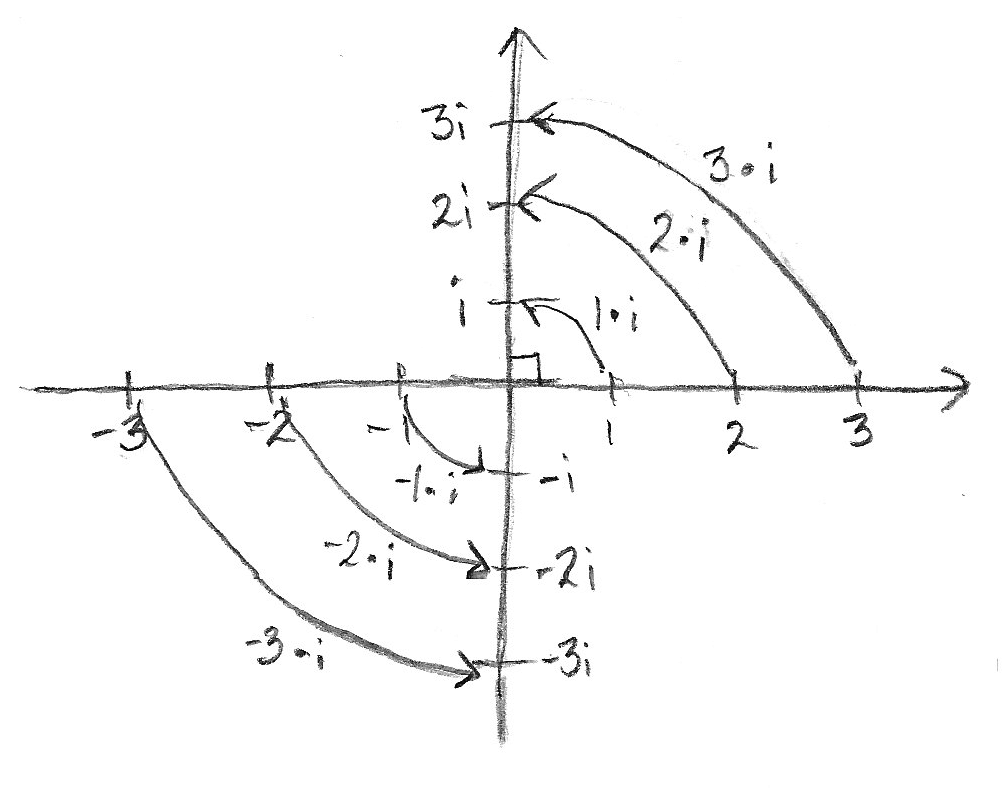
Pose this interesting thought question to students: Is there a number we can multiply by that corresponds to a rotation?

Students may find that this is a strange question. First, such a number *does not* take the number line to itself, so we have to *imagine* another number line that is a rotation of the original:

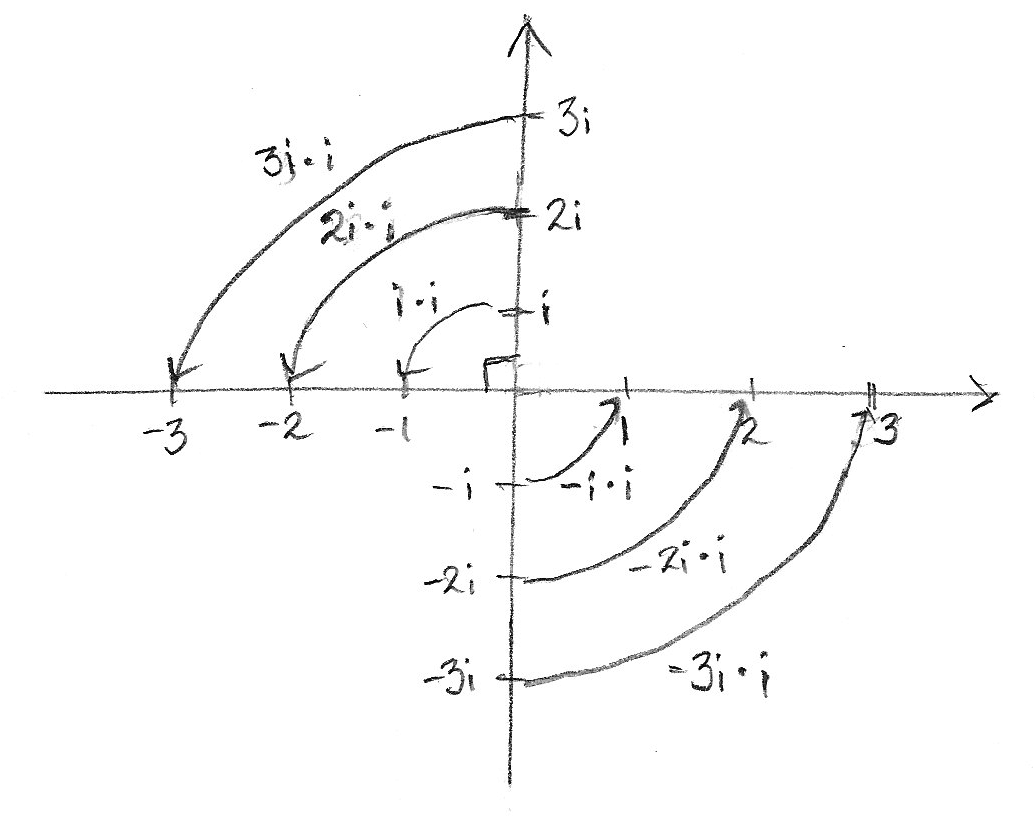


This is like the coordinate plane. However, how should we label the points on the vertical axis?

Well, since we *imagined* such a number existed, let’s call it the imaginary axis and subdivide it into units of something called . Then, the point on the number line rotates to on the rotated number line and so on, as follows:



* What happens if we multiply a point on the vertical number line by ?
  + *We rotate that point by counterclockwise:*

**

When we perform two rotations, it is the same as performing a rotation, so multiplying by twice results in the same rotation as multiplying by . Since two rotations by is the same as a single rotation by , two rotations by is equivalent to multiplication by twice, and one rotation by is equivalent to multiplication by we have

**MP.2**

for any real number thus,

* Why might this new number be useful?
  + *Recall from the Opening Exercise that there are no real solutions to the equation  
    However, this new number is a solution.*

In fact, “solving” the equation , we get

However, because we know from above that , and , we have two solutions to the quadratic equation , which are and .

These result suggests that “.” That seems a little weird, but this new imagined number already appears to solve problems we could not solve before.

For example, in Algebra I, when we applied the quadratic formula to  
we found that

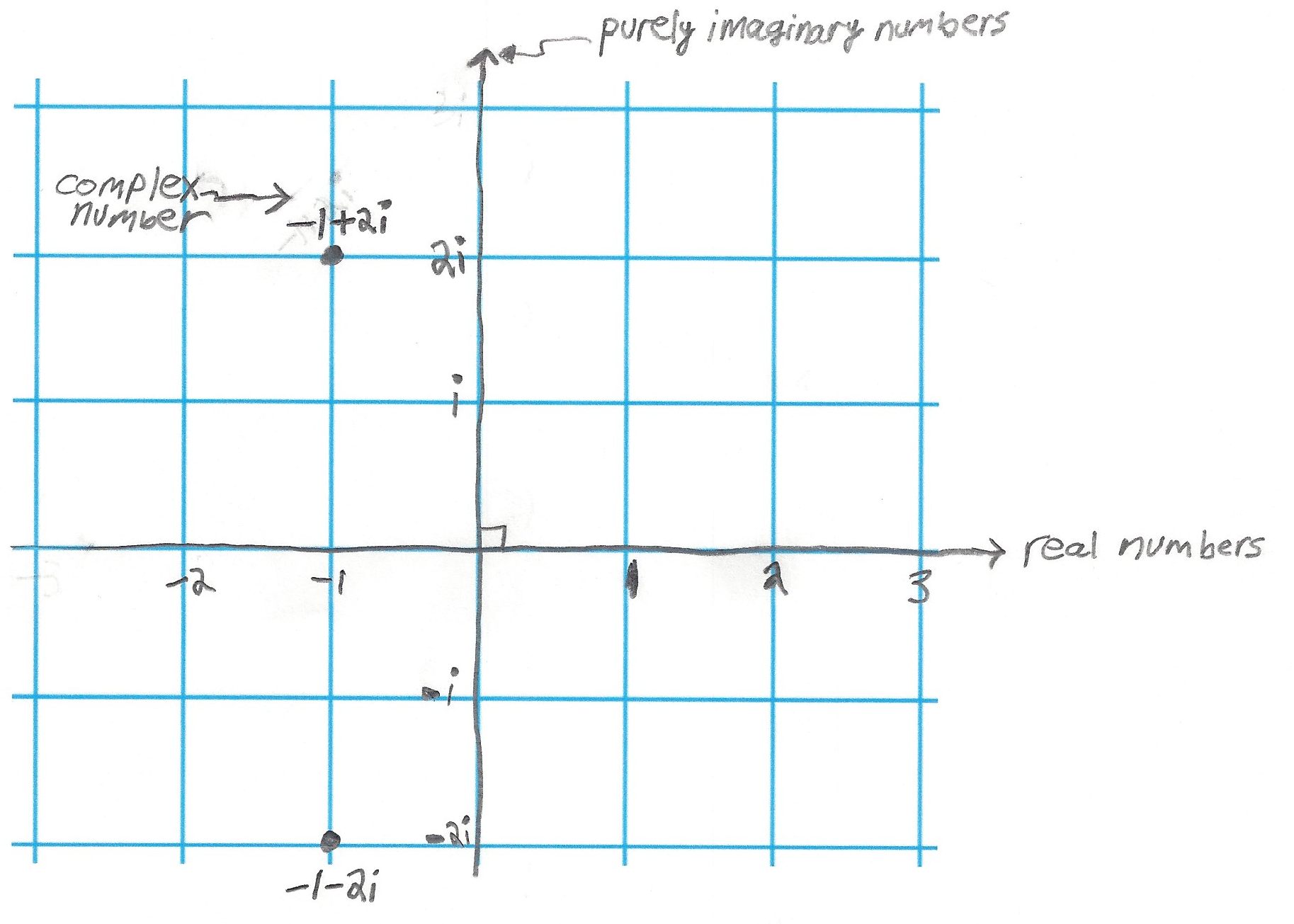
or

or.

Recognizing the negative number under the square root, we reported that the equation has no real solutions. Now, however, we can write

Therefore, or , which means and are the solutions to

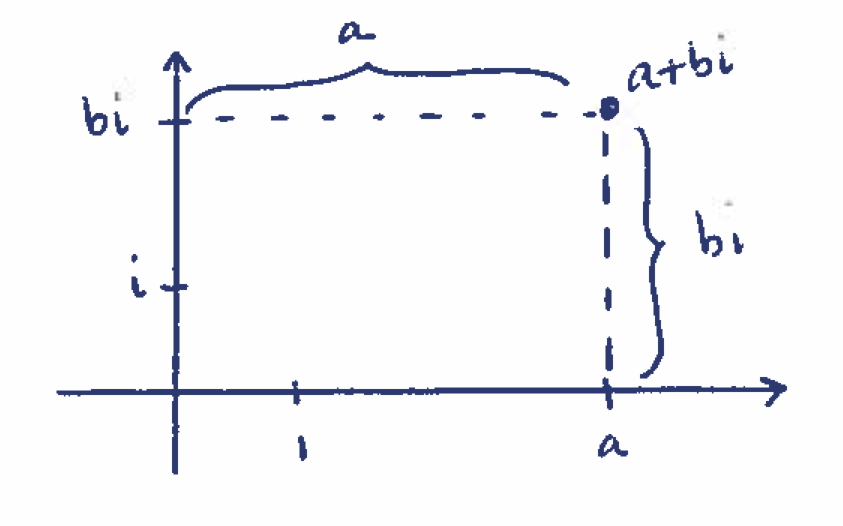
The solutions and are numbers called complex numbers, which we can locate in the complex plane.



*Scaffolding:*

* Name a few complex numbers for students to plot on their graph paper. This will build an understanding of their locations in this coordinate system. For example, consider , , , , and . Make sure students are also cognizant of the fact that real numbers are also complex numbers, e.g., , , , .

In fact, all complex numbers can be written in the form

where and are real numbers. Just as we can represent real numbers on the number line, we can represent complex numbers in the complex plane. Each complex number can be located in the complex plane in the same way we locate the point in the Cartesian plane. From the origin, translate units horizontally along the real axis and units vertically along the imaginary axis.

Since complex numbers are built from real numbers, we should be able to add, subtract, multiply, and divide them. They should also satisfy the commutative, associative, and distributive properties, just as real numbers do.

Let’s check how some of these operations work for complex numbers.

Examples 1–2 (4 minutes): Addition and Subtraction with Complex Numbers

Addition of variable expressions is a matter of re-arranging terms according to the properties of operations. Often, we call this “combining like terms.” These properties of operations apply to complex numbers.

**MP.7**

Example 1: Addition with Complex Numbers

Compute .

Example 2: Subtraction with Complex Numbers

Compute .

*Scaffolding:*

If necessary, further examples of addition and multiplication with complex numbers are as follows:

Examples 3-4 (6 minutes): Multiplication with Complex Numbers

Multiplication uses the properties of operations and the fact that . It is analogous to polynomial multiplication.

**MP.7**

Example 3: Multiplication with Complex Numbers

Compute .

Example 4: Multiplication with Complex Numbers

Verify that and are solutions to .

:

:

So, both complex numbers and are solutions to the quadratic equation .

Closing (4 minutes)

Close by asking students to write or discuss with a neighbor some reasons for defining the set of complex numbers in the first place. Have them explain the importance of complex numbers satisfying the arithmetic properties of real numbers. How does geometry help explain ?

The Lesson Summary box presents key findings from today’s lesson.

Lesson Summary

Multiplying by rotates every complex number in the complex plane by about the origin.

Every complex number is in the form , where is the real part and is the imaginary part of the number. Real numbers are also complex numbers; the real number can be written as the complex number .

Adding two complex numbers is analogous to combining like terms in a polynomial expression.

Multiplying two complex numbers is like multiplying two binomials, except one can use to further write the expression in simpler form.

Complex numbers satisfy the associative, commutative, and distributive properties.

Complex numbers can now allow us to find solutions to equations that previously had no real number solutions.

Exit Ticket (5 minutes)

In this Exit Ticket, students reduce a complex expression into its form and then locate the corresponding point on the complex plane.

Name Date

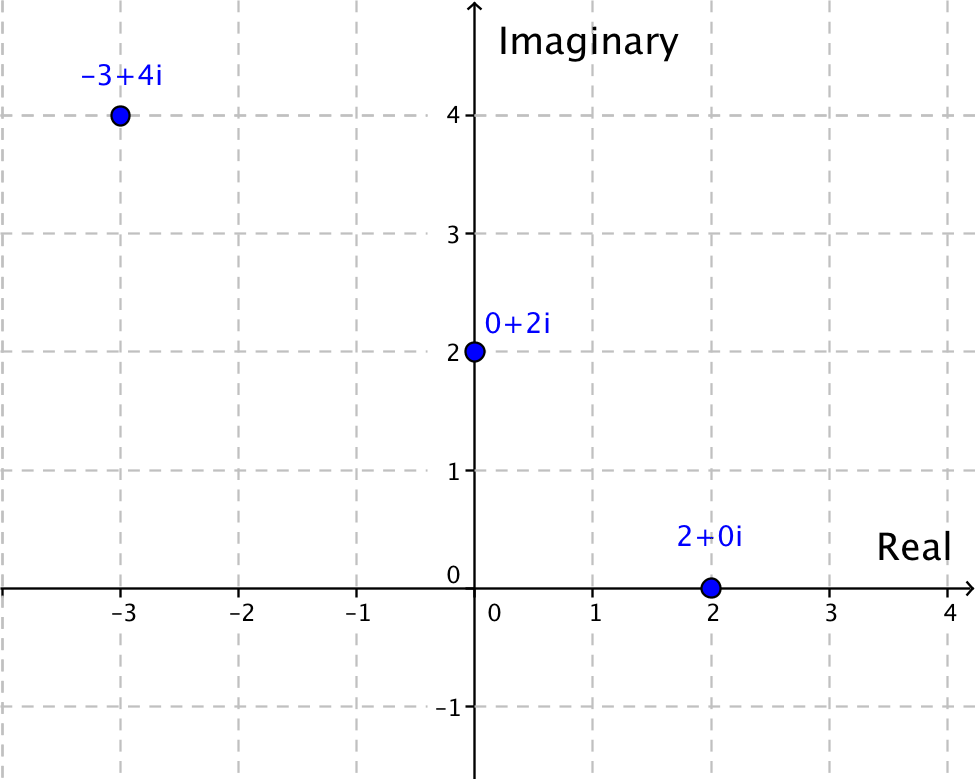
Lesson 37: A Surprising Boost from Geometry

Exit Ticket

Express the quantities below in form, and graph the corresponding points on the complex plane. If you use one set of axes, be sure to label each point appropriately.

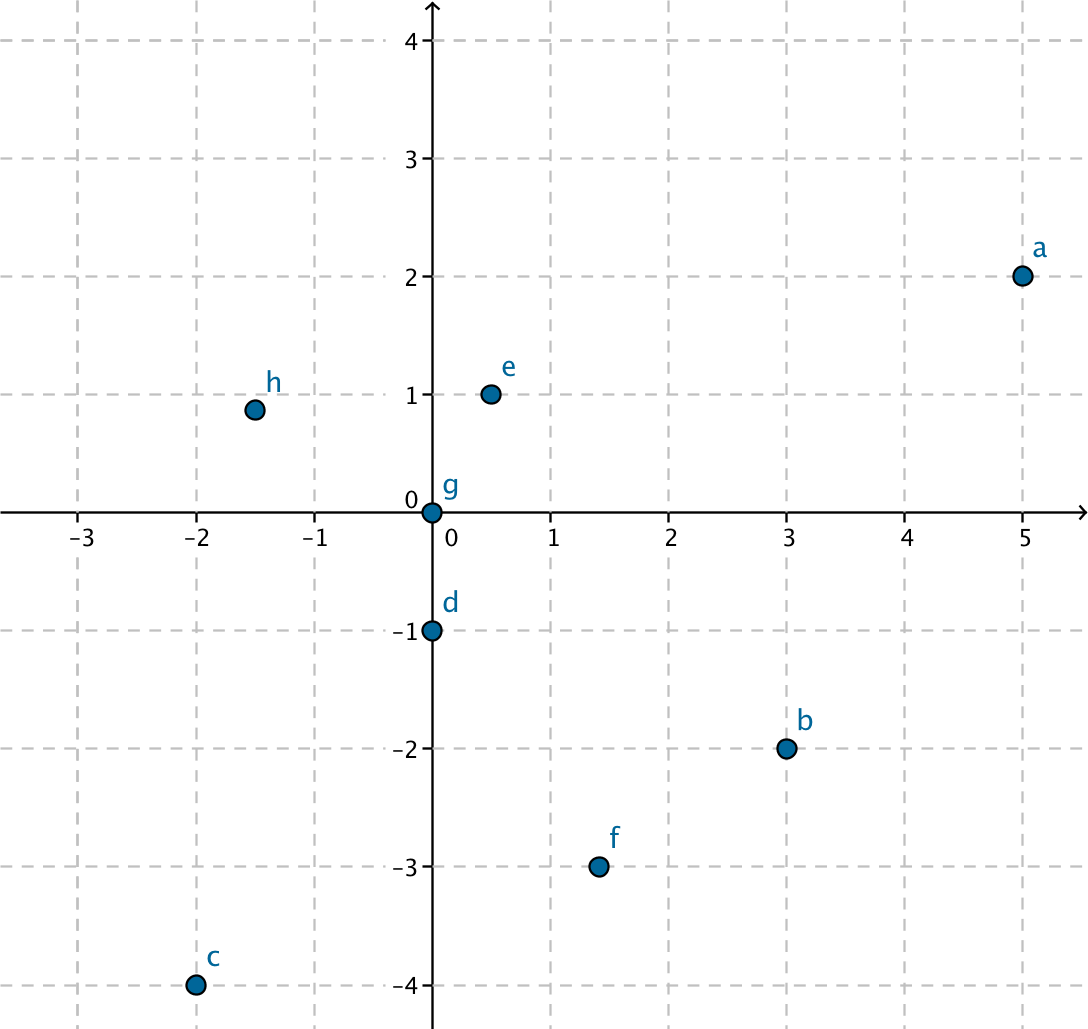
Exit Ticket Sample Solutions

Express the quantities below in form, and graph the corresponding points on the complex plane. If you use one set of axes, be sure to label each point appropriately.



Problem Set Sample Solutions

This problem set offers your students an opportunity to practice and gain facility with complex numbers and complex number arithmetic.

1.  Locate the point on the complex plane corresponding to the complex number given in parts (a)–(h). On one set of axes, label each point by its identifying letter. For example, the point corresponding to should be labeled “a.”
2. Express each of the following in form.
3. Express each of the following in form.
4. Find the real values of and in each of the following equations using the fact that if , then and .

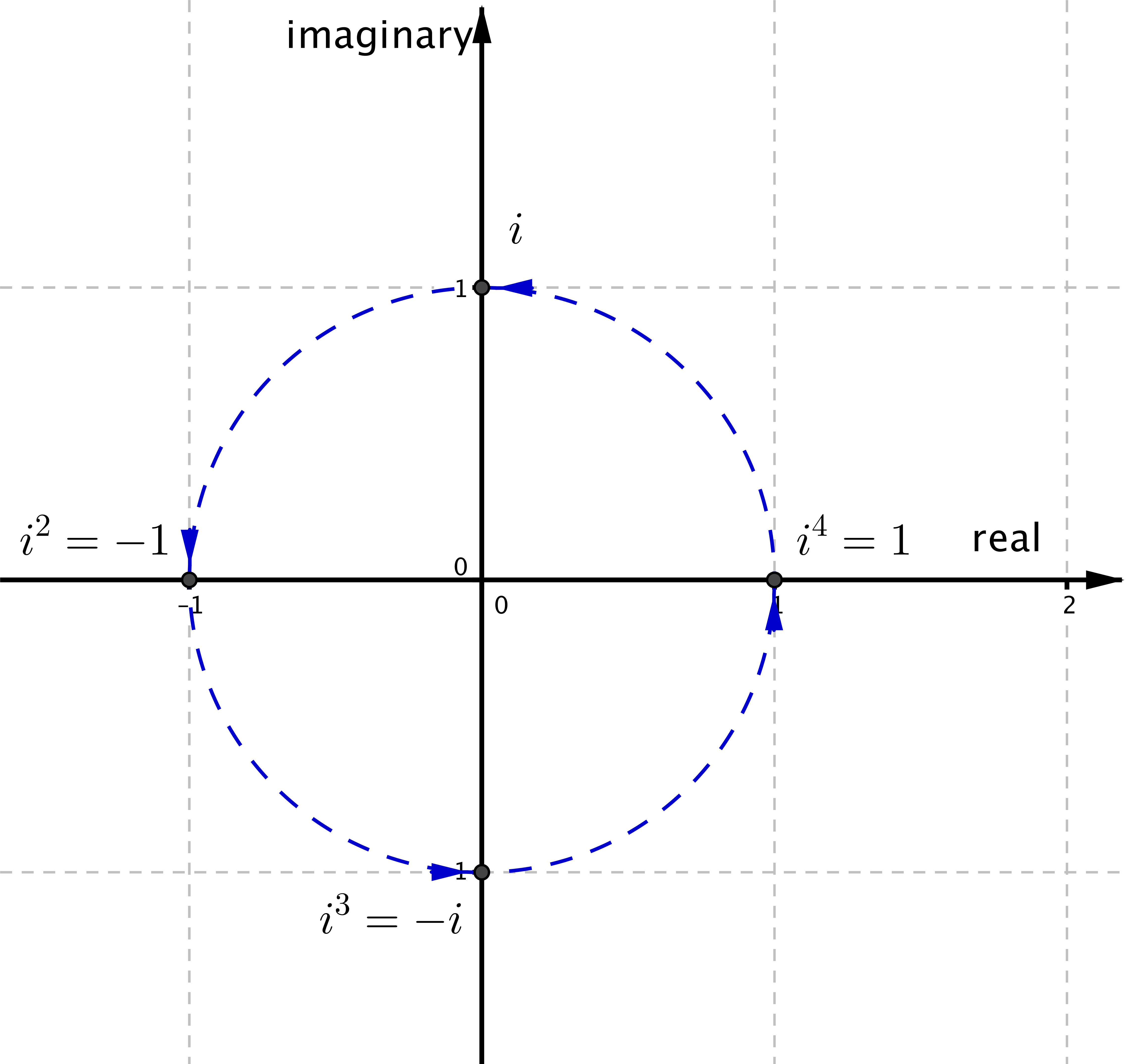
**MP.7**

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**MP.7**

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1. Since , we see that  
   lot , , , and on the complex plane and describe how multiplication by each rotates points in the complex plane.

Multiplying by rotates points by counterclockwise around . Multiplying by rotates points by about . Multiplying by rotates points counterclockwise by about the origin, which is equivalent to rotation by clockwise about the origin. Multiplying by rotates points counterclockwise by , which is equivalent to not rotating at all. The points , , , and are plotted below on the complex plane.

1. Express each of the following in form.

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A simple approach is to notice that every multiplications by result in four rotations, which takes back to . Therefore, divide by , which is with a remainder . So, rotations will take onto .

**MP.8**

1. Express each of the following in form.

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1. Evaluate when .
2. Evaluate when .
3. Show by substitution that is a solution to .
4. a. Evaluate the four products below.
   1. Evaluate .
   2. Evaluate
   3. Evaluate

**MP.7**

* 1. Evaluate
  2. Suppose and are positive real numbers. Determine whether the following quantities are equal or not equal.
     1. and **not equal**
     2. and  **equal**