Lesson 30: Linear Systems in Three Variables

Student Outcomes

- Students solve linear systems in three variables algebraically.

Lesson Notes

Students solved systems of linear equations in two variables using substitution and elimination in Grade 8 and then encountered the topic again in Algebra I when talking about solving systems of linear equalities and inequalities. In this lesson, we begin with a quick review of elimination to solve a linear system in two variables along with one application problem before moving onto solving a system of equations in three variables using algebraic techniques.

Classwork

Opening (2 minutes)

This lesson transitions from solving linear 2-by-2 equations as in Algebra I to solving systems of equations involving linear and nonlinear equations in two variables in the next two lessons. These nonlinear systems will be solved algebraically using substitution or by graphing each equation and finding points of intersection, if any. This lesson helps remind students how to solve linear systems of equations and introduces them to 3-by-3 systems of linear equations (which will be later analyzed using matrices in Precalculus).

Exercises 1–3 (8 minutes)

Exercises 1–3

Determine the value of x and y in the following systems of equations.

1. $$2x + 3y = 7$$
   $$2x + y = 3$$

   $x = \frac{1}{2}, y = 2$

2. $$5x - 2y = 4$$
   $$-2x + y = 2$$

   $x = 8, y = 18$

After this review of using elimination to solve a system, guide students through the set-up of the following problem, and then let them solve using the techniques reviewed in Exercises 1 and 2.

3. A scientist wants to create 120 ml of a solution that is 30% acidic. To create this solution, she has access to a 20% solution and a 45% solution. How many milliliters of each solution should she combine to create the 30% solution?

Solve this problem using a system of two equations in two variables.

Solution:

Milliliters of 20% solution: x ml
Milliliters of 45% solution: y ml

Write one equation to represent the total amounts of each solution needed:

$$x + y = 120$$

Since 30% of 120 ml is 36, we can write one equation to model the acidic portion:

$$0.20x + 0.45y = 36$$

Writing these two equations as a system:

$x + y = 120$

$0.20x + 0.45y = 36$

To solve, multiply both sides of the top equation by either 0.20 to eliminate x or 0.45 to eliminate y. The following work is for eliminating x:

$$0.20(x + y) = 0.20(120)$$
$$0.20x + 0.45y = 40$$

which gives

$$0.20x + 0.20y = 24$$
$$0.20x + 0.45y = 36.$$

Replacing the top equation with the difference between the bottom equation and top equation results in a new system with the same solutions:

$$0.25y = 12$$
$$0.20x + 0.45y = 36$$

The top equation can quickly be solved for y,

$$y = 48,$$

and substituting $y = 48$ back into the original first equation allows us to find x:

$$x + 48 = 120$$
$$x = 72$$

Thus, we need 48 ml of the 45% solution and 72 ml of the 20% solution.

Discussion (5 minutes)

MP.1

- In the previous examples we see how to solve a system of linear equations in two variables using elimination methods. However, what if we have three variables? For example, what are the solutions to the following system of equations?

$$2x + 3y - z = 5$$
$$4x - y - z = -1$$

Allow students time to work together and struggle with this system and realize that they cannot find a unique solution. Include the following third equation and ask students if they can solve it now.

Scaffolding:

To help students, ask them if they can eliminate two of the variables from either equation (they cannot). Have a discussion around what that means (the graph of the solution set is a line, not a

$$x + 4y + z = 12$$

Give students an opportunity to consider solutions or other ideas on how to begin the process of solving this system. After considering their suggestions and providing feedback, guide them through the process in the example below.

Example 1 (9 minutes)

Example 1

Determine the values for x, y, and z in the following system:

$2x + 3y - z = 5$ **(1)**

$4x - y - z = -1$ **(2)**

$x + 4y + z = 12$ **(3)**

Suggest numbering the equations as shown above to help organize the process.

- Eliminate z from equations (1) and (2) by subtraction:

  $2x + 3y - z = 5$

  $\underline{4x - \quad y - z = -1}$

  $-2x + 4y = 6$

- Our goal is to find two equations in two unknowns. Thus, we will also eliminate z from equations (2) and (3) by adding as follows:

  $4x - y - z = -1$

  $\underline{x + 4y + z = 12}$

  $5x + 3y = 11$

- Our new system of three equations in three variables has two equations with only two variables in them:

  $-2x + 4y = 6$

  $4x - y - z = -1$

  $5x + 3y = 11$

- These two equations now give us two equations in two variables, which we reviewed how to solve in Exercises 1–2.

$$-2x + 4y = 6$$
$$5x + 3y = 11$$

At this point, you can let students solve this individually or with partners, or guide them through the process if necessary.

- To get matching coefficients, we need to multiply both equations by a constant:

$$5(-2x + 4y) = 5(6) \quad \rightarrow \quad -10x + 20y = 30$$
$$2(5x + 3y) = 2(11) \quad \rightarrow \quad 10x + 6y = 22$$

- Replacing the top equation with the sum of the top and bottom equations together gives the following:

$$26y = 52$$
$$10x + 6y = 22$$

- The new top equation can be solved for y:

$$y = 2$$

- Replace $y = 2$ in one of the equations to find x:

$$5x + 3(2) = 11$$
$$5x + 6 = 11$$
$$5x = 5$$
$$x = 1$$

- Replace $x = 1$ and $y = 2$ in any of the original equations to find z:

$$2(1) + 3(2) - z = 5$$
$$2 + 6 - z = 5$$
$$8 - z = 5$$
$$z = 3$$

- The solution, $x = 1$, $y = 2$, and $z = 3$, can be written compactly as an ordered triple of numbers $(1, 2, 3)$.

You might want to point out to your students that the point $(1, 2, 3)$ can be thought of as a point in a three-dimensional coordinate plane, and that it is, like a two-by-two system of equations, the intersection point in three-space of the three planes given by the graphs of each equation. These concepts are not the point of this lesson, so addressing them is optional.

Point out that a linear system involving three variables requires three equations in order for the solution to possibly be a single point.

The following problems provide examples of situations that require solving systems of equations in three variables.

Exercise 4 (8 minutes)

Exercise 4

Given the system below, determine the values of r, s, and u that satisfy all three equations.

$$r + 2s - u = 8$$
$$s + u = 4$$
$$r - s - u = 2$$

Adding the second and third equation together produces the equation $r = 6$. Substituting this into the first equation and adding it to the second gives $6 + 3s = 12$, so that $s = 2$. Replacing s with 2 in the second equation gives $u = 2$. The solution to this system of equations is $(6, 2, 2)$.

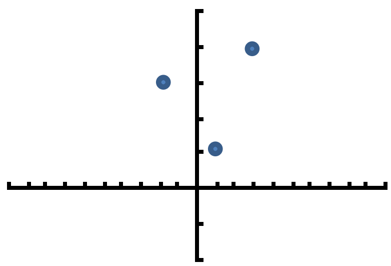
Exercise 5 (6 minutes)

Exercise 5

Find the equation of the form $y = ax^2 + bx + c$ that satisfies the points $(1, 6)$, $(3, 20)$, and $(-2, 15)$.

$a = 2$, $b = -1$, $c = 5$; therefore, the quadratic equation is $y = 2x^2 - x + 5$.

Students may need help setting this up. A graph of the points may help.



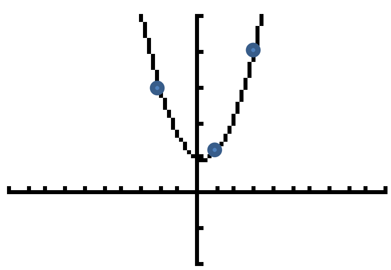
MP.7

- Since we know three ordered pairs, we can create three equations.

$$6 = a + b + c$$
$$20 = 9a + 3b + c$$
$$15 = 4a - 2b + c$$

Ask students to explain where the three equations came from. Then have them use the technique from Example 1 to solve this system.

Have students use a graphing utility to graph the equation using the coefficient solutions to confirm the answer.



Closing (2 minutes)

- Having solved systems of two linear equations, we see in the lesson that in order to solve a linear system in three variables, we need three equations. How many equations might we need to solve a system with four variables? Five?

Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 30: Linear Systems in Three Variables

Exit Ticket

For the following system, determine the values of p, q, and r that satisfy all three equations:

$$\begin{aligned} 2p + q - r &= 8 \\ q + r &= 4 \\ p - q \quad &= 2 \end{aligned}$$

Exit Ticket Sample Solutions

For the following system, determine the values of p, q, and r that satisfy all three equations:

$$\begin{aligned} 2p + q - r &= 8 \\ q + r &= 4 \\ p - q \quad &= 2 \end{aligned}$$

$p = 4$, $q = 2$, $r = 2$, or equivalently $(4, 2, 2)$

Problem Set Sample Solutions

Solve the following systems.

1. $$\begin{aligned} x + y &= 3 \\ y + z &= 6 \\ x + z &= 5 \end{aligned}$$

   $x = 1$, $y = 2$, $z = 4$ or $(1, 2, 4)$

2. $$\begin{aligned} r &= 2(s - t) \\ 2t &= 3(s - r) \\ r + t &= 2s - 3 \end{aligned}$$

   $r = 2$, $s = 4$, $t = 3$, or $(2, 4, 3)$

3. $$\begin{aligned} 2a + 4b + c &= 5 \\ a - 4b &= -6 \\ 2b + c &= 7 \end{aligned}$$

   $a = -2$, $b = 1$, $c = 5$ or $(-2, 1, 5)$

4. $$\begin{aligned} 2x + y - z &= -5 \\ 4x - 2y + z &= 10 \\ 2x + 3y + 2z &= 3 \end{aligned}$$

   $x = \frac{1}{2}$, $y = -2$, $z = 4$ or $\left(\frac{1}{2}, -2, 4\right)$

5. $$\begin{aligned} r + 3s + t &= 3 \\ 2r - 3s + 2t &= 3 \\ -r + 3s - 3t &= 1 \end{aligned}$$

   $r = 3$, $s = \frac{1}{3}$, $t = -1$ or $\left(3, \frac{1}{3}, -1\right)$

6. $$\begin{aligned} x - y &= 1 \\ 2y + z &= -4 \\ x - 2z &= -6 \end{aligned}$$

   $x = -2$, $y = -3$, $z = 2$ or $(-2, -3, 2)$

7. $$\begin{aligned} x &= 3(y - z) \\ y &= 5(z - x) \\ x + y &= z + 4 \end{aligned}$$

   $x = 3$, $y = 5$, $z = 4$ or $(3, 5, 4)$

8. $$\begin{aligned} p + q + 3r &= 4 \\ 2q + 3r &= 7 \\ p - q - r &= -2 \end{aligned}$$

   $p = 2$, $q = 5$, $r = -1$ or $(2, 5, -1)$

9. $$\begin{aligned} \frac{1}{x} + \frac{1}{y} + \frac{1}{z} &= 5 \\ \frac{1}{x} + \frac{1}{y} &= 2 \\ \frac{1}{x} - \frac{1}{z} &= -2 \end{aligned}$$

   $x = 1$, $y = 1$, $z = \frac{1}{3}$ or $\left(1, 1, \frac{1}{3}\right)$

10. $$\begin{aligned} \frac{1}{a} + \frac{1}{b} + \frac{1}{c} &= 6 \\ \frac{1}{b} + \frac{1}{c} &= 5 \\ \frac{1}{a} - \frac{1}{b} &= -1 \end{aligned}$$

    $a = 1$, $b = \frac{1}{2}$, $c = \frac{1}{3}$ or $\left(1, \frac{1}{2}, \frac{1}{3}\right)$

11. Find the equation of the form $y = ax^2 + bx + c$ whose graph passes through the points $(1, -1)$, $(3, 23)$, and $(-1, 7)$.

$y = 4x^2 - 4x - 1$

12. Show that for any number t, the values $x = t + 2$, $y = 1 - t$, and $z = t + 1$ are solutions to the system of equations below.

$$x + y = 3$$
$$y + z = 2$$

(in this situation, we say that t *parameterizes the solution set of the system.*)

$$x + y = (t + 2) + (1 - t) = 3$$
$$y + z = (1 - t) + (t + 1) = 2$$

13. Some rational expressions can be written as the sum of two or more rational expressions whose denominators are the factors of its denominator (called a *partial fraction decomposition*). Find the partial fraction decomposition for the following example by filling in the blank to make the equation true for all n except 0 and -1.

$$\frac{1}{n(n+1)} = \frac{\_\_\_\_}{n} - \frac{1}{n+1}$$

Adding $\frac{1}{n+1}$ to both sides of the equations, we have $\frac{1}{n(n+1)} + \frac{1}{(n+1)} = \frac{1}{n(n+1)} + \frac{n}{n(n+1)} = \frac{(n+1)}{n(n+1)} = \frac{1}{n}$, so $\frac{1}{n(n+1)} + \frac{1}{(n+1)} = \frac{1}{n}$, and equivalently we have $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{(n+1)}$. Thus, the blank should contain a 1.

14. A chemist needs to make 40 ml of a 15% acid solution. He has a 5% acid solution and a 30% acid solution on hand. If he uses the 5% and 30% solutions to create the 15% solution, how many ml of each will he need?

He will need 24 ml of the 5% solution and 16 ml of the 30% solution.

15. An airplane makes a 400 mile trip against a head wind in 4 hours. The return trip takes 2.5 hours, the wind now being a tail wind. If the plane maintains a constant speed with respect to still air, and the speed of the wind is also constant and does not vary, find the still-air speed of the plane and the speed of the wind.

The speed of the plane in still wind is 130 mph, and the speed of the wind is 30 mph.

16. A restaurant owner estimates that she needs in small change the same number of dimes as pennies and nickels together and the same number of pennies as nickels. If she gets $\$26$ worth of pennies, nickels, and dimes, how should they be distributed?

She will need 200 dimes ($\$20$ worth), 100 nickels ($\$5$ worth), and 100 pennies ($\$1$ worth) for a total of $\$26$.