



Lesson 28: A Focus on Square Roots

Student Outcomes

- Students solve simple radical equations and understand the possibility of extraneous solutions. They understand that care must be taken with the role of square roots so as to avoid apparent paradoxes.
- Students explain and justify the steps taken in solving simple radical equations.

Lesson Notes

In the next two lessons, students work with radical expressions and equations. They extend their understanding of the idea that not all operations are invertible, which was explored in Algebra I and continued in the previous lessons on solving rational equations. Squaring both sides of an equation in some cases produces an extraneous solution. They also continue to work with fractional expressions and equations as seen in the previous lessons, but those expressions now contain radicals. This lesson highlights standards **A-REI.A.1**, which calls for students to be able to explain each step required to solve an equation, and **A-REI.A.2**, which calls for students to solve a radical equation and show how extraneous solutions might arise. It also addresses the standard MP.3 by building and analyzing arguments used in solving radical equations. In Example 2, students consider the difference between working with an *expression*, whose value should be preserved, and working with an *equation*, whose sides can be changed in value as long as the equality is preserved. This difference addresses the standard MP.7 because students are stepping back to get an overview of expressions and equations as objects subject to different rules.

Classwork

Opening (1 minute)

Recall that working with radicals can be tricky, especially when negative numbers are involved. When solving a radical equation, one must always check the answers found to verify that they are indeed valid solutions to the equation. In some cases, extraneous solutions appear and must be eliminated. Recall that an *extraneous solution* is one that satisfies a transformed equation but not the original one.

Exercises 1–4 (7 minutes)

Give students a few minutes to work through the first four exercises, and then discuss the results as a whole class. Circulate the room to assess the students' understanding.

Scaffolding:

- If students are struggling, show a few simpler examples such as solving $\sqrt{x} = 5$ or $\sqrt{x} = -5$.
- Another option would be to provide the following alternative model for students to complete.

Fill in the blanks to fully show and explain the solution process.

$$\begin{array}{rcl}
 \sqrt{x} - 6 & = & 4 \\
 \underline{\sqrt{x} = 10} & & \text{Added 6 to both sides to isolate the radical} \\
 \sqrt{x}^2 & = & 10^2 \\
 x & = & 100
 \end{array}$$

Square both sides

Exercises 1–4

For Exercises 1–4, describe each step taken to solve the equation. Then, check the solution to see if it is valid. If it is not a valid solution, explain why.

1. $\sqrt{x} - 6 = 4$

$\sqrt{x} = 10$ Add 6 to both sides

$x = 100$ Square both sides

Check: $\sqrt{100} - 6 = 10 - 6 = 4$

So 100 is a valid solution.

2. $\sqrt[3]{x} - 6 = 4$

$\sqrt[3]{x} = 10$ Add 6 to both sides

$x = 1000$ Cube both sides

Check: $\sqrt[3]{1000} - 6 = 10 - 6 = 4$

So 1000 is a valid solution.

3. $\sqrt{x} + 6 = 4$

$\sqrt{x} = -2$

$x = 4$

Check: $\sqrt{4} + 6 = 2 + 6 = 8$, and $8 \neq 4$, so 4 is not a valid solution.

4. $\sqrt[3]{x} + 6 = 4$

$\sqrt[3]{x} = -2$

$x = -8$

Check: $\sqrt[3]{-8} + 6 = -2 + 6 = 4$, so -8 is a valid solution.

Discussion

Consider each of the following questions, one at a time.

- What was the first step taken in solving the radical equations in Exercises 1 and 2?
 - The radical was isolated.
- What was the second step taken?
 - Both sides were squared or cubed to eliminate the radical.
- What happened in Exercise 3?
 - The same steps were used to solve the equation as were used in Exercise 1, but this time the solution found did not work. There is no solution to the equation.
- Why did that happen?

This is one of the focal points of the lesson. Ask students to answer in writing or discuss with a partner before sharing their answers with the rest of the class. In the discussion, emphasize that 4 is an *extraneous solution*; it is the solution to $x = 4$ but not to the original equation.

- For 4 to be a solution, $\sqrt{4}$ would need to equal -2 . Even though $(-2)^2 = 4$, we define $\sqrt{4} = 2$ so that $f(x) = \sqrt{x}$ takes on only one value for $x \geq 0$ and is thus a function. As a result, the square root of a positive number is only the positive value. Therefore, 4 is an extraneous solution.

- What other types of equations sometimes have extraneous solutions?
 - Rational equations can have extraneous solutions that create zero in the denominator.
- Why did the solution process work in Exercise 4?
 - The cube root of a negative number is negative, so a cube root equation does not have the same issues with the negative numbers as a square root does.

MP.3

Example 1 (5 minutes)

Work through the example to solidify the steps in solving a radical equation. Depending on how the students did with the first four exercises, you may want them to continue working with a partner, or you may want to work through this example with the whole class at once. Be sure the students can explain and justify the steps they are taking.

Example 1

Solve the radical equation. Be sure to check your solutions.

$$\sqrt{3x+5} - 2 = -1$$

Solution: $\sqrt{3x+5} = 1$

$$3x + 5 = 1$$

$$3x = -4$$

$$x = -\frac{4}{3}$$

Check: $\sqrt{3\left(-\frac{4}{3}\right) + 5} - 2 = \sqrt{-4 + 5} - 2 = \sqrt{1} - 2 = -1$, so $-\frac{4}{3}$ is a valid solution.

Discussion

- What was the first step you took?
 - *Isolated the radical*
- Why did you do that first?
 - *Isolating the radical allows it to be eliminated by squaring or cubing both sides of the equation.*
- What was the next step?
 - *Squared both sides*
- Why did you do that?
 - *The purpose was to eliminate the radical from the equation.*
- Even though we are solving a new type of equation, does this feel like a familiar process?
 - *Yes. When solving an equation, we work on undoing any operation by doing the inverse. To undo a square root we use the inverse, so we square the expression.*
- How do the steps we are following relate to your previous experiences with solving other types of equations?
 - *We are still following the basic process to solve an equation, which is to undo any operation on the same side as the variable by using the inverse operation.*
- Why is it important to check the solution?
 - *Sometimes extraneous solutions appear because the square root of a positive number or zero is never negative.*

Summarize (in writing or with a partner) what you have learned about solving radical equations. Be sure that you explain what to do when you get an invalid solution.

Exercises 5–15 (15 minutes)

Allow students time to work the problems individually and then check with a partner. Circulate around the room. Make sure students are checking for extraneous solutions.

Exercises 5–15

Solve each radical equation. Be sure to check your solutions.

5. $\sqrt{2x - 3} = 11$

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6. $\sqrt[3]{6 - x} = -3$

33

7. $\sqrt{x + 5} - 9 = -12$

No solution

8. $\sqrt{4x - 7} = \sqrt{3x + 9}$

16

9. $-12\sqrt{x - 6} = 18$

No solution

10. $3\sqrt[3]{x + 2} = 12$

62

11. $\sqrt{x^2 - 5} = 2$

3 and -3

12. $\sqrt{x^2 + 8x} = 3$

-9 and 1

- Which exercises produced extraneous solutions?
 - Exercises 7 and 9
- Which exercises produced more than one solution? Why?
 - Exercises 11 and 12 because after eliminating the radical, the equation became a quadratic equation. Both solutions were valid when checked.
- Write an example of a radical equation that has an extraneous solution. Exchange with a partner and confirm that the example does in fact have an extraneous solution.

Multiply each expression.

13. $(\sqrt{x} + 2)(\sqrt{x} - 2)$

 $x - 4$

14. $(\sqrt{x} + 4)(\sqrt{x} + 4)$

 $x + 8\sqrt{x} + 16$

15. $(\sqrt{x - 5})(\sqrt{x - 5})$

 $x - 5$

In the next example and exercises, we are working with rational expressions and equations that contain radicals. The purpose of these problems is to continue to build fluency working with radicals, to build on the work done in the previous lessons on rational expressions and equations, and to highlight MP.7, which calls for students to recognize and make use of structure in an expression.

MP.3

Example 2 (5 minutes)

Work through the two examples as a class, making sure students are clear on the differences between working with an *expression* and working with an *equation*.

Example 2

Rationalize the denominator in each expression. That is, rewrite each expression so that the fraction has a rational expression in the denominator.

a. $\frac{x-9}{\sqrt{x-9}}$

$$\begin{aligned} &\frac{x-9}{\sqrt{x-9}} \cdot \frac{\sqrt{x-9}}{\sqrt{x-9}} \\ &\frac{(x-9)\sqrt{x-9}}{x-9} \\ &\frac{\sqrt{x-9}}{\sqrt{x-9}} \end{aligned}$$

b. $\frac{x-9}{\sqrt{x}+3}$

$$\begin{aligned} &\frac{x-9}{\sqrt{x}+3} \cdot \frac{\sqrt{x}-3}{\sqrt{x}-3} \\ &\frac{(x-9)(\sqrt{x}-3)}{x-9} \\ &\frac{\sqrt{x}-3}{\sqrt{x}-3} \end{aligned}$$

- What do the directions mean by “rationalize the denominator”?
 - *To remove the radical from the denominator so that the denominator will be a rational expression.*
- How can we accomplish this goal in part (a)?
 - *Multiply the numerator and denominator by $\sqrt{x-9}$.*
- Why not just square the expression?
 - *We are working with an expression, not an equation. You cannot square the expression because you would be changing its value. You can multiply the numerator and denominator by $\sqrt{x-9}$ because that is equivalent to multiplying by 1. It does not change the value of the expression.*
- Can we take the same approach in part (b)?
 - *No, multiplying by $\sqrt{x}+3$ would not remove the radical from the denominator.*
- Based on Exercise 13, what number should we multiply numerator and denominator by in part (b) in order to make the denominator rational?
 - $\sqrt{x}-3$
- In these examples, what was accomplished by rationalizing the denominator?
 - *It allowed us to create an equivalent expression that is simpler.*
- Why would that be advantageous?
 - *It would be easier to work with if we were evaluating it for a particular value of x .*

Exercises 16–18 (5 minutes)

Allow students time to work on the three problems and then debrief. Students may have taken different approaches on Exercise 17, such as squaring both sides first or rationalizing the denominator. Share a few different approaches and compare.

MP.7

Exercises 16–18

16. Rewrite $\frac{1}{\sqrt{x}-5}$ in an equivalent form with a rational expression in the denominator.

$$\frac{\sqrt{x} + 5}{x - 25}$$

17. Solve the radical equation $\frac{3}{\sqrt{x+3}} = 1$. Be sure to check for extraneous solutions.

$$x = 6$$

18. Without solving the radical equation $\sqrt{x+5} + 9 = 0$, how could you tell that it has no real solution?

The radical expression $\sqrt{x+5}$ is positive or zero. In either case, adding 9 to it cannot give zero.

Scaffolding:

- If students are struggling with Exercise 17, have them approach the equation logically first rather than algebraically. If the output must equal 1 and the numerator is 3, what must the denominator equal?

Closing (2 minutes)

Ask students to respond to these questions in writing or with a partner. Use this as an opportunity to informally assess the students' understanding.

- Explain to your neighbor how to solve a radical equation. What steps do you take and why?
 - *Isolate the radical and then eliminate it by raising both sides to an exponent. The radical is isolated so that both sides can be squared or cubed as a means of eliminating the radical.*
- How is solving a radical equation similar to solving other types of equations we have solved?
 - *We are isolating the variable by undoing any operation on the same side.*
- Why is it important to check the solutions?
 - *Remember that the square root of a number takes on only the positive value. When solving a radical equation involving a square root, squaring both sides of the equation in the process of solving may make the negative "disappear" and may create an extraneous solution.*

Exit Ticket (5 minutes)

Name _____

Date _____

Lesson 28: A Focus on Square Roots

Exit Ticket

Consider the radical equation $3\sqrt{6-x} + 4 = -8$.

1. Solve the equation. Next to each step, write a description of what is being done.
2. Check the solution.
3. Explain why the calculation in Problem 1 does not produce the solution to the equation.

Exit Ticket Sample Solutions

Consider the radical equation $3\sqrt{6-x} + 4 = -8$

1. Solve the equation. Next to each step, write a description of what is being done.

$$3\sqrt{6-x} = -12$$

Subtract 4 from both sides.

$$\sqrt{6-x} = -4$$

Divide both sides by 3 in order to isolate the radical.

$$6-x = 16$$

Square both sides to eliminate the radical.

$$x = -10$$

Subtract 6 from both sides and divide by -1 .

2. Check the solution.

$$3\sqrt{6-(-10)} + 4 = 3\sqrt{16} + 4 = 3(4) + 4 = 16, \text{ and } 16 \neq -8, \text{ so } -10 \text{ is not a valid solution.}$$

3. Explain why the calculation in Problem 1 does not produce the solution to the equation.

Because the square root of a positive number is positive, $3\sqrt{6-x}$ will be positive. A positive number added to 4 cannot be -8 .

Problem Set Sample Solutions

1. a. If $\sqrt{x} = 9$, then what is the value of x ?

$$x = 81$$

- b. If $x^2 = 9$, then what is the value of x ?

$$x = 3 \text{ or } x = -3$$

- c. Is there a value of x such that $\sqrt{x+5} = 0$? If yes, what is the value? If no, explain why not.

$$\text{Yes, } x = -5$$

- d. Is there a value of x such that $\sqrt{x} + 5 = 0$? If yes, what is the value? If no, explain why not.

No, $\sqrt{x}$ will be a positive value or zero for any value of x , so the sum cannot equal 0. If $x = 25$, then $\sqrt{25} + 5 = 10$.

2. a. Is the statement $\sqrt{x^2} = x$ true for all x -values? Explain.

No, this statement is only true for $x \geq 0$. If $x < 0$, it is not true. For example, if $x = -5$, $\sqrt{(-5)^2} = \sqrt{25} = 5$, then $\sqrt{(-5)^2} \neq -5$.

- b. Is the statement $\sqrt[3]{x^3} = x$ true for all x -values? Explain.

Yes, this statement is true for all x -values. For example, if $x = 2$, then $\sqrt[3]{2^3} = 2$. If $x = -2$, then $\sqrt[3]{(-2)^3} = -2$. Since the cube root of a positive number is positive, and the cube root of a negative number is negative, this statement is true for any value of x .

Rationalize the denominator in each expression.

$$3. \frac{4-x}{2+\sqrt{x}}$$

$$2 - \sqrt{x}$$

$$4. \frac{2}{\sqrt{x-12}}$$

$$\frac{2\sqrt{x-12}}{x-12}$$

$$5. \frac{1}{\sqrt{x+3}-\sqrt{x}}$$

$$\frac{\sqrt{x+3}+\sqrt{x}}{3}$$

Solve each equation and check the solutions.

$$6. \sqrt{x+6} = 3$$

$$x = 3$$

$$7. 2\sqrt{x+3} = 6$$

$$x = 6$$

$$8. \sqrt{x+3} + 6 = 3$$

$$\text{No solution}$$

$$9. \sqrt{x+3} - 6 = 3$$

$$x = 78$$

$$10. 16 = 8 + \sqrt{x}$$

$$x = 64$$

$$11. \sqrt{3x-5} = 7$$

$$x = 18$$

$$12. \sqrt{2x-3} = \sqrt{10-x}$$

$$x = \frac{13}{3}$$

$$13. 3\sqrt{x+2} + \sqrt{x-4} = 0$$

$$\text{No solution}$$

$$14. \frac{\sqrt{x+9}}{4} = 3$$

$$x = 135$$

$$15. \frac{12}{\sqrt{x+9}} = 3$$

$$x = 7$$

$$16. \sqrt{x^2+9} = 5$$

$$x = 4 \text{ or } x = -4$$

$$17. \sqrt{x^2-6x} = 4$$

$$x = 8 \text{ or } x = -2$$

$$18. \frac{5}{\sqrt{x-2}} = 5$$

$$x = 3$$

$$19. \frac{5}{\sqrt{x-2}} = 5$$

$$x = 9$$

$$20. \sqrt[3]{5x-3} + 8 = 6$$

$$x = -1$$

$$21. \sqrt[3]{9-x} = 6$$

$$x = -207$$

22. Consider the inequality $\sqrt{x^2+4x} > 0$. Determine whether each x -value is a solution to the inequality.

a. $x = -10$
Yes

b. $x = -4$
No

c. $x = 10$
Yes

d. $x = 4$
Yes

23. Show that $\frac{a-b}{\sqrt{a}-\sqrt{b}} = \sqrt{a} + \sqrt{b}$ for all values of a and b such that $a > 0$ and $b > 0$ and $a \neq b$.

If we multiply the numerator and denominator of $\frac{a-b}{\sqrt{a}-\sqrt{b}}$ by $\sqrt{a} + \sqrt{b}$ to rationalize the denominator, then we have

$$\frac{a-b}{\sqrt{a}-\sqrt{b}} = \frac{a-b}{\sqrt{a}-\sqrt{b}} \cdot \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}} = \frac{(a-b)(\sqrt{a}+\sqrt{b})}{a-b} = \sqrt{a} + \sqrt{b}.$$

24. Without actually solving the equation, explain why the equation $\sqrt{x+1} + 2 = 0$ has no solution.

The value of $\sqrt{x+1}$ must be positive which is then added to 2. The sum of two positive numbers is positive; therefore, the sum cannot equal 0.