Lesson 27: Word Problems Leading to Rational Equations

Student Outcomes

- Students solve word problems using models that involve rational expressions.

Lesson Notes

In the preceding lessons, students learned to add, subtract, multiply, and divide rational expressions and solve rational equations in order to develop the tools needed for solving application problems involving rational equations in this lesson (**A-REI.A.2**). Students will develop their problem-solving and modeling abilities by carefully reading the problem description and converting information into equations (MP.1), thus creating a mathematical model of the problem (MP.4).

Classwork

Exercise 1 (13 minutes)

We now turn to some applied problems that we can model with rational equations, strengthening students' problem-solving and modeling experience in alignment with standards MP.1 and MP.4. We can solve these equations using the skills developed in previous lessons. Have students work in small groups to answer this set of four questions. At the end of the work time, ask different groups to present their solutions to the class. Suggest to students that they should: (a) read the problem aloud, (b) paraphrase and summarize the problem in their own words, (c) find an equation that models the situation, and (d) say how it represents the quantities involved. Check to make sure that students understand the problem before they begin trying to solve it.

You may want to encourage students, especially with this first problem, by suggesting that they use m for the unknown quantity and asking if they can arrive at an equation that relates this unknown quantity to the known quantities.

Exercise 1

1. **Anne and Maria play tennis almost every weekend. So far, Anne has won 12 out of 20 matches.**

   a. **How many matches will Anne have to win in a row to improve her winning percentage to 75%?**

   Suppose that Anne has already won 12 of 20 matches, and let m represent the number of additional matches she must win to raise her winning percentage to 75%. After playing and winning all of those additional m matches, she has won $12 + m$ matches out of a total of $20 + m$ matches played. Her winning percentage is then $\frac{12+m}{20+m}$, and we want to find the value of m that solves the equation

$$\frac{12+m}{20+m} = 0.75.$$

MP.1 & MP.4

Scaffolding:

Students may benefit from having the problem read aloud and summarized. They should be encouraged to restate the problem in their own words to a partner.

If students are struggling, present the equation $\frac{12+m}{20+m} = 0.75$, and ask students how this models the situation.

Students who may be above grade level could be challenged to write their own word problems that result in rational equations.

Multiply both sides by $20 + m$.

$$12 + m = 0.75(20 + m)$$
$$12 + m = 15 + 0.75m$$

Solving for m:

$$0.25m = 3$$
$$m = 12$$

So, Anne would need to win 12 matches in a row in order to improve her winning percentage to 75%.

MP.1 & MP.4

b. How many matches will Anne have to win in a row to improve her winning percentage to 90%?

This situation is similar to that for part (a), except that we want a winning percentage of 0.90, instead of 0.75. Again, we let m represent the number of matches Anne must win consecutively to bring her winning percentage up to 90%.

$$\frac{12 + m}{20 + m} = 0.90$$

Solving for m:

$$12 + m = 0.90(20 + m)$$
$$12 + m = 18 + 0.90m$$
$$0.10m = 6$$
$$m = 60$$

In order for Anne to bring her winning percentage up to 90%, she would need to win the next 60 consecutive matches.

MP.2

c. Can Anne reach a winning percentage of 100%?

Allow students to come to the conclusion that Anne will never reach a winning percentage of 100% because she has already lost 8 matches.

d. After Anne has reached a winning percentage of 90% by winning consecutive matches as in part (b), how many matches can she now lose in a row to have a winning percentage of 50%?

Recall from part (b) that she had won 72 matches out of 80 to reach a winning percentage of 90%. We will now assume that she loses the next k matches in a row. Then, she will have won 72 matches out of $80 + k$ matches, and we want to know the value of k that makes this a 50% win rate.

$$\frac{72}{80 + k} = 0.50$$

Solving the equation:

$$72 = 0.50(80 + k)$$
$$72 = 40 + 0.50k$$
$$32 = 0.50k$$
$$64 = k$$

Thus, after reaching a 90% winning percentage in 80 matches, Anne can lose 64 matches in a row to drop to a 50% winning percentage.

Example 1 (5 minutes)

Work this problem at the front of the room, but allow the class to provide input and steer the discussion. Depending on how students did with the first exercise, the teacher may lead a discussion of this problem as a class, ask students to work in groups, or ask students to work independently while targeting instruction with a small group that struggled on the first example.

MP.4

Example 1

Working together, it takes Sam, Jenna, and Francisco two hours to paint one room. When Sam is working alone, he can paint one room in 6 hours. When Jenna works alone, she can paint one room in 4 hours. Determine how long it would take Francisco to paint one room on his own.

Consider how much can be accomplished in one hour. Sam, Jenna, and Francisco together can paint half a room in one hour. If Sam can paint one room in 6 hours on his own, then in one hour he can paint $\frac{1}{6}$ of the room. Similarly, Jenna can paint $\frac{1}{4}$ of the room in one hour. We do not yet know how much Francisco can paint in one hour, so we will say he can paint $\frac{1}{f}$ of the room. So, in one hour, Sam has painted $\frac{1}{6}$ of the room, Jenna has painted $\frac{1}{4}$ of the room, and all three together can paint $\frac{1}{2}$ the room, leading to the following equation for how much can be painted in one hour:

$$\frac{1}{6}+\frac{1}{4}+\frac{1}{f}=\frac{1}{2}.$$

A common multiple of the denominators is $12f$. Multiplying both sides by $12f$ gives us:

$$\frac{12f}{6}+\frac{12f}{4}+\frac{12f}{f}=\frac{12f}{2}$$
$$2f+3f+12=6f,$$

which leads us to the value of f:

$$f=12.$$

So, Francisco can paint the room in 12 hours on his own.

Exercise 2 (5 minutes)

Remind students that distance equals rate times time ($d = r \cdot t$) before having them work on this exercise in pairs or small groups. Be sure to have groups share their results before continuing to the next exercise.

Exercises 2–4

2. Melissa walks 3 miles to the house of a friend and returns home on a bike. She averages 4 miles per hour faster when cycling than when walking, and the total time for both trips is two hours. Find her walking speed.

Using the relationship $d = r \cdot t$, we have $t = \frac{d}{r}$. The time it takes for Melissa to walk to her friend's house is $\frac{3}{r}$, and the time to cycle back is $\frac{3}{r+4}$. Thus, we can write an equation that describes the combined time for both trips:

$$\frac{3}{r} + \frac{3}{r+4} = 2.$$

A common multiple of the denominators is $r(r+4)$, so we multiply both sides of the equation by $r(r+4)$.

$$3(r+4) + 3r = 2r(r+4)$$
$$3r + 12 + 3r = 2r^2 + 8r$$
$$2r^2 + 2r - 12 = 0$$
$$2(r-2)(r+3) = 0$$

Thus, $r = -3$ or $r = 2$. Since r represents Melissa's speed, it does not make sense for r to be negative. So, the only solution is 2, which means that Melissa's walking speed is 2 miles per hour.

MP.1 & MP.4

Exercise 3 (10 minutes)

3. You have 10 liters of a juice blend that is 60% juice.

a. How many liters of pure juice need to be added in order to make a blend that is 75% juice?

We start off with 10 liters of a blend containing 60% juice. Then, this blend contains $0.60(10) = 6$ liters of juice in the 10 liters of mixture. If we add A liters of pure juice, then the concentration of juice in the blend is $\frac{6+A}{10+A}$. We want to know which value of A makes this blend 75% juice.

$$\frac{6+A}{10+A} = 0.75$$
$$6 + A = 0.75(10 + A)$$
$$6 + A = 7.5 + 0.75A$$
$$0.25A = 1.5$$
$$A = 6$$

Thus, if we add 6 liters of pure juice, we have 16 liters of a blend that contains 12 liters of juice, meaning that the concentration of juice in this blend is 75%.

b. How many liters of pure juice need to be added in order to make a blend that is 90% juice?

$$\frac{6+A}{10+A} = 0.90$$
$$6 + A = 0.9(10 + A)$$
$$6 + A = 9 + 0.9A$$
$$3 = 0.1A$$
$$A = 30$$

Thus, if we add 30 liters of pure juice, we will have 40 liters of a blend that contains 36 liters of pure juice, meaning that the concentration of juice in this blend is 90%.

MP.4

c. Write a rational equation that relates the desired percentage p to the amount A of pure juice that needs to be added to make a blend that is $p\%$ juice, where $0 < p < 100$. What is a reasonable restriction on the set of possible values of p? Explain your answer.

$$\frac{6+A}{10+A} = \frac{p}{100}$$

We need to have $60 < p < 100$ for the problem to make sense. We already have 60% juice; the percentage cannot decrease by adding more juice, and we can never have a mixture that is more than 100 percent juice.

d. Suppose that you have added 15 liters of juice to the original 10 liters. What is the percentage of juice in this blend?

$$\frac{p}{100} = \frac{6+15}{10+15} = 0.84$$

So, the new blend contains 84% pure juice.

MP.2

e. Solve your equation in part (c) for the amount A. Are there any excluded values of the variable p? Does this make sense in the context of the problem?

$$\begin{aligned}
\frac{6+A}{10+A} &= \frac{p}{100} \\
100(6+A) &= p(10+A) \\
600+100A &= 10p+pA \\
100A-pA &= 10p-600 \\
A(100-p) &= 10p-600 \\
A &= \frac{10p-600}{100-p}
\end{aligned}$$

We see from the equation for A that $p \neq 100$. This makes sense because we can never make a 100% juice solution since we started with a diluted amount of juice.

Exercise 4 (5 minutes)

Allow students to work together in pairs or small groups for this exercise. This exercise is a bit different from the previous example in that the amount of acid comes from a diluted solution, not a pure solution. Be sure that students get the numerator set up correctly. (If there is not enough time to do the entire problem, have students set up the equations in class and finish solving them for homework.)

4. You have a solution containing 10% acid and a solution containing 30% acid.

a. How much of the 30% solution must you add to 1 liter of the 10% solution to create a mixture that is 22% acid?

If we add A liters of the 30% solution, then the new mixture is $1+A$ liters of solution that contains $0.1+0.3A$ liters of acid. We want the final mixture to be 22% acid, so we need to solve the equation:

$$\frac{0.1+0.3A}{1+A} = 0.22.$$

Solving this gives

$$\begin{aligned}
0.1+0.3A &= 0.22(1+A) \\
0.1+0.3A &= 0.22+0.22A \\
0.08A &= 0.12 \\
A &= 1.5.
\end{aligned}$$

Thus, if we add 1.5 liters of 30% acid solution to 1 liter of 10% acid solution, the result is 2.5 liters of 22% acid solution.

MP.4

b. Write a rational equation that relates the desired percentage p to the amount A of 30% acid solution that needs to be added to 1 liter of 10% acid solution to make a blend that is $p\%$ acid, where $0 < p < 100$. What is a reasonable restriction on the set of possible values of p? Explain your answer.

$$\frac{0.1 + 0.3A}{1 + A} = \frac{p}{100}$$

We must have $10 < p < 30$ because if we blend a 10% acid solution and a 30% acid solution, the blend will contain an acid percentage between 10% and 30%.

MP.2

c. Solve your equation in part (b) for A. Are there any excluded values of p? Does this make sense in the context of the problem?

$$A = \frac{10 - p}{p - 30}$$

We need to exclude 30 from the possible range of values of p, which makes sense in context because we can never reach a 30% acid solution since we started with a solution that was 10% acid.

d. If you have added some 30% acid solution to 1 liter of 10% acid solution to make a 26% acid solution, how much of the stronger acid did you add?

The formula in part (c) gives $A = \frac{10-26}{26-30}$; therefore, $A = 4$. We added 4 liters of the 30% acid solution to the 1 liter of 10% acid solution to make a 26% acid mixture.

Closing (2 minutes)

Ask students to summarize the important parts of the lesson in writing, to a partner, or as a class. Use this as an opportunity to informally assess understanding of the lesson.

Lesson Summary

In this lesson, we developed the students' problem solving skills by asking them to carefully read a problem, rephrase it in a form comfortable for their own understanding, and convert fact sentences about unknown quantities into algebraic equations. Specifically, they used rational equations to model and solve some application problems and further developed their skills in working with rational expressions.

Exit Ticket (5 minutes)

Name _____ Date _____

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Exit Ticket

Bob can paint a fence in 5 hours, and working with Jen, the two of them painted the fence in 2 hours. How long would it have taken Jen to paint the fence alone?

Exit Ticket Sample Solutions

Bob can paint a fence in 5 hours, and working with Jen, the two of them painted the fence in 2 hours. How long would it have taken Jen to paint the fence alone?

Let x represent the time it would take Jen to paint the fence alone. Then, Bob can paint the entire fence in 5 hours; therefore, in one hour he can paint $\frac{1}{5}$ of the fence. Similarly, Jen can paint $\frac{1}{x}$ of the fence in one hour. We know that it took them two hours to complete the job, and together they can paint $\frac{1}{2}$ of the fence in one hour. We then have to solve the equation:

$$\frac{1}{5}+\frac{1}{x}=\frac{1}{2}$$
$$\frac{2x}{10x}+\frac{10}{10x}=\frac{5x}{10x}$$
$$2x+10=5x$$
$$x=\frac{10}{3}.$$

Thus, it would have taken Jen 3 hours and 20 minutes to paint the fence alone.

Problem Set Sample Solutions

1. **If 2 inlet pipes can fill a pool in one hour and 30 minutes, and one pipe can fill the pool in two hours and 30 minutes on its own, how long would the other pipe take to fill the pool on its own?**

$$\frac{1}{2.5}+\frac{1}{x}=\frac{1}{1.5}$$

We find that $x=3.75$; therefore, it takes 3 hours and 45 minutes for the second pipe to fill the pool by itself.

2. **If one inlet pipe can fill the pool in 2 hours with the outlet drain closed, and the same inlet pipe can fill the pool in 2.5 hours with the drain open, how long does it take the drain to empty the pool if there is no water entering the pool?**

$$\frac{1}{2}-\frac{1}{x}=\frac{1}{2.5}$$

We find that $x=10$; therefore, it takes 10 hours for the drain to empty the pool by itself.

3. **It takes 36 minutes less time to travel 120 miles by car at night than by day because the lack of traffic allows the average speed at night to be 10 miles per hour faster than in the daytime. Find the average speed in the daytime.**

$$\frac{120}{t-36}=\frac{120}{t}+\frac{1}{6}$$

We find that $t=180$ minutes, which is 3 hours; therefore, the average speed in the daytime is $\frac{120}{3}=40$ miles per hour.

4. The difference in the average speed of two trains is 16 miles per hour. The slower train takes 2 hours longer to travel 170 miles than the faster train takes to travel 150 miles. Find the speed of the slower train.

$$\frac{150}{t}-\frac{170}{t+2}=16$$

We find that $t=3$ hours. The average speed of the slower train is 34 miles per hour.

5. A school library spends $\$80$ a month on magazines. The average price for magazines bought in January was 70 cents more than the average price in December. Because of the price increase, the school library was forced to subscribe to 7 fewer magazines. How many magazines did the school library subscribe to in December?

$$\frac{80}{x+0.70}=\frac{80}{x}-7$$

The average price in December is $x=\$2.50$, so the school subscribed to 32 magazines in December.

6. An investor bought a number of shares of stock for $\$1,600$. After the price dropped by $\$10$ per share, the investor sold all but 4 of her shares for $\$1,120$. How many shares did she originally buy?

$$\frac{1600}{x}=\frac{1120}{x-4}+10$$

Two possible answers: $x=32$ shares or $x=20$ shares

7. Newton's law of universal gravitation, $F=\frac{Gm_1m_2}{r^2}$, measures the force of gravity between two masses m_1 and m_2, where r is the distance between the centers of the masses, and G is the universal gravitational constant. Solve this equation for G.

$$G=\frac{Fr^2}{m_1m_2}$$

8. Suppose that $=\frac{x+y}{1-xy}$.

   a. Show that when $x=\frac{1}{a}$ and $y=\frac{2a-1}{a+2}$, the value of t does not depend on the value of a.

   When simplified, we find that $t=2$; therefore, the value of t does not depend on the value of a.

   b. For which values of a do these relationships have no meaning?

   If a is 0, then x has no meaning. If $a=-2$, then y has no meaning.

9. Consider the rational equation $\frac{1}{R}=\frac{1}{x}+\frac{1}{y}$.

   a. Find the value of R when $x=\frac{2}{5}$ and $y=\frac{3}{4}$.

$$\frac{1}{R}=\frac{1}{2/5}+\frac{1}{3/4}$$

   So $R=\frac{6}{23}$.

b. Solve this equation for R and simplify.

There are two approaches to solve this equation for R.

The first way is to perform the addition on the right:

$$\begin{aligned}\frac{1}{R} &= \frac{1}{x}+\frac{1}{y}\\ &= \frac{y}{xy}+\frac{x}{xy}\\ &= \frac{x+y}{xy}.\end{aligned}$$

The second way is to take reciprocals of both sides, and then simplify:

$$\begin{aligned}R &= \frac{1}{1/x+1/y}\\ &= \frac{1}{y/xy+x/xy}\\ &= \frac{1}{(x+y)/xy}.\end{aligned}$$

In either case, we find that $R=\frac{xy}{x+y}$.

10. Consider an ecosystem of rabbits in a park that starts with 10 rabbits and can sustain up to 60 rabbits. An equation that roughly models this scenario is

$$P=\frac{60}{1+\frac{5}{t+1}},$$

where P represents the rabbit population in year t of the study.

a. What is the rabbit population in year 10? Round your answer to the nearest whole rabbit.

If $t=10$, then $P=41.25$; therefore, there are 41 rabbits in the park.

b. Solve this equation for t. Describe what this equation represents in the context of this problem.

$$t=\frac{60-6P}{P-60}$$

This equation represents the relationship between the number of rabbits, P, and the year, t. If we know how many rabbits we have, $10<P<60$, we will know how long it took for the rabbit population to grow that much. If the population is 10, then this equation says we are in year 0 of the study, which fits with the given scenario.

c. At what time does the population reach 50 rabbits?

If $P=50$, then $t=\frac{60-300}{50-60}=-\frac{240}{-10}=24$; therefore, the rabbit population is 50 in year 24 of the study.

Extension:

11. Suppose that Huck Finn can paint a fence in 5 hours. If Tom Sawyer helps him paint the fence, they can do it in 3 hours. How long would it take for Tom to paint the fence by himself?

Huck paints the fence in 5 hours, so his rate of fence painting is $\frac{1}{5}$ fence per hour. Let T denote the percentage of the fence Tom can paint in an hour. Then

$$1 \text{ fence} = \left(\left(\frac{1}{5}+T\right)\text{ fence per hour}\right)\cdot(3 \text{ hours}).$$

$$\begin{aligned}3 &= \frac{1}{\frac{1}{5}+T}=\frac{1}{\frac{1}{5}+\frac{5T}{5}}=\frac{5}{1+5T}\\ 3(1+5T) &= 5\\ 15T &= 2\\ T &= \frac{2}{15}\end{aligned}$$

So, Tom can paint $\frac{2}{15}$ of the fence in an hour. Thus, Tom would take $\frac{15}{2}=7.5$ hours to paint the fence by himself.

12. Huck Finn can paint a fence in 5 hours. After some practice, Tom Sawyer can now paint the fence in 6 hours.

a. How long would it take Huck and Tom to paint the fence together?

The amount of fence that Huck can paint per hour is $\frac{1}{5}$, and the amount that Tom can paint per hour is $\frac{1}{6}$. So, together they can paint $\frac{1}{5}+\frac{1}{6}$ of the fence per hour. Suppose the entire job of painting the fence takes h hours. Then, the amount of the fence that is painted is $h\left(\frac{1}{5}+\frac{1}{6}\right)$. Since the entire fence is painted, we need to solve the equation $h\left(\frac{1}{5}+\frac{1}{6}\right)=1$.

$$h\left(\frac{1}{5}+\frac{1}{6}\right)=1$$
$$h=\frac{1}{\frac{1}{5}+\frac{1}{6}}=\frac{30}{6+5}=\frac{30}{11}$$

So, together they can paint the fence in $\frac{30}{11}$ hours, which is 2 hours and 44 minutes.

b. Tom demands a half hour break while Huck continues to paint, and they finish the job together. How long does it take them to paint the fence?

Suppose the entire job of painting the fence takes h hours. Then, Huck paints at a rate of $\frac{1}{5}$ of the fence per hour for h hours, so he paints $\frac{h}{5}$ of the fence. Tom paints at a rate of $\frac{1}{6}$ of the fence per hour for $h-\frac{1}{2}$ hours, so he paints $\frac{1}{6}\left(h-\frac{1}{2}\right)$ of the fence. Together, they paint the whole fence; so, we need to solve the following equation for h:

$$\frac{1}{5}h+\frac{1}{6}\left(h-\frac{1}{2}\right)=1$$
$$\frac{1}{5}h+\frac{1}{6}h-\frac{1}{12}=1$$
$$\frac{1}{5}h+\frac{1}{6}h=\frac{13}{12}$$
$$60\left(\frac{1}{5}h+\frac{1}{6}h\right)=60\cdot\frac{13}{12}$$
$$12h+10h=65$$
$$h=\frac{65}{22}.$$

Thus, it takes $\frac{65}{22}$ hours, which is 2 hours 57 minutes, to paint the fence with Tom taking a $\frac{1}{2}$ hour break.

c. Suppose that they have to finish the fence in $3\frac{1}{2}$ hours. What's the longest break that Tom can take?

Suppose the entire job of painting the fence takes $\frac{7}{2}$ hours, and Tom stops painting for b hours for his break. Then, Huck paints at a rate of $\frac{1}{5}$ of the fence per hour for $\frac{7}{2}$ hours, so he paints $\frac{7}{10}$ of the fence. Tom paints at a rate of $\frac{1}{6}$ of the fence per hour for $\left(\frac{7}{2}-b\right)$ hours, so he paints $\frac{1}{6}\left(\frac{7}{2}-b\right)$ of the fence. Together, they paint the whole fence; so, we need to solve the following equation for b:

$$\frac{7}{10}+\frac{1}{6}\left(\frac{7}{2}-b\right)=1$$
$$\frac{7}{10}+\frac{7}{12}-\frac{b}{6}=1$$
$$60\left(\frac{7}{10}+\frac{7}{12}-\frac{b}{6}\right)=60$$
$$42+35-10b=60$$
$$42+35-60=10b$$
$$b=\frac{17}{10}.$$

Thus, if Tom takes a break for $\frac{17}{10}$ hours, which is 1 hour and 42 minutes, the fence will be painted in $3\frac{1}{2}$ hours.