Lesson 25: Adding and Subtracting Rational Expressions

Student Outcomes

- Students perform addition and subtraction of rational expressions.

Lesson Notes

In this lesson, we review addition and subtraction of fractions using the familiar number line technique that students have seen in earlier grades. This leads to an algebraic explanation of how to add and subtract fractions and an opportunity to practice MP.7. We then move to the process for adding and subtracting rational expressions by converting to equivalent rational expressions with a common denominator. As in the past three lessons, we continue to draw parallels between arithmetic of rational numbers and arithmetic of rational expressions.

Classwork

The four basic arithmetic operations are addition, subtraction, multiplication, and division. The previous lesson showed how to multiply and divide rational expressions. This lesson tackles the remaining operations of addition and subtraction of rational expressions, which are skills needed to address **A-APR.C.6**. As discussed in the previous lesson, we operate with rational expressions in the same way we work with rational numbers expressed as fractions. First, we will review the theory behind addition and subtraction of rational numbers.

Exercise 1 (8 minutes)

First, remind students how to add fractions with the same denominator. Allow them to work through the following sum individually. The solution should be presented to the class either by the teacher or by a student because we are going to extend the process of adding fractions to the new process of adding rational expressions.

Scaffolding:

If students need practice adding and subtracting fractions with a common denominator, have them compute the following.

- $\frac{2}{5}+\frac{1}{5}$
- $\frac{5}{7}-\frac{3}{7}$
- $\frac{17}{24}-\frac{12}{24}$

Exercises 1–4

1. **Calculate the following sum: $\frac{3}{10}+\frac{6}{10}$.**

One approach to this calculation is to factor out $\frac{1}{10}$ from each term.

$$\begin{aligned}\frac{3}{10}+\frac{6}{10}&=3\cdot\frac{1}{10}+6\cdot\frac{1}{10}\\&=(3+6)\cdot\frac{1}{10}\\&=\frac{9}{10}\end{aligned}$$

Ask students for help in stating the rule for adding and subtracting rational numbers with the same denominator.

> If a, b, and c are integers with $b \neq 0$, then
> $$\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b} \quad \text{and} \quad \frac{a}{b}-\frac{c}{b}=\frac{a-c}{b}.$$

The result in the box above is also valid for real numbers a, b, and c.

- But what if the fractions have different denominators? Let's examine a technique to add the fractions $\frac{2}{5}$ and $\frac{1}{3}$.
- Recall that when we first learned to add fractions, we represented them on a number line. Let's first look at $\frac{2}{5}$.

- And we want to add to this the fraction $\frac{1}{3}$.

- If we try placing these two segments next to each other, the exact location of the endpoint is difficult to identify.

- The units on the two original graphs do not match. We need to identify a common unit in order to identify the endpoint of the combined segments. We need to identify a number into which both denominators divide without remainder and write each fraction as an equivalent fraction with that number as denominator; such a number is known as a *common denominator*.
- Since 15 is a common denominator of $\frac{2}{5}$ and $\frac{1}{3}$, we divide the interval $[0, 1]$ into 15 parts of equal length. Now when we look at the segments of length $\frac{2}{5}$ and $\frac{1}{3}$ placed next to each other on the number line, we can see that the combined segment has length $\frac{11}{15}$.

- How can we do this without using the number line every time? The fraction $\frac{2}{5}$ is equivalent to $\frac{6}{15}$, and the fraction $\frac{1}{3}$ is equivalent to $\frac{5}{15}$. We then have

$$\begin{aligned}\frac{2}{5}+\frac{1}{3}&=\frac{6}{15}+\frac{5}{15}\\&=\frac{11}{15}.\end{aligned}$$

- Thus, when adding rational numbers, we have to find a common multiple for the two denominators and write each rational number as an equivalent rational number with the new common denominator. Then we can add the numerators together.

Have students discuss how to rewrite the original fraction as an equivalent fraction with the chosen common denominator. Discuss how the identity property of multiplication allows you to multiply the top and the bottom by the same number so that product of the original denominator and the number gives the chosen common denominator.

- Generalizing, let's add together two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$. The first step is to rewrite both fractions as equivalent fractions with the same denominator. A simple common denominator that could be used is the product of the original two denominators:
$$\frac{a}{b}+\frac{c}{d}=\frac{ad}{bd}+\frac{bc}{bd}.$$
- Once we have a common denominator, we can add the two expressions together, using our previous rule for adding two expressions with the same denominator:
$$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}.$$
- We could use the same approach to develop a process for subtracting rational numbers:
$$\frac{a}{b}-\frac{c}{d}=\frac{ad-bc}{bd}.$$
- Now that we know to find a common denominator before adding or subtracting, we can state the general rule for adding and subtracting rational numbers. Notice that one common denominator that will always work is the product of the two original denominators.

> If a, b, c, and d are integers with $b \neq 0$ and $d \neq 0$, then
> $$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd} \quad \text{and} \quad \frac{a}{b}-\frac{c}{d}=\frac{ad-bc}{bd}.$$

As with our other rules developed in this and the previous lesson, the rule summarized in the box above is also valid for real numbers.

Exercises 2–4 (5 minutes)

Ask the students to work in groups to write what they have learned in their notebooks or journals. Check in to assess their understanding. Then, have students work in pairs to quickly work through the following review exercises. Allow them to think about how to approach Exercise 4, which involves adding three rational expressions. There are multiple ways to approach this problem. They could generalize the process for two rational expressions, rearrange terms using the commutative property to combine the terms with the same denominator, and then add using the above process, or they could group the addends using the associative property and perform addition twice.

2. $\frac{3}{20}-\frac{4}{15}$

$$\frac{3}{20}-\frac{4}{15}=\frac{9}{60}-\frac{16}{60}=-\frac{7}{60}$$

3. $\frac{\pi}{4} + \frac{\sqrt{2}}{5}$

$\frac{\pi}{4} + \frac{\sqrt{2}}{5} = \frac{5\pi}{20} + \frac{4\sqrt{2}}{20} = \frac{5\pi + 4\sqrt{2}}{20}$

4. $\frac{a}{m} + \frac{b}{2m} - \frac{c}{m}$

$\frac{a}{m} + \frac{b}{2m} - \frac{c}{m} = \frac{2a}{2m} + \frac{b}{2m} - \frac{2c}{2m} = \frac{2a + b - 2c}{2m}$

Discussion (2 minutes)

- Before we can add rational numbers or rational expressions, we need to convert to equivalent rational expressions with the same denominators. Finding such a denominator involves finding a common multiple of the original denominators. For example, 60 is a common multiple of 20 and 15. There are other common multiples, such as 120, 180, and 300, but smaller numbers are easier to work with.
- To add and subtract rational expressions, we follow the same procedure as when adding and subtracting rational numbers. First, we find a denominator that is a common multiple of the other denominators, and then we rewrite each expression as an equivalent rational expression with this new common denominator. We then apply the rule for adding or subtracting with the same denominator.

If a, b, and c are rational expressions with $b \neq 0$, then

$$\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b} \quad \text{and} \quad \frac{a}{b} - \frac{c}{b} = \frac{a-c}{b}.$$

Example 1 (10 minutes)

Work through these examples as a class, getting input from the students at each step.

Example 1

Perform the indicated operations below and simplify.

a. $\frac{a+b}{4} + \frac{2a-b}{5}$

A common multiple of 4 and 5 is 20, so we can write each expression as an equivalent rational expression with denominator 20. We have $\frac{a+b}{4} = \frac{5a+5b}{20}$ and $\frac{2a-b}{5} = \frac{8a-4b}{20}$, so that

$\frac{a+b}{4} + \frac{2a-b}{5} = \frac{5a+5b}{20} + \frac{8a-4b}{20} = \frac{13a+b}{20}$.

b. $\frac{4}{3x} - \frac{3}{5x^2}$

A common multiple of $3x$ and $5x^2$ is $15x^2$, so we can write each expression as an equivalent rational expression with denominator $15x^2$. We have $\frac{4}{3x} - \frac{3}{5x^2} = \frac{20x}{15x^2} - \frac{9}{15x^2} = \frac{20x-9}{15x^2}$.

c. $\frac{3}{2x^2+2x}+\frac{5}{x^2-3x-4}$

Since $2x^2+2x=2x(x+1)$ and $x^2-3x-4=(x-4)(x+1)$, a common multiple of $2x^2+2x$ and x^2-3x-4 is $2x(x+1)(x-4)$. Then we have $\frac{3}{2x^2+2x}+\frac{5}{x^2-3x-4}=\frac{3(x-4)}{2x(x+1)(x-4)}+\frac{5\cdot 2x}{2x(x+1)(x-4)}=\frac{13x-12}{2x(x+1)(x-4)}$.

Exercises 5–8 (8 minutes)

Have students work on these exercises in pairs or small groups.

Exercises 5–8

Perform the indicated operations for each problem below.

5. $\frac{5}{x-2}+\frac{3x}{4x-8}$

A common multiple is $4(x-2)$.

$$\frac{5}{x-2}+\frac{3x}{4x-8}=\frac{20}{4(x-2)}+\frac{3x}{4(x-2)}=\frac{3x+20}{4(x-2)}$$

6. $\frac{7m}{m-3}+\frac{5m}{3-m}$

Notice that $(3-m)=-(m-3)$.

A common multiple is $(m-3)$.

$$\frac{7m}{m-3}+\frac{5m}{3-m}=\frac{7m}{m-3}+\frac{-5m}{m-3}=\frac{7m}{m-3}-\frac{5m}{m-3}=\frac{2m}{m-3}$$

7. $\frac{b^2}{b^2-2bc+c^2}-\frac{b}{b-c}$

A common multiple is $(b-c)(b-c)$.

$$\frac{b^2}{b^2-2bc+c^2}-\frac{b}{b-c}=\frac{b^2}{(b-c)(b-c)}-\frac{b^2-bc}{(b-c)(b-c)}=\frac{bc}{(b-c)^2}$$

8. $\frac{x}{x^2-1}-\frac{2x}{x^2+x-2}$

A common multiple is $(x-1)(x+1)(x+2)$.

$$\frac{x}{x^2-1}-\frac{2x}{x^2+x-2}=\frac{x}{(x-1)(x+1)}-\frac{2x}{(x-1)(x+2)}=\frac{x(x+2)}{(x-1)(x+1)(x+2)}-\frac{2x(x+1)}{(x-1)(x+1)(x+2)}=\frac{-x^2}{(x-1)(x+1)(x+2)}$$

Example 2 (5 minutes)

Complex fractions were introduced in the previous lesson with multiplication and division of rational expressions, but these examples require performing addition and subtraction operations prior to doing the division. Remind students that when rewriting a complex fraction as division of rational expressions, they should add parentheses to the expressions both in the numerator and denominator. Then they should work inside the parentheses first following the standard order of operations.

MP.7

Example 2

Simplify the following expression.

$$\frac{\frac{b^2+b-1}{2b-1}-1}{4-\frac{8}{(b+1)}}$$

First, we can rewrite the complex fraction as a division problem, remembering to add parentheses.

$$\frac{\frac{b^2+b-1}{2b-1}-1}{4-\frac{8}{(b+1)}}=\left(\frac{b^2+b-1}{2b-1}-1\right)\div\left(4-\frac{8}{(b+1)}\right)$$

Remember that to divide rational expressions, we multiply by the reciprocal of the quotient. However, we first need to write each rational expression in an equivalent $\frac{P}{Q}$ form. For this, we need to find common denominators.

$$\begin{aligned}\frac{b^2+b-1}{2b-1}-1&=\frac{b^2+b-1}{2b-1}-\frac{2b-1}{2b-1}\\&=\frac{b^2-b}{2b-1}\end{aligned}$$

$$\begin{aligned}4-\frac{8}{(b+1)}&=\frac{4(b+1)}{b+1}-\frac{8}{(b+1)}\\&=\frac{4b-4}{(b+1)}\\&=\frac{4(b-1)}{b+1}\end{aligned}$$

Now, we can substitute these equivalent expressions into our calculation above and continue to perform the division as we did in Lesson 24.

$$\begin{aligned}\frac{\frac{b^2+b-1}{2b-1}-1}{4-\frac{8}{(b+1)}}&=\left(\frac{b^2+b-1}{2b-1}-1\right)\div\left(4-\frac{8}{(b+1)}\right)\\&=\left(\frac{b^2-b}{2b-1}\right)\div\left(\frac{4(b-1)}{b+1}\right)\\&=\left(\frac{b(b-1)}{2b-1}\right)\cdot\left(\frac{(b+1)}{4(b-1)}\right)\\&=\frac{b(b+1)}{4(2b-1)}\end{aligned}$$

Closing (2 minutes)

Ask students to summarize the important parts of the lesson in writing, to a partner, or as a class. Use this opportunity to informally assess their understanding of the lesson. In particular, ask students to verbally or symbolically articulate the processes for adding and subtracting rational expressions.

Lesson Summary

In this lesson, we extended addition and subtraction of rational numbers to addition and subtraction of rational expressions. The process for adding or subtracting rational expressions can be summarized as follows:

- **Find a common multiple of the denominators to use as a common denominator.**
- **Find equivalent rational expressions for each expression using the common denominator.**
- **Add or subtract the numerators as indicated and simplify if needed.**

Exit Ticket (5 minutes)

Name ______________________________ Date ______________

Lesson 25: Adding and Subtracting Rational Expressions

Exit Ticket

Perform the indicated operation.

1. $\frac{3}{a+2} + \frac{4}{a-5}$

2. $\frac{4r}{r+3} - \frac{5}{r}$

Exit Ticket Sample Solutions

Perform the indicated operation.

1. $\frac{3}{a+2}+\frac{4}{a-5}$

$$\frac{3}{a+2}+\frac{4}{a-5}=\frac{3a-15}{(a+2)(a-5)}+\frac{4a+8}{(a+2)(a-5)}$$
$$=\frac{7a-7}{(a+2)(a-5)}$$

2. $\frac{4r}{r+3}-\frac{5}{r}$

$$\frac{4r}{r+3}-\frac{5}{r}=\frac{4r^2}{r(r+3)}-\frac{5r+15}{r(r+3)}$$
$$=\frac{4r^2-5r-15}{r(r+3)}$$

Problem Set Sample Solutions

1. **Write each sum or difference as a single rational expression.**

   a. $\frac{7}{8}-\frac{\sqrt{3}}{5}$

   $\frac{35-8\sqrt{3}}{40}$

   b. $\frac{\sqrt{5}}{10}+\frac{\sqrt{2}}{6}+2$

   $\frac{3\sqrt{5}+5\sqrt{2}+60}{30}$

   c. $\frac{4}{x}+\frac{3}{2x}$

   $\frac{11}{2x}$

2. **Write as a single rational expression.**

   a. $\frac{1}{x}-\frac{1}{x-1}$

   $-\frac{1}{x(x-1)}$

   b. $\frac{3x}{2y}-\frac{5x}{6y}+\frac{x}{3y}$

   $\frac{x}{y}$

   c. $\frac{a-b}{a^2}+\frac{1}{a}$

   $\frac{2a-b}{a^2}$

   d. $\frac{1}{p-2}-\frac{1}{p+2}$

   $\frac{4}{(p-2)(p+2)}$

   e. $\frac{1}{p-2}+\frac{1}{2-p}$

   0

   f. $\frac{1}{b+1}-\frac{b}{1+b}$

   $\frac{1-b}{b+1}$

g. $1-\frac{1}{1+p}$

$\frac{p}{1+p}$

h. $\frac{p+q}{p-q}-2$

$\frac{3q-p}{p-q}$

i. $\frac{r}{s-r}+\frac{s}{r+s}$

$\frac{r^2+s^2}{(s-r)(r+s)}$

j. $\frac{3}{x-4}+\frac{2}{4-x}$

$\frac{1}{x-4}$

k. $\frac{3n}{n-2}+\frac{3}{2-n}$

$\frac{3n-3}{n-2}$

l. $\frac{8x}{3y-2x}+\frac{12y}{2x-3y}$

-4

m. $\frac{1}{2m-4n}-\frac{1}{2m+4n}-\frac{m}{m^2-4n^2}$

$-\frac{1}{m+2n}$

n. $\frac{1}{(2a-b)(a-c)}+\frac{1}{(b-c)(b-2a)}$

$\frac{b-a}{(a-c)(b-c)(2a-b)}$

o. $\frac{b^2+1}{b^2-4}+\frac{1}{b+2}+\frac{1}{b-2}$

$\frac{b^2+2b+1}{(b-2)(b+2)}$

3. Simplify the following expressions.

a. $\dfrac{\frac{1}{a}-\frac{1}{2a}}{\frac{4}{a}}$

$\frac{1}{8}$

b. $\dfrac{\frac{5x}{2}+1}{\frac{5x}{4}-\frac{1}{5x}}$

$\frac{10x}{5x-2}$

c. $\dfrac{1+\frac{4x+3}{x^2+1}}{1-\frac{x+7}{x^2+1}}$

$\frac{x+2}{x-3}$

EXTENSION:

4. Suppose that $x \neq 0$ and $y \neq 0$. We know from our work in this section that $\frac{1}{x}\cdot\frac{1}{y}$ is equivalent to $\frac{1}{xy}$. Is it also true that $\frac{1}{x}+\frac{1}{y}$ is equivalent to $\frac{1}{x+y}$? Provide evidence to support your answer.

No, the rational expressions $\frac{1}{x}+\frac{1}{y}$ and $\frac{1}{x+y}$ are not equivalent. Consider $x=2$ and $y=1$. Then $\frac{1}{x+y}=\frac{1}{2+1}=\frac{1}{3}$, but $\frac{1}{x}+\frac{1}{y}=\frac{1}{2}+1=\frac{3}{2}$. Since $\frac{1}{3}\neq\frac{3}{2}$, the expressions $\frac{1}{x}+\frac{1}{y}$ and $\frac{1}{x+y}$ are not equivalent.

5. Suppose that $x=\frac{2t}{1+t^2}$ and $y=\frac{1-t^2}{1+t^2}$. Show that the value of x^2+y^2 does not depend on the value of t.

$$\begin{aligned}
x^2+y^2 &= \left(\frac{2t}{1+t^2}\right)^2+\left(\frac{1-t^2}{1+t^2}\right)^2 \\
&= \frac{4t^2}{(1+t^2)^2}+\frac{(1-t^2)^2}{(1+t^2)^2} \\
&= \frac{4t^2+(1-2t^2+t^4)}{(1+t^2)^2} \\
&= \frac{1+2t^2+t^4}{1+2t^2+t^4} \\
&= 1
\end{aligned}$$

Since $x^2+y^2=1$, the value of x^2+y^2 does not depend on the value of t.

6. Show that for any real numbers a and b, and any integers x and y so that $x \neq 0$, $y \neq 0$, $x \neq y$, and $x \neq -y$,

$$\left(\frac{y}{x}-\frac{x}{y}\right)\left(\frac{ax+by}{x+y}-\frac{ax-by}{x-y}\right)=2(a-b).$$

$$\begin{aligned}\left(\frac{y}{x}-\frac{x}{y}\right)\left(\frac{ax+by}{x+y}-\frac{ax-by}{x-y}\right) &= \left(\frac{y^2}{xy}-\frac{x^2}{xy}\right)\left(\frac{(ax+by)(x-y)}{(x+y)(x-y)}-\frac{(ax-by)(x+y)}{(x-y)(x+y)}\right)\\ &= \left(\frac{y^2-x^2}{xy}\right)\left(\frac{ax^2-axy+bxy-by^2}{x^2-y^2}-\frac{(ax^2+axy-bxy-by^2}{x^2-y^2}\right)\\ &= -\left(\frac{x^2-y^2}{xy}\right)\left(\frac{-2axy+2bxy}{x^2-y^2}\right)\\ &= -\left(\frac{1}{xy}\right)\left(\frac{-2xy(a-b)}{1}\right)\\ &= 2(a-b)\end{aligned}$$

7. Suppose that n is a positive integer.

a. Simplify the expression $\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n+1}\right)$.

$$\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n+1}\right)=\left(\frac{n+1}{n}\right)\left(\frac{n+2}{n+1}\right)=\left(\frac{n+2}{n}\right)$$

b. Simplify the expression $\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n+1}\right)\left(1+\frac{1}{n+2}\right)$.

$$\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n+1}\right)\left(1+\frac{1}{n+2}\right)=\left(\frac{n+1}{n}\right)\left(\frac{n+2}{n+1}\right)\left(\frac{n+3}{n+2}\right)=\left(\frac{n+3}{n}\right)$$

c. Simplify the expression $\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n+1}\right)\left(1+\frac{1}{n+2}\right)\left(1+\frac{1}{n+3}\right)$.

$$\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n+1}\right)\left(1+\frac{1}{n+2}\right)\left(1+\frac{1}{n+3}\right)=\left(\frac{n+1}{n}\right)\left(\frac{n+2}{n+1}\right)\left(\frac{n+3}{n+2}\right)\left(\frac{n+4}{n+3}\right)=\left(\frac{n+4}{n}\right)$$

MP.7

d. If this pattern continues, what is the product of n of these factors?

If we have n of these factors, then the product will be

$$\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n+1}\right)\cdots\left(1+\frac{1}{n+(n-1)}\right)=\frac{n+n}{n}=2.$$