Lesson 22: Equivalent Rational Expressions

Student Outcomes

- Students define rational expressions and write them in equivalent forms.

Lesson Notes

In this module, we have been working with polynomials and polynomial functions. In elementary school, students mastered arithmetic operations with integers before advancing to performing arithmetic operations with rational numbers. Just as a rational number is built from integers, a rational expression is built from polynomial expressions. A precise definition of a rational expression is included at the end of the lesson.

Informally, a rational expression is any expression that is made by a finite sequence of addition, subtraction, multiplication, and division operations on polynomials. After algebraic manipulation, a rational expression can always be written as $\frac{P}{Q}$, where P is any polynomial and Q is any polynomial except the zero polynomial. Remember that constants, such as 2, and variables, such as x, count as polynomials, so the rational numbers are also considered to be rational expressions. Standard **A-APR.C.6** focuses on rewriting rational expressions in equivalent forms, and in the next three lessons, we apply that standard to write complicated rational expressions in the simplified form $\frac{P}{Q}$. However, the prompt "simplify the rational expression" does not only mean putting expressions in the form $\frac{P}{Q}$ but also any form that is conducive to solving the problem at hand. The skills developed in Lessons 22–25 are necessary prerequisites for addressing standard **A-REI.A.2**, solving rational equations, which is the focus of Lessons 26 and 27.

Classwork

Opening Exercise (8 minutes)

The Opening Exercise serves two purposes: (1) to reactivate prior knowledge of equivalent fractions, and (2) as a review for students who struggle with fractions. We want students to see that the process they use to reduce a fraction to lowest terms is the same they will use to reduce a rational expression to lowest terms. To begin, pass out 2–3 notecard-sized slips of paper to each student or pair of students.

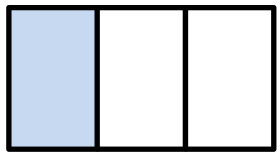
- We are going to start with a review of how to visualize equivalent fractions.

Opening Exercise

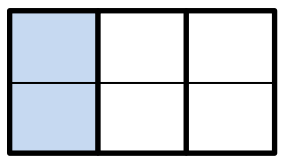
On your own or with a partner, write two fractions that are equivalent to $\frac{1}{3}$, and use the slips of paper to create visual models to justify your response.

Use the following to either walk through the exercise for scaffolding or as an example of likely student responses.

- We can use the following area model to represent the fraction $\frac{1}{3}$. Because the three boxes have the same area, shading one of the three boxes shows that $\frac{1}{3}$ of the area in the figure is shaded.



- Now, if we draw a horizontal line dividing the columns in half, we have six congruent rectangles, two of which are shaded so that $\frac{2}{6}$ of the area in the figure is shaded.



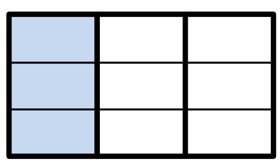
Scaffolding:

Students who are already comfortable with fractions can instead reduce the following rational expressions to lowest terms.

$$\frac{5}{15}, \frac{27}{36}, \frac{\sqrt{75}}{5}, \frac{\pi^4}{\pi^2}$$

In any case, do not spend too much time on these exercises; instead, use them as a bridge to reducing rational expressions that contain variables.

- In the figure below, we have now divided the original rectangle into nine congruent sub-rectangles, three of which are shaded so that $\frac{3}{9}$ of the area in the figure is shaded.



- Let's suppose that the area of the original rectangle is 1. In walking the class through the example, point out that the shaded area in the first figure is $\frac{1}{3}$, the shaded area in the second figure is $\frac{2}{6}$, and the shaded area in the third figure is $\frac{3}{9}$. Since the area of the shaded regions are the same in all three figures, we see that $\frac{1}{3} = \frac{2}{6} = \frac{3}{9}$. Thus, $\frac{1}{3}$, $\frac{2}{6}$, and $\frac{3}{9}$ are equivalent fractions.

If students have come up with different equivalent fractions, then incorporate those into the discussion of equivalent areas, noting that the shaded regions are the same for every student.

MP.8

- Now, what if we were to choose any positive integer n and draw lines across our figure so that the columns are divided into n pieces of the same size? What is the area of the shaded region?

Give students time to think and write, and ask them to share their answers with a partner. Anticipate that students will express the generalization in words or suggest either $\frac{1}{3}$ or $\frac{n}{3n}$. Both are correct and, ideally, both will be suggested.

- Thus, we have the rule:

> If a, b, and n are integers with $n \neq 0$ and $b \neq 0$, then
> $$\frac{na}{nb} = \frac{a}{b}.$$

> *Scaffolding:*
> Students may also express the generalization in words.

The result summarized in the box above is also true for real numbers a, b, and n as well as for polynomial and rational expressions.

- Then $\frac{2}{6} = \frac{2(1)}{2(3)} = \frac{1}{3}$ and $\frac{3}{9} = \frac{3(1)}{3(3)} = \frac{1}{3}$.
- We say that a rational number is simplified, or reduced to lowest terms, when the numerator and denominator do not have a factor in common. Thus, while $\frac{1}{3}, \frac{2}{6}$, and $\frac{3}{9}$ are equivalent, only $\frac{1}{3}$ is in lowest terms.

Discussion (10 minutes)

- Which of the following are rational numbers, and which are not?

$$\frac{3}{4}, 3.14, \pi, \frac{5}{0}, -\sqrt{17}, 23, \frac{1+\sqrt{5}}{2}, -1, 6.022 \times 10^{23}, 0$$

  - *Rational:* $\frac{3}{4}, 3.14, 23, -1, 6.022 \times 10^{23}, 0$
  - *Not rational:* $\pi, \frac{5}{0}, -\sqrt{17}, \frac{1+\sqrt{5}}{2}$

Of the numbers that were not rational, were they all irrational numbers?

  - *No. Since division by zero is undefined,* $\frac{5}{0}$ *is neither rational nor irrational.*

- Today we learn about rational expressions, which are related to the polynomials we've been studying. Just as the integers are the foundational building blocks of rational numbers, polynomial expressions are the foundational building blocks for rational expressions. Based on what we know about rational numbers, give an example of what you think a rational expression is.

> *Scaffolding:*
> Relate the new ideas of rational expressions back to the more familiar ideas of rational numbers throughout this lesson.

Ask students to write down an example and share it with their partner or small group. Allow groups to debate and present one of the group's examples to the class.

- Recall that a rational number is a number that we can write as $\frac{p}{q}$, where p and q are integers and q is nonzero. We can consider a new type of expression, called a *rational expression*, which is made from polynomials by adding, subtracting, multiplying, and dividing them. Any rational expression can be expressed as $\frac{P}{Q}$, where P and Q are polynomial expressions and Q is not zero, even though it may not be presented in this form originally.

Remind students that numbers are also polynomial expressions, which means that rational numbers are included in the set of rational expressions.

- The following are examples of rational expressions. Notice that we need to exclude values of the variables that make the denominators zero so that we do not divide by zero.
- $\frac{31}{47}$
  - *The denominator is never zero, so we do not need to exclude any values.*
- $\frac{ab^2}{3a-2b}$
  - *We need $3a \neq 2b$.*
- $\frac{5x+1}{3x^2+4}$
  - *The denominator is never zero, so we do not need to exclude any values.*
- $\frac{3}{b^2-7}$
  - *We need $b \neq \sqrt{7}$ and $b \neq -\sqrt{7}$.*

Have students create a Frayer model in their notebooks, such as the one provided. Circulate around the classroom to informally assess student understanding. Since a formal definition of rational expressions has not yet been given, there is some leeway on the description and characteristics sections, but make sure that nothing they have written is incorrect. Ask students to share their characteristics, examples, and non-examples to populate a class model on the board.

Description	Characteristics
An expression that can be written as $\frac{P}{Q}$, where P and Q are polynomials and Q is not zero.	*Follows similar rules as rational numbers do.*
Rational Expression	
$\frac{3}{5}$, $\frac{x^2-4x}{x+1}$ *with* $x \neq -1$ $\frac{x^2y}{2}$, $\frac{a^2+b^2}{(a+2)(a-1)}$, *with* $a \neq -2, 1$	$\frac{x+1}{0}$ *(cannot divide by* 0*)* $\frac{2^x}{3x}$ *(*2^x *is not a polynomial)*
Examples	Non-examples

It is important to note that the excluded values of the variables remain even after simplification. This is because the two expressions would not be equal if the variables were allowed to take on these values. Discuss with a partner when the following are not equivalent and why:

- $\frac{2x}{3x}$ and $\frac{2}{3}$
  - *These are equivalent everywhere except at $x = 0$. At $x = 0$, $\frac{2x}{3x}$ is undefined, whereas $\frac{2}{3}$ is equal to $\frac{2}{3}$.*

> *Scaffolding:*
>
> Encourage struggling students to plug in various values of the variables to see that the expressions *are* equivalent for almost all values of the variables. But for values in which the denominator of one expression is equal to zero, they are not equivalent.

- $\frac{3x(x-5)}{4(x-5)}$ and $\frac{3x}{4}$
  - *At* $x = 5$, $\frac{3x(x-5)}{4(x-5)}$ *is undefined, whereas* $\frac{3x}{4} = \frac{3(5)}{4} = \frac{15}{4}$.
- $\frac{x-3}{x^2-x-6}$ and $\frac{1}{x+2}$
  - *At* $x = 3$, $\frac{x-3}{x^2-x-6}$ *is undefined, whereas* $\frac{1}{x+2} = \frac{1}{3+2} = \frac{1}{5}$.
- Summarize with your partner or in writing any conclusions you can draw about equivalent rational expressions. Circulate around the classroom to assess understanding.

Example 1 (6 minutes)

Example 1

Consider the following rational expression: $\frac{2(a-1)-2}{6(a-1)-3a}$. Turn to your neighbor and discuss the following: For what values of a is the expression undefined?

Sample the students' answers. When they suggest that the denominator cannot be zero, give the class a minute to work out that the denominator is zero when $a = 2$.

$$\frac{2(a-1)-2}{6(a-1)-3a}$$

$$\begin{aligned} 6(a-1)-3a &= 0 \\ 6a-6-3a &= 0 \\ 3a-6 &= 0 \\ a &= 2 \end{aligned}$$

- Let's reduce the rational expression $\frac{2(a-1)-2}{6(a-1)-3a}$ with $a \neq 2$ to lowest terms. Since no common factor is visible in the given form of the expression, we first simplify the numerator and denominator by distributing and combining like terms.

$$\begin{aligned} \frac{2(a-1)-2}{6(a-1)-3a} &= \frac{2a-2-2}{6a-6-3a} \\ &= \frac{2a-4}{3a-6} \end{aligned}$$

- Next, we factor the numerator and denominator, and divide both by any common factors. This step shows clearly why we had to specify that $a \neq 2$.

$$\begin{aligned} \frac{2(a-1)-2}{6(a-1)-3a} &= \frac{2a-4}{3a-6} \\ &= \frac{2(a-2)}{3(a-2)} \\ &= \frac{2}{3} \end{aligned}$$

Scaffolding:

Students may need to be reminded that although $(a-1)$ appears in the numerator and denominator, it is not a common *factor* to the numerator and denominator, and thus, we cannot simplify the expression by dividing by $(a-1)$.

- As long as $a \neq 2$, we see that $\frac{2(a-1)-2}{6(a-1)-3a}$ and $\frac{2}{3}$ are equivalent rational expressions.

If we allow a to take on the value of 2, then $\frac{2(a-1)-2}{6(a-1)-3a}$ is undefined. However, the expression $\frac{2}{3}$ is always defined so these expressions are not equivalent.

Exercise 1 (10 minutes)

Allow students to work on the following exercises in pairs.

Exercise 1

Reduce the following rational expressions to lowest terms, and identify the values of the variable(s) that must be excluded to prevent division by zero.

a. $\frac{2(x+1)+2}{(2x+3)(x+1)-1}$

$\frac{2(x+1)+2}{(2x+3)(x+1)-1} = \frac{2x+4}{2x^2+5x+2} = \frac{2(x+2)}{(2x+1)(x+2)} = \frac{2}{2x+1}$, *for $x \neq -2$ and $x \neq -\frac{1}{2}$.*

b. $\frac{x^2-x-6}{5x^2+10x}$

$\frac{x^2-x-6}{5x^2+10x} = \frac{(x+2)(x-3)}{5x(x+2)} = \frac{x-3}{5x}$, *for $x \neq 0$ and $x \neq -2$.*

c. $\frac{3-x}{x^2-9}$

$\frac{3-x}{x^2-9} = \frac{-(x-3)}{(x-3)(x+3)} = -\frac{1}{x+3}$, *for $x \neq 3$ and $x \neq -3$.*

d. $\frac{3x-3y}{y^2-2xy+x^2}$

$\frac{3x-3y}{y^2-2xy+x^2} = \frac{-3(y-x)}{(y-x)(y-x)} = -\left(\frac{3}{y-x}\right)$, *for $y \neq x$.*

Closing (5 minutes)

The precise definition of a rational expression is presented here for teacher reference and may be shared with students at your discretion. Discussion questions for closing this lesson follow the definition. Notice the similarity between the definition of a rational expression given here and the definition of a polynomial expression given in the closing of Lesson 5 earlier in this module.

> **Rational Expression:** A *rational expression* is either a numerical expression or a variable symbol or the result of placing two previously generated rational expressions into the blanks of the addition operator (__+__), the subtraction operator (__−__), the multiplication operator (__×__), or the division operator (__÷__).

Have students discuss the following questions with a partner and write down their conclusions. Circulate around the room to assess their understanding.

- How do you reduce a rational expression of the form $\frac{P}{Q}$ to lowest terms?
  - *Factor the polynomial expressions in the numerator and denominator, and divide any common factors from both the numerator and denominator.*
- How do you know which values of the variable(s) to exclude for a rational expression?
  - *Any value of the variable(s) that makes the denominator zero at any point of the process must be excluded.*

Lesson Summary

- **If a, b, and n are integers with $n \neq 0$ and $b \neq 0$, then**
  $$\frac{na}{nb} = \frac{a}{b}.$$
- **The rule for rational expressions is the same as the rule for integers but requires the domain of the rational expression to be restricted (i.e., no value of the variable that can make the denominator of the original rational expression zero is allowed).**

Exit Ticket (6 minutes)

Name ______________________________ Date ____________________

Lesson 22: Equivalent Rational Expressions

Exit Ticket

1. Find an equivalent rational expression in lowest terms, and identify the value(s) of the variables that must be excluded to prevent division by zero.

$$\frac{x^2 - 7x + 12}{6 - 5x + x^2}$$

2. Determine whether or not the rational expressions $\frac{x+4}{(x+2)(x-3)}$ and $\frac{x^2+5x+4}{(x+1)(x+2)(x-3)}$ are equivalent for $x \neq -1$, $x \neq -2$, and $x \neq 3$. Explain how you know.

Exit Ticket Sample Solutions

1. **Find an equivalent rational expression in lowest terms, and identify the value(s) of the variables that must be excluded to prevent division by zero.**

   If $x \neq 3$ and $x \neq 2$, then we have

   $$\frac{x^2 - 7x + 12}{6 - 5x + x^2} = \frac{(x-4)(x-3)}{(x-3)(x-2)} = \frac{x-4}{x-2}.$$

2. **Determine whether or not the rational expressions $\frac{x+4}{(x+2)(x-3)}$ and $\frac{x^2+5x+4}{(x+1)(x+2)(x-3)}$ are equivalent for $x \neq -1$, $x \neq -2$, and $x \neq 3$. Explain how you know.**

   Since $\frac{x^2+5x+4}{(x+1)(x+2)(x-3)} = \frac{(x+1)(x+4)}{(x+1)(x+2)(x-3)} = \frac{x+4}{(x+2)(x-3)}$ as long as $x \neq -1$, $x \neq -2$, and $x \neq 3$, the rational expressions $\frac{x+4}{(x+2)(x-3)}$ and $\frac{x^2+5x+4}{(x+1)(x+2)(x-3)}$ are equivalent.

Problem Set Sample Solutions

1. **Find an equivalent rational expression in lowest terms, and identify the value(s) of the variable that must be excluded to prevent division by zero.**

   a. $\frac{16n}{20n}$ — $\frac{4}{5}$; $n \neq 0$

   b. $\frac{x^3y}{y^4x}$ — $\frac{x^2}{y^3}$; $x \neq 0$ *and* $y \neq 0$

   c. $\frac{2n-8n^2}{4n}$ — $\frac{1-4n}{2}$; $n \neq 0$

   d. $\frac{db+dc}{db}$ — $\frac{b+c}{b}$; $b \neq 0$ *and* $d \neq 0$.

   e. $\frac{x^2-9b^2}{x^2-2xb-3b^2}$ — $\frac{x+3b}{x+b}$; $x \neq 3b$ *and* $x \neq -b$

   f. $\frac{3n^2-5n-2}{2n-4}$ — $\frac{3n+1}{2}$; $n \neq 2$

   g. $\frac{3x-2y}{9x^2-4y^2}$ — $\frac{1}{3x+2y}$; $y \neq \frac{3}{2}x$ *and* $y \neq -\frac{3}{2}x$

   h. $\frac{4a^2-12ab}{a^2-6ab+9b^2}$ — $\frac{4a}{a-3b}$; $a \neq 3b$

   i. $\frac{y-x}{x-y}$ — -1; $x \neq y$

   j. $\frac{a^2-b^2}{b+a}$ — $a - b$; $a \neq -b$

   k. $\frac{4x-2y}{3y-6x}$ — $-\frac{2}{3}$; $y \neq 2x$

   l. $\frac{9-x^2}{(x-3)^3}$ — $-\frac{3+x}{(x-3)^2}$; $x \neq 3$

m. $\frac{x^2-5x+6}{8-2x-x^2}$ $-\frac{x-3}{4+x}$; $x \neq 2$ *and* $x \neq -4$

n. $\frac{a-b}{xa-xb-a+b}$ $\frac{1}{x-1}$; $x \neq 1$ *and* $a \neq b$

o. $\frac{(x+y)^2-9a^2}{2x+2y-6a}$ $\frac{x+y+3a}{2}$; $a \neq \frac{x+y}{3}$

p. $\frac{8x^3-y^3}{4x^2-y^2}$ $\frac{4x^2+2xy+y^2}{2x+y}$; $y \neq 2x$ *and* $y \neq -2x$

2. Write a rational expression with denominator $6b$ that is equivalent to

a. $\frac{a}{b}$.

$\frac{6a}{6b}$

b. one-half of $\frac{a}{b}$.

$\frac{3a}{6b}$

c. $\frac{1}{3}$.

$\frac{2b}{6b}$

3. Remember that algebra is just another way to perform arithmetic but with variables replacing numbers.

a. Simplify the following rational expression: $\frac{(x^2y)^2(xy)^3z^2}{(xy^2)^2yz}$.

$$\frac{(x^2y)^2(xy)^3z^2}{(xy^2)^2yz} = \frac{x^4y^2 \cdot x^3y^3 \cdot z^2}{x^2y^4 \cdot yz} = \frac{x^7y^5z^2}{x^2y^5z} = x^5z$$

b. Simplify the following rational expression without using a calculator: $\frac{12^2 \cdot 6^3 \cdot 5^2}{18^2 \cdot 15}$.

$$\frac{12^2 \cdot 6^3 \cdot 5^2}{18^2 \cdot 15} = \frac{4^2 \cdot 3^2 \cdot 6^3 \cdot 5^2}{2^2 \cdot 9^2 \cdot 3 \cdot 5} = \frac{2^4 \cdot 3^2 \cdot 2^3 \cdot 3^3 \cdot 5^2}{2^2 \cdot 3^4 \cdot 3 \cdot 5} = \frac{2^7 \cdot 3^5 \cdot 5^2}{2^2 \cdot 3^5 \cdot 5} = 2^5 \cdot 5 = 32 \cdot 5 = 160$$

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c. How are the calculations in parts (a) and (b) similar? How are they different? Which expression was easier to simplify?

Both simplifications relied on using the rules of exponents. It was easier to simplify the algebraic expression in part (a) because we did not have to factor any numbers, such as 18, 15, and 12. However, if we substitute $x = 2$, $y = 3$, and $z = 5$, these two expressions have the exact same structure. Algebra allows us to do this calculation more quickly.