Lesson 14: Graphing Factored Polynomials

Classwork

Opening Exercise

An engineer is designing a rollercoaster for younger children and has tried some functions to model the height of the rollercoaster during the first 300 yards. She came up with the following function to describe what she believes would make a fun start to the ride:

$$H(x) = -3x^4 + 21x^3 - 48x^2 + 36x,$$

where $H(x)$ is the height of the rollercoaster (in yards) when the rollercoaster is $100x$ yards from the beginning of the ride. Answer the following questions to help determine at which distances from the beginning of the ride the rollercoaster is at its lowest height.

a. Does this function describe a roller coaster that would be fun to ride? Explain.

b. Can you see any obvious x-values from the equation where the rollercoaster is at height 0?

c. Using a graphing utility, graph the function H on the interval $0 \le x \le 3$, and identify when the rollercoaster is 0 yards off the ground.

d. What do the x-values you found in part (c) mean in terms of distance from the beginning of the ride?

e. Why do roller coasters always start with the largest hill first?

f. Verify your answers to part (c) by factoring the polynomial function H.

g. How do you think the engineer came up with the function for this model?

h. What is wrong with this rollercoaster model at distance 0 yards and 300 yards? Why might this not initially bother the engineer when she is first designing the track?

Example 1

Graph each of the following polynomial functions. What are the function's zeros (counting multiplicities)? What are the solutions to $f(x) = 0$? What are the x-intercepts to the graph of the function? How does the degree of the polynomial function compare to the x-intercepts of the graph of the function?

a. $f(x) = x(x - 1)(x + 1)$

b. $f(x) = (x + 3)(x + 3)(x + 3)(x + 3)$

c. $f(x) = (x-1)(x-2)(x+3)(x+4)(x+4)$

d. $f(x) = (x^2+1)(x-2)(x-3)$

Example 2

Consider the function $f(x) = x^3 - 13x^2 + 44x - 32$.

a. Use the fact that $x - 4$ is a factor of f to factor this polynomial.

b. Find the x-intercepts for the graph of f.

c. At which x-values can the function change from being positive to negative or from negative to positive?

d. To sketch a graph of f, we need to consider whether the function is positive or negative on the four intervals $x < 1$, $1 < x < 4$, $4 < x < 8$, and $x > 8$. Why is that?

e. How can we tell if the function is positive or negative on an interval between x-intercepts?

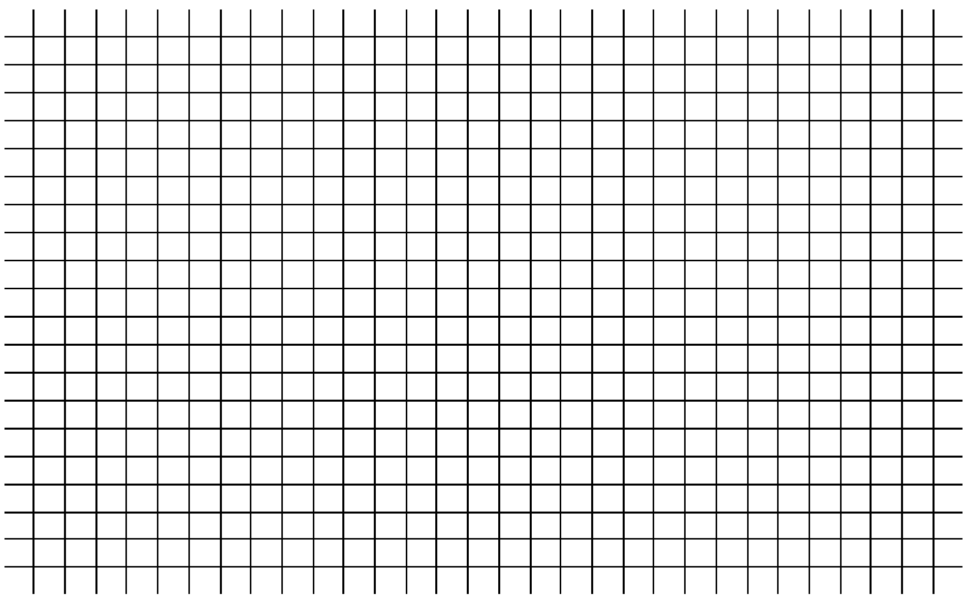
f. For $x < 1$, is the graph above or below the x-axis? How can you tell?

g. For $1 < x < 4$, is the graph above or below the x-axis? How can you tell?

h. For $4 < x < 8$, is the graph above or below the x-axis? How can you tell?

i. For $x > 8$, is the graph above or below the x-axis? How can you tell?

j. Use the information generated in parts (f)–(i) to sketch a graph of f.



k. Graph $y = f(x)$ on the interval from $[0,9]$ using a graphing utility, and compare your sketch with the graph generated by the graphing utility.

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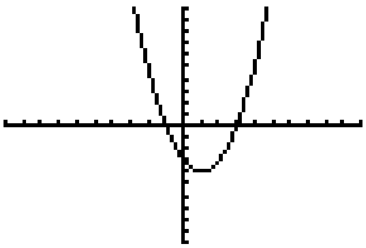
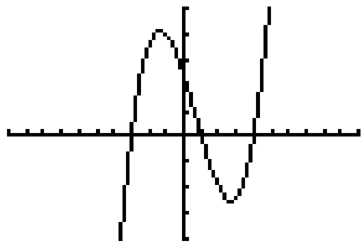
Discussion

For any particular polynomial, can we determine how many relative maxima or minima there are? Consider the following polynomial functions in factored form and their graphs.

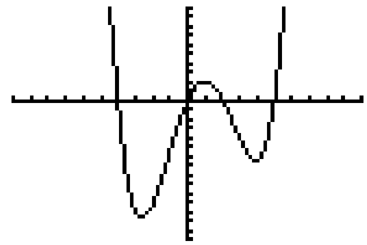
$f(x) = (x+1)(x-3)$



$g(x) = (x+3)(x-1)(x-4)$



$h(x) = (x)(x+4)(x-2)(x-5)$



Degree of each polynomial:

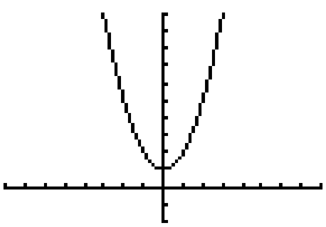
Number of x-intercepts in each graph:

Number of relative maxima or minima in each graph:

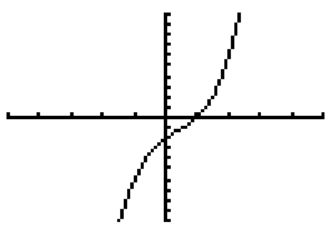
What observations can we make from this information?

Is this true for every polynomial? Consider the examples below.

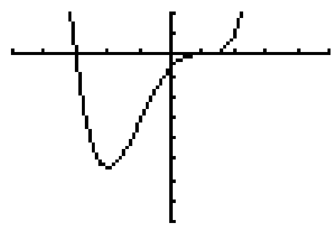
$\boldsymbol{r(x) = x^2 + 1}$



$\boldsymbol{s(x) = (x^2+2)(x-1)}$



$\boldsymbol{t(x) = (x+3)(x-1)(x-1)(x-1)}$



Degree of each polynomial:

Number of x-intercepts in each graph:

Number of relative maximums or minimums in each graph:

What observations can we make from this information?

Relevant Vocabulary

<u>Increasing/Decreasing</u>: Given a function f whose domain and range are subsets of the real numbers and I is an interval contained within the domain, the function is called *increasing on the interval* I if

$$f(x_1) < f(x_2) \text{ whenever } x_1 < x_2 \text{ in } I.$$

It is called decreasing on the interval I if

$$f(x_1) > f(x_2) \text{ whenever } x_1 < x_2 \text{ in } I.$$

<u>Relative Maximum</u>: Let f be a function whose domain and range are subsets of the real numbers. The function has a *relative maximum at* c if there exists an open interval I of the domain that contains c such that

$$f(x) \leq f(c) \text{ for all } x \text{ in the interval } I.$$

If c is a relative maximum, then the value $f(c)$ is called the *relative maximum value*.

<u>Relative Minimum</u>: Let f be a function whose domain and range are subsets of the real numbers. The function has a *relative minimum at* c if there exists an open interval I of the domain that contains c such that

$$f(x) \geq f(c) \text{ for all } x \text{ in the interval } I.$$

If c is a relative minimum, then the value $f(c)$ is called the *relative minimum value*.

<u>Graph of f</u>: Given a function f whose domain D and the range are subsets of the real numbers, the graph of f is the set of ordered pairs in the Cartesian plane given by

$$\{(x, f(x)) \mid x \in D\}.$$

<u>Graph of $y = f(x)$</u>: Given a function f whose domain D and the range are subsets of the real numbers, the *graph of* $y = f(x)$ is the set of ordered pairs (x, y) in the Cartesian plane given by

$$\{(x, y) \mid x \in D \text{ and } y = f(x)\}.$$

Lesson Summary

A polynomial of degree n may have up to n x-intercepts and up to $n - 1$ relative maximum/minimum points.

A relative maximum is the x-value c that produces the highest point on a graph of f in a circle around $(c, f(c))$. That highest value $f(c)$ is a relative maximum value.

A relative minimum is the x-value d that produces the lowest point on a graph of f in a circle around $(d, f(d))$. That lowest value $f(d)$ is a relative minimum value.

Problem Set

1. For each function below, identify the largest possible number of x-intercepts and the largest possible number of relative maximum and minimum points based on the degree of the polynomial. Then use a calculator or graphing utility to graph the function and find the actual number of x-intercepts and relative maximum/minimum points.

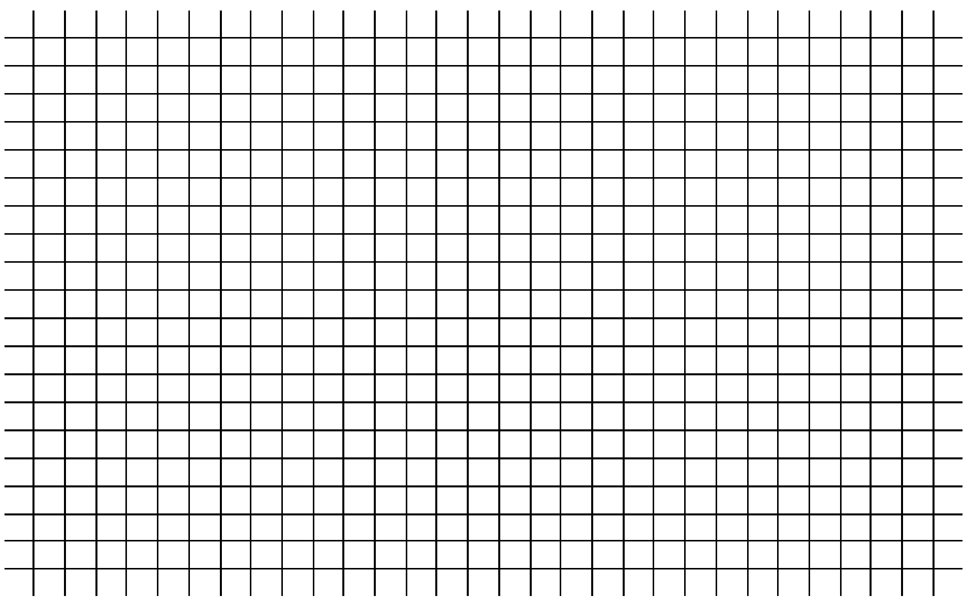
    a. $f(x) = 4x^3 - 2x + 1$

    b. $g(x) = x^7 - 4x^5 - x^3 + 4x$

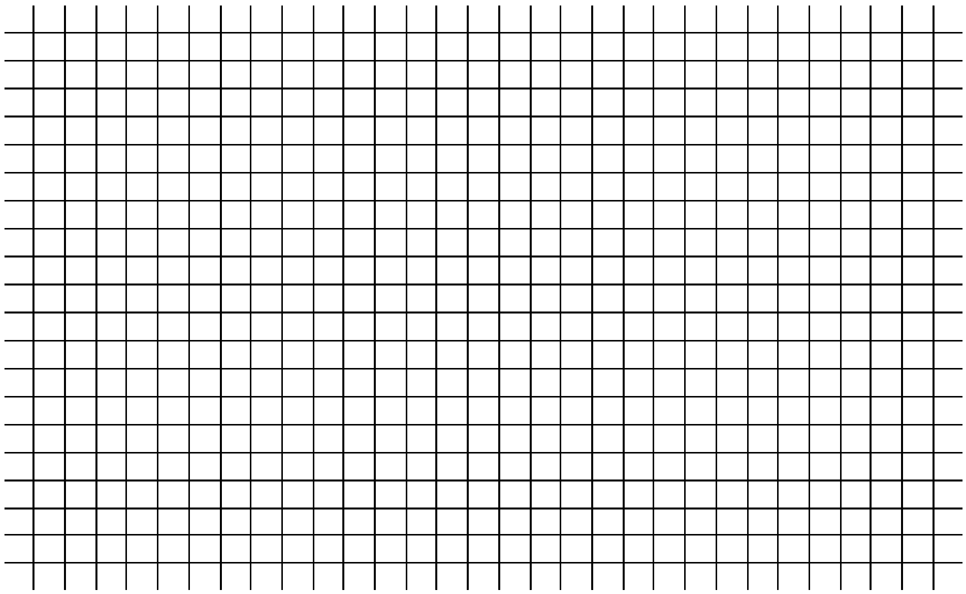
    c. $h(x) = x^4 + 4x^3 + 2x^2 - 4x + 2$

Function	Largest number of x-intercepts	Largest number of relative max/mins	Actual number of x-intercepts	Actual number of relative max/mins
a. $f(x)$				
b. $g(x)$				
c. $h(x)$				

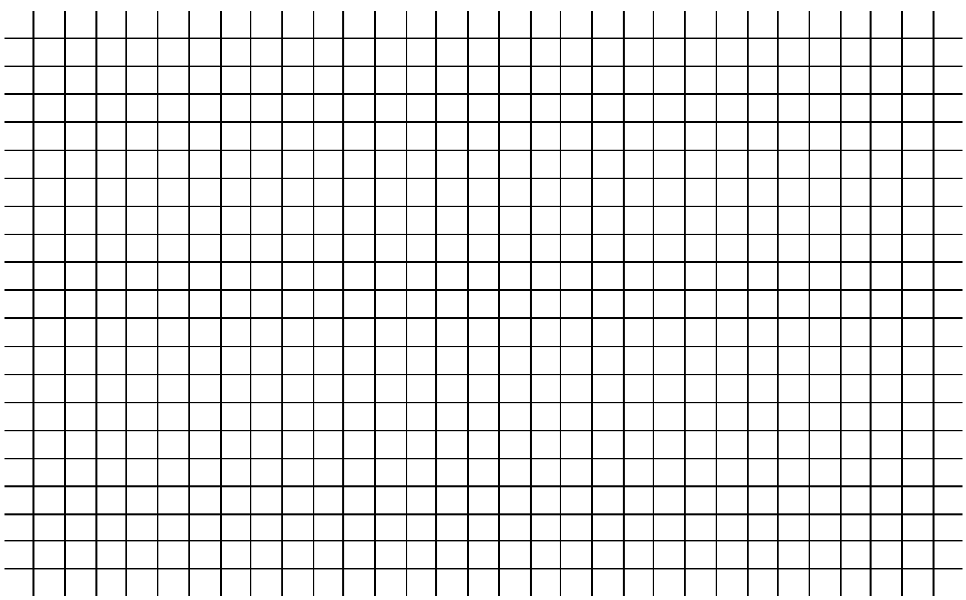
2. Sketch a graph of the function $f(x) = \frac{1}{2}(x+5)(x+1)(x-2)$ by finding the zeros and determining the sign of the values of the function between zeros.



3. Sketch a graph of the function $f(x) = -(x+2)(x-4)(x-6)$ by finding the zeros and determining the sign of the values of the function between zeros.



4. Sketch a graph of the function $f(x) = x^3 - 2x^2 - x + 2$ by finding the zeros and determining the sign of the values of the function between zeros.



5. Sketch a graph of the function $f(x) = x^4 - 4x^3 + 2x^2 + 4x - 3$ by determining the sign of the values of the function between the zeros -1, 1, and 3.

6. A function f has zeros at -1, 3, and 5. We know that $f(-2)$ and $f(2)$ are negative, while $f(4)$ and $f(6)$ are positive. Sketch a graph of f.

7. The function $h(t) = -16\mathrm{t}^2 + 33\mathrm{t} + 45$ represents the height of a ball tossed upward from the roof of a building 45 feet in the air after t seconds. Without graphing, determine when the ball will hit the ground.