Lesson 12: Overcoming Obstacles in Factoring

Student Outcomes

- Students will factor certain forms of polynomial expressions by using the structure of the polynomials.

Lesson Notes

Students have factored polynomial expressions in earlier lessons and in earlier courses. In this lesson, we explore further techniques for factoring polynomial expressions, including factoring by completing the square, by applying the quadratic formula and by grouping. We apply these techniques to solve polynomial equations.

The idea of the greatest common factor (GCF) is important to this lesson. The teacher may want to consider displaying a GCF poster on the classroom wall for reference. Consider using some of the problem set exercises during the lesson to supplement the examples included here.

Classwork

Opening (4 minutes)

Consider the following polynomial equation.

$$(x^2 - 4x + 3)(x^2 + 4x - 5) = 0$$

Discuss the following questions in pairs or small groups:

1. What is the degree of this polynomial? How do you know?
2. How many solutions to this equation should there be? How do you know?
3. How might you begin to solve this equation?

Scaffolding:
Ask struggling students to first solve the equation $(x-3)(x-1) = 0$. Point out that they have used the zero product property, and ask how that property applies to the given problem.

We can solve this equation by factoring because we can solve each of the equations

$$x^2 - 4x + 3 = 0$$
$$x^2 + 4x - 5 = 0.$$

There is no need to solve the whole way through; students completed a problem like this in Lesson 11. The idea is that students see that this can be done relatively quickly. The factored form of the original equation is

$$(x^2 - 4x + 3)(x^2 + 4x - 5) = (x-1)(x-3)(x-1)(x+5) = (x-1)^2(x-3)(x+5) = 0,$$

and the three solutions are 1, 3, and -5.

However, consider the next example.

Example 1 (8 minutes)

Example 1

Find all real solutions to the equation $(x^2 - 6x + 3)(2x^2 - 4x - 7) = 0$.

Allow students the opportunity to struggle with factoring these expressions, discuss with their neighbors, and reach the conclusion that neither expression can be factored with integer coefficients.

- We have discovered an obstacle to factoring. The expressions $x^2 - 6x + 3$ and $2x^2 - 4x - 7$ do not factor as readily as the examples from the previous lesson. Does anybody recall how we might factor them?

Students have completed the square in both Geometry and Algebra I, so give them an opportunity to recall the process.

- When a quadratic expression is not easily factorable, we can either apply a technique called *completing the square,* or we can use the *quadratic formula*. Let's factor the first expression by completing the square.
- We first create some space between the x term and the constant term:

$$x^2 - 6x + \_\_\_\_ - \_\_\_\_ + 3 = 0.$$

- The next step is the key step. Take half of the coefficient of the x term, square that number, and add and subtract it in the space we created:

$$x^2 - 6x + (-3)^2 - (-3)^2 + 3 = 0$$
$$x^2 - 6x + 9 - 9 + 3 = 0.$$

Discuss the following questions with the class, and give them the opportunity to justify this step.

- Why did we choose 9? Why did we both add and subtract 9? How does this help us solve the equation?
  - *Adding* 9 *creates a perfect square trinomial in the first three terms.*
  - *Adding and subtracting* 9 *means that we have not changed the value of the expression on the left side of the equation.*
  - *Adding and subtracting the* 9 *creates a perfect square trinomial* $x^2 - 6x + 9 = (x-3)^2$.
- We cannot just add a number to an expression without changing its value. By adding 9 and subtracting 9, we have essentially added 0 using the additive identity property, which leads to an equivalent expression on the left hand side of the equation and thus preserves solutions of the equations.
- This process creates a structure that allows us to factor the first three terms of the expression on the left side of the equation and then solve for the variable.

$$\underbrace{x^2 - 6x + 9}_{\text{Perfect square trinomial}} - 9 \; + 3 = 0$$

$$(x-3)^2 - 6 = 0$$

- Solving for x:

$$(x-3)^2 = 6$$

$$x - 3 = \sqrt{6} \quad \text{or} \quad x - 3 = -\sqrt{6}$$

$$x = 3 + \sqrt{6} \quad \text{or} \quad x = 3 - \sqrt{6}\,.$$

- Thus, we have found two solutions by setting the first quadratic expression equal to zero, completing the square, and solving the factored equation. Since the leading coefficient of $x^2 - 6x + 3$ is 1, we know from our work in Algebra I that the factored form is

$$x^2 - 6x + 3 = \left(x - \left(3 + \sqrt{6}\right)\right)\left(x - \left(3 - \sqrt{6}\right)\right).$$

Scaffolding:

We can use the tabular method to give a visual representation of the process of completing the square. For example, the polynomial $x^2 - 6x$ can be represented as follows to illustrate the "missing" term.

	x	-3
x	x^2	$-3x$
-3	$-3x$	?

- Let's repeat the process with the second equation. What is the first step to completing the square?

$$2x^2 - 4x - 7 = 0$$

Allow students an opportunity to suggest the first step to completing the square.

- We can only complete the square when the leading coefficient is 1, so our first step is to factor out the 2.

$$2\left(x^2 - 2x - \frac{7}{2}\right) = 0$$

- Now we can complete the square with the expression inside the parentheses.

$$2\left(x^2 - 2x + \_\_\_\_ - \_\_\_\_ + \frac{7}{2}\right) = 0$$
$$2\left(x^2 - 2x + (-1)^2 - (-1)^2 - \frac{7}{2}\right) = 0$$
$$2\left(x^2 - 2x + 1 - \frac{9}{2}\right) = 0$$
$$2\left((x-1)^2 - \frac{9}{2}\right) = 0$$

- Next, we divide both sides by 2.

$$(x-1)^2 - \frac{9}{2} = 0$$

- Finally, we solve for x.

$$(x-1)^2 = \frac{9}{2}$$
$$x = 1 + \sqrt{\frac{9}{2}} \quad \text{or} \quad x = 1 - \sqrt{\frac{9}{2}}$$
$$x = 1 + \frac{3\sqrt{2}}{2} \quad \text{or} \quad x = 1 - \frac{3\sqrt{2}}{2}$$

- Thus, we have found two more solutions to our original fourth degree equation. We then have

$$\begin{aligned} 2x^2 - 4x - 7 &= 2\left(x^2 - 2x - \frac{7}{2}\right) \\ &= 2\left(x - \left(1 + \frac{3\sqrt{2}}{2}\right)\right)\left(x - \left(1 - \frac{3\sqrt{2}}{2}\right)\right). \end{aligned}$$

- Notice that we needed to multiply the factors by 2 to make the leading coefficients match.
- Finally, we have the factored form of our original polynomial equation:

$$(x^2 - 6x + 3)(2x^2 - 4x - 7) = 0 \text{ in factored form}$$
$$2\left(x - (3+\sqrt{6})\right)\left(x - (3-\sqrt{6})\right)\left(x - \left(1 + \frac{3\sqrt{2}}{2}\right)\right)\left(x - \left(1 - \frac{3\sqrt{2}}{2}\right)\right) = 0.$$

- Thus, the solutions to the equation $(x^2-6x+3)(2x^2-4x-7)=0$ are the four values $3+\sqrt{6}$, $3-\sqrt{6}$, $1+\frac{3\sqrt{2}}{2}$, and $1-\frac{3\sqrt{2}}{2}$.
- Similarly, we could have applied the quadratic formula to find the solutions to each quadratic equation in the previous example. Recall the quadratic formula.
  - *The two solutions to the quadratic equation $ax^2+bx+c=0$ are $\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\frac{-b-\sqrt{b^2-4ac}}{2a}$.*

Exercise 1 (6 minutes)

Exercise 1

Factor and find all real solutions to the equation $(x^2-2x-4)(3x^2+8x-3)=0$.

Ask half of the students to apply the quadratic formula to solve $x^2-2x-4=0$ and the other half to apply the quadratic formula to solve $3x^2+8x-3=0$.

The quadratic formula gives solutions $1+\sqrt{5}$ and $1-\sqrt{5}$ for the first equation and -3 and $\frac{1}{3}$ for the second equation.

Since $1+\sqrt{5}$ and $1-\sqrt{5}$ are the two solutions to $x^2-2x-4=0$ found by the quadratic formula, we know from work in Algebra I that $\left(x-\left(1+\sqrt{5}\right)\right)\left(x-\left(1-\sqrt{5}\right)\right)=x^2-2x-4$. However, we need to be more careful when using the solutions to factor the second quadratic expression. The leading coefficient of $(x+3)\left(x-\frac{1}{3}\right)=x^2+\frac{8}{3}x-1$ is 1, and the leading coefficient of $3x^2+8x-3$ is 3, so we need to multiply our factors by 3:

$$3x^2+8x-3=3(x+3)\left(x-\frac{1}{3}\right).$$

Thus, the factored form of the original equation is

$$(x^2-2x-4)(3x^2+8x-3)=3\left(x-\left(1+\sqrt{5}\right)\right)\left(x-\left(1-\sqrt{5}\right)\right)(x+3)\left(x-\frac{1}{3}\right)=0,$$

and the four solutions to $(x^2-2x-4)(3x^2+8x-3)=0$ are $1+\sqrt{5}$, $1-\sqrt{5}$, -3, and $\frac{1}{3}$.

To summarize, if we have a fourth degree polynomial already factored into two quadratic expressions, we can try to factor the entire polynomial by completing the square on one or both quadratic expressions, or by using the quadratic formula to find the roots of the quadratic polynomials and then constructing the factored form of each quadratic polynomial.

Discussion (6 minutes)

- We have overcome the obstacle of difficult-to-factor quadratic expressions. Let's look next at the obstacles encountered when attempting to solve a third degree polynomial equation such as the following:

$$x^3+3x^2-9x-27=0.$$

- How might we begin to solve this equation?

Scaffolding:

Allow students to generate ideas. Advanced students may benefit from significant independent time to attempt to solve the equation (MP.1).

Allow students an opportunity to brainstorm as a class, in pairs, or in table groups. Students may note that coefficients are powers of 3 but may not be sure how that helps. Let them know they are seeing something important that we may be able to use. Stronger students might even try to group the components.

- While we have made some interesting observations, we have not quite found a way to factor this expression. What if we know that $x + 3$ is one factor?

If the students do not come up with polynomial division, you can point them in that direction through a numerical example: Suppose we want the factors of 210, and we know that one factor is 3. How do we find the other factors?

Have the students perform the polynomial division

$$x+3\overline{\big)x^3+3x^2-9x-27}$$

to find additional factors. Students may also use the tabular method discussed in earlier lessons in this module.

$$\begin{array}{r} x^2+0x-9 \\ x+3\overline{\big)x^3+3x^2-9x-27} \\ \underline{x^3+3x^2} \\ -9x-27 \\ \underline{-9x-27} \\ 0 \end{array}$$

So $\dfrac{x^3+3x^2-9x-27}{x+3} = x^2-9$

and $x^3+3x^2-9x-27 = (x+3)(x^2-9)$

- Since $x^3 + 3x^3 - 9x - 27 = (x + 3)(x^2 - 9)$, we know that
$$x^3 + 3x^2 - 9x - 27 = (x + 3)(x - 3)(x + 3) = (x + 3)^2(x - 3).$$
- By the zero product property, the solutions to $x^3 + 3x^2 - 9x - 27 = 0$ are -3 and 3.
- But, how do we start if we don't know any of the factors in advance?

Example 2 (6 minutes)

Example 2

Find all solutions to $x^3 + 3x^2 - 9x - 27 = 0$ by factoring the equation.

Scaffolding:

If needed, have students compute the GCF for some numerical examples such as

GCF $75, 100$ is 25
GCF $72, 16, 24$ is 8
GCF $10, 35, 63$ is 1.

- Let's start with our original equation $x^3 + 3x^2 - 9x - 27 = 0$. Is there a greatest common factor (GCF) for all four terms on the left hand side we can factor out?
  - *No, the GCF is* 1.

- Let's group the terms of the left hand side as follows:

$$x^3 + 3x^2 - 9x - 27 = (x^3 + 3x^2) - (9x + 27).$$

- Can we factor out a GCF from each set of parentheses independently?
  - *Yes, x^2 can be factored out of the first piece and 9 out of the second.*

Factor the GCF out of each part. Have students do as much of this work as possible.

$$x^3 + 3x^2 - 9x - 27 = x^2(x + 3) - 9(x + 3).$$

MP.7

- Do you notice anything interesting about the right side of the above equation?
  - *I noticed that $x + 3$ is a common factor.*
- Since both terms have a factor of $(x + 3)$, we have found a quantity that can be factored out.

$$x^3 + 3x^2 - 9x - 27 = (x + 3)(x^2 - 9)$$

- And as we saw above, we can take this one step further.

$$x^3 + 3x^2 - 9x - 27 = (x + 3)(x + 3)(x - 3)$$

- Because of the zero property, the original problem is now easy to solve because $x^3 + 3x^2 - 9x - 27 = 0$ exactly when $(x + 3)^2(x - 3) = 0$. What are the solutions to the original equation?
  - *The solutions to $x^3 + 3x^2 - 9x - 27 = 0$ are $x = -3$ and $x = 3$.*
- The process you just completed is often called *factoring by grouping,* and it works only on certain 3rd degree polynomial expressions, such as $x^3 + 3x^2 - 9x - 27$.

Exercise 2 (4 minutes)

Allow students to work in pairs or small groups on these exercises. Realize that there are two ways to group the terms that will result in the same factored expression. Circulate around the room while students are working, and take note of any groups that are using a different approach. At the end of these exercises, ask students who grouped differently to share their method, and discuss as a class.

Exercise 2

Find all real solutions to $x^3 - 5x^2 - 4x + 20 = 0$.

$$\begin{aligned} x^3 - 5x^2 - 4x + 20 &= 0 \\ x^2(x - 5) - 4(x - 5) &= 0 \\ (x - 5)(x^2 - 4) &= 0 \\ (x - 5)(x - 2)(x + 2) &= 0 \end{aligned}$$

Thus, the solutions are 5, 2, and -2.

$$\begin{aligned} x^3 - 5x^2 - 4x + 20 &= 0 \\ x(x^2 - 4) - 5(x^2 - 4) &= 0 \\ (x - 5)(x^2 - 4) &= 0 \\ (x - 5)(x - 2)(x + 2) &= 0 \end{aligned}$$

Exercise 3 (4 minutes)

Exercise 3

Find all real solutions to $x^3 - 8x^2 - 2x + 16 = 0$.

$$x^3 - 8x^2 - 2x + 16 = 0$$
$$x^2(x-8) - 2(x-8) = 0$$
$$(x-8)(x^2-2) = 0$$

$$x^3 - 8x^2 - 2x + 16 = 0$$
$$x(x^2-2) - 8(x^2-2) = 0$$
$$(x-8)(x^2-2) = 0$$

Thus, the solutions are 8, $\sqrt{2}$, and $-\sqrt{2}$.

Closing (2 minutes)

Ask students to summarize the important parts of the lesson in writing, to a partner, or as a class. Use this as an opportunity to informally assess understanding of the lesson. The following are some important summary elements.

Lesson Summary

In this lesson, we learned some techniques to use when faced with factoring polynomials and solving polynomial equations.

- **If a fourth degree polynomial can be factored into two quadratic expressions, then each quadratic expression might be factorable either using the quadratic formula or by completing the square.**
- **Some third degree polynomials can be factored using the technique of factoring by grouping.**

Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 12: Overcoming Obstacles in Factoring

Exit Ticket

Solve the following equation and explain your solution method.

$$x^3 + 7x^2 - x - 7 = 0$$

Exit Ticket Sample Solutions

Solve the following equation and explain your solution method.

$$x^3 + 7x^2 - x - 7 = 0$$

$$x^2(x+7) - (x+7) = 0$$
$$(x+7)(x^2-1) = 0$$
$$(x+7)(x-1)(x+1) = 0$$

The solutions are -7, 1, and -1. The equation was solved by factoring by grouping. I grouped the four terms into two groups, and then factored the GCF from each group. I then factored out the common term $(x+7)$ from each group to find the factored form of the equation. I then applied the zero product property to find the solutions to the equation.

Problem Set Sample Solutions

1. **Solve each of the following equations by completing the square.**
   a. $x^2 - 6x + 2 = 0$ — $3+\sqrt{7},\ 3-\sqrt{7}$
   b. $x^2 - 4x = -1$ — $2+\sqrt{3},\ 2-\sqrt{3}$
   c. $x^2 + x - \frac{3}{4} = 0$ — $\frac{1}{2}, -\frac{3}{2}$
   d. $3x^2 - 9x = -6$ — $2,\ 1$
   e. $(2x^2 - 5x + 2)(3x^2 - 4x + 1) = 0$ — $2, \frac{1}{2}, 1, \frac{1}{3}$
   f. $x^4 - 4x^2 + 2 = 0$ — $\sqrt{2+\sqrt{2}}, -\sqrt{2+\sqrt{2}}, \sqrt{2-\sqrt{2}}, -\sqrt{2-\sqrt{2}}$

2. **Solve each of the following equations using the quadratic formula.**
   a. $x^2 - 5x - 3 = 0$ — $\frac{5}{2}+\frac{\sqrt{37}}{2}, \frac{5}{2}-\frac{\sqrt{37}}{2}$
   b. $(6x^2 - 7x + 2)(x^2 - 5x + 5) = 0$ — $\frac{1}{2}, \frac{2}{3}, \frac{1}{2}(5+\sqrt{5}), \frac{1}{2}(5-\sqrt{5})$
   c. $(3x^2 - 13x + 14)(x^2 - 4x + 1) = 0$ — $2, \frac{7}{3}, 2+\sqrt{3}, 2-\sqrt{3}$

3. **Not all of the expressions in the equations below can be factored using the techniques discussed so far in this course. First, determine if the expression can be factored with real coefficients. If so, factor the expression and find all real solutions to the equation.**

a. $x^2 - 5x - 24 = 0$

Can be factored: $(x-8)(x+3) = 0.$

Solutions: $8,\ -3$

b. $3x^2 + 5x - 2 = 0$

Can be factored: $(3x-1)(x+2) = 0.$

Solutions: $\frac{1}{3}, -2$

c. $x^2 + 2x + 4 = 0$

Cannot be factored with real number coefficients.

d. $x^3 + 3x^2 - 2x + 6 = 0$

Cannot be factored with real number coefficients.

e. $x^3 + 3x^2 + 2x + 6 = 0$

Can be factored: $(x+3)(x^2+2) = 0.$

Solution: -3

f. $2x^3 + x^2 - 6x - 3 = 0$

Can be factored: $(2x+1)(x-\sqrt{3})(x+\sqrt{3}) = 0.$

Solutions: $-\frac{1}{2}, \sqrt{3}, -\sqrt{3}$

g. $8x^3 - 12x^2 + 2x - 3 = 0$

Can be factored: $(2x-3)(4x^2+1) = 0.$

Solution: $\frac{3}{2}$

h. $6x^3 + 8x^2 + 15x + 20 = 0$

Can be factored: $(3x+4)(2x^2+5) = 0.$

Solution: $-\frac{4}{3}$

i. $4x^3 + 2x^2 - 36x - 18 = 0$

Can be factored: $2(2x+1)(x-3)(x+3) = 0.$

Solutions: $-\frac{1}{2}, 3, -3$

j. $x^2 - \frac{1}{2}x - \frac{15}{2} = 0$

Can be factored: $\left(x+\frac{5}{2}\right)(x-3) = 0.$

Solutions: $-\frac{5}{2}, 3$

4. **Solve the following equations by bringing all terms to one side of the equation and factoring out the greatest common factor.**

a. $(x-2)(x-1) = (x-2)(x+1)$

$$\begin{aligned}(x-2)(x+1)-(x-2)(x-1) &= 0\\ (x-2)\big(x+1-(x-1)\big) &= 0\\ (x-2)(2) &= 0\\ x &= 2\end{aligned}$$

So, the only solution to $(x-2)(x-1) = (x-2)(x+1)$ *is* 2.

b. $(2x+3)(x-4) = (2x+3)(x+5)$

$$\begin{aligned}(2x+3)(x-4)-(2x+3)(x+5) &= 0\\ (2x+3)\big(x-4-(x+5)\big) &= 0\\ (2x+3)(-9) &= 0\\ x &= -\frac{3}{2}\end{aligned}$$

So, the only solution to $(2x+3)(x-4) = (2x+3)(x+5)$ *is* $-\frac{3}{2}$.

c. $(x-1)(2x+3)=(x-1)(x+2)$

$$(x-1)(2x+3)-(x-1)(x+2)=0$$
$$(x-1)\big(2x+3-(x+2)\big)=0$$
$$(x-1)(x+1)=0$$
$$x=1 \text{ or } x=-1$$

The solutions to $(x-1)(2x+3)=(x-1)(x+2)$ *are* 1 *and* -1.

d. $(x^2+1)(3x-7)=(x^2+1)(3x+2)$

$$(x^2+1)(3x-7)-(x^2+1)(3x+2)=0$$
$$(x^2+1)\big(3x-7-(3x+2)\big)=0$$
$$(x^2+1)(-9)=0$$
$$x^2+1=0$$

There are no real number solutions to $(x^2+1)(3x-7)=(x^2+1)(3x+2)$.

e. $(x+3)(2x^2+7)=(x+3)(x^2+8)$

$$(x+3)(2x^2+7)-(x+3)(x^2+8)=0$$
$$(x+3)\big(2x^2+7-(x^2+8)\big)=0$$
$$(x+3)(x^2-1)=0$$
$$(x+3)(x-1)(x+1)=0$$

The three solutions to $(x+3)(2x^2+7)=(x+3)(x^2+8)$ *are* -3, -1, *and* 1.

5. Consider the expression x^4+1. Since x^2+1 does not factor with real number coefficients, we might expect that x^4+1 also does not factor with real number coefficients. In this exercise, we will investigate the possibility of factoring x^4+1.

a. Simplify the expression $(x^2+1)^2-2x^2$.

$$(x^2+1)^2-2x^2=x^4+1$$

b. Factor $(x^2+1)^2-2x^2$ as a difference of squares.

$$(x^2+1)^2-2x^2=\left((x^2+1)-\sqrt{2}\,x\right)\left((x^2+1)+\sqrt{2}\,x\right)$$

c. Is it possible to factor x^4+1 with real number coefficients? Explain.

Yes. $x^4+1=\left((x^2+1)-\sqrt{2}\,x\right)\left((x^2+1)+\sqrt{2}\,x\right)$

In an equivalent but more conventional form, we have

$$x^4+1=(x^2-\sqrt{2}\,x+1)(x^2+\sqrt{2}\,x+1).$$