Lesson 11: The Special Role of Zero in Factoring

Student Outcomes

- Students find solutions to polynomial equations where the polynomial expression is not factored into linear factors.
- Students construct a polynomial function that has a specified set of zeros with stated multiplicity.

Lesson Notes

This lesson focuses on the first part of standard **A-APR.B.3**, identifying zeros of polynomials presented in factored form. Although the terms root and zero are interchangeable, for consistency only the term zero is used throughout this lesson and in later lessons. The second part of the standard, using the zeros to construct a rough graph of a polynomial function, is delayed until Lesson 14. The ideas that begin in this lesson continue in Lesson 19, in which students will be able to associate a zero of a polynomial function to a factor in the factored form of the associated polynomial as a consequence of the Remainder Theorem, and culminate in Lesson 39, in which students apply the Fundamental Theorem of Algebra to factor polynomial expressions completely over the complex numbers.

Classwork

Opening Exercise (12 minutes)

> **Opening Exercise**
>
> **Find all solutions to the equation $(x^2 + 5x + 6)(x^2 - 3x - 4) = 0$.**

Scaffolding:

Here is an alternative opening activity that may better illuminate the special role of zero.

- For each equation, list some possible values for x and y.

  $xy = 10$, $xy = 1$,
  $xy = -1$, $xy = 0$

- What do you notice? Does one equation tell you more information than others?

The main point of this opening exercise is for students to recognize and then formalize that the statement "If $ab = 0$, then $a = 0$ or $b = 0$" applies not only when a and b are numbers or linear functions (which we used when solving a quadratic equation), but also applies to cases where a and b are polynomial functions of any degree.

In small groups, let students discuss ways to solve this equation. Walk around the room and offer advice such as, "Have you considered factoring each quadratic expression? What do you get?" As soon as one group factors both quadratic expressions, or when three minutes have passed, show, or let that group show, the factorization on the board.

$$\underbrace{(x+2)(x+3)}_{x^2+5x+6} \cdot \underbrace{(x-4)(x+1)}_{x^2-3x-4} = 0$$

- What are the solutions to this equation?
  - $-2, -3, 4, -1$

- Why?
  - *If x is any number other than $-2, -3, 4, -1$, then each factor is a nonzero number, i.e., $x + 2 \neq 0$, $x + 3 \neq 0$, etc. However, the multiplication of four nonzero numbers is nonzero, so that value of x cannot be a solution. Therefore, the only possible solutions are $-2, -3, 4$, and -1. It is easy to confirm that these are indeed solutions by substituting them each into the equation individually.*
- Why are these numbers also solutions to the original equation?
  - *Because the expression $(x + 2)(x + 3)(x - 4)(x + 1)$ is equivalent to $(x^2 + 5x + 6)(x^2 - 3x - 4)$.*

Now let's study the solutions to $x^2 + 5x + 6 = 0$ and $x^2 - 3x - 4 = 0$ separately.

- What are the solutions to $x^2 + 5x + 6 = 0$?
  - $-2, -3$
- What are the solutions to $x^2 - 3x - 4 = 0$?
  - $4, -1$
- Relate the solutions of the equation $(x^2 + 5x + 6)(x^2 - 3x - 4) = 0$ to the solutions of the compound statement, "$x^2 + 5x + 6 = 0$ or $x^2 - 3x - 4 = 0$."
  - *They are the same.*
- Given two polynomial functions p and q of any degree, the solution set of the equation $p(x)q(x) = 0$ is the union of the solution set of $p(x) = 0$ and the solution set of $q(x) = 0$. Can you explain why?

Lead students in a discussion of the following proof:

- Suppose a is a solution to the equation $p(x)q(x) = 0$; that is, it is a number that satisfies $p(a)q(a) = 0$. Since $p(a)$ is a number and $q(a)$ is a number, one or both of them must be zero, by the zero product property that states, "If the product of two numbers is zero, then at least one of the numbers is zero." Therefore, $p(a) = 0$ or $q(a) = 0$, which means a is a solution to the compound statement, "$p(x) = 0$ or $q(x) = 0$."
- Now let's prove the other direction and show that if a is a solution to the compound statement, then it is a solution to the equation $p(x)q(x) = 0$. This direction is also easy: Suppose a is a number such that either $p(a) = 0$ or $q(a) = 0$. In the first case, $p(a)q(a) = 0 \cdot q(a) = 0$. In the second case, $p(a)q(a) = p(a) \cdot 0 = 0$. Hence, in either case, a is a solution to the equation $p(x)q(x) = 0$.

Students may have difficulty understanding the distinction between the equations $p(x)q(x) = 0$ and $p(a)q(a) = 0$. Help students understand that $p(x)q(x) = 0$ is an equation in a variable x, while $p(a)$ is the value of the function p when it is evaluated at the number. Thus, $p(a)q(a)$ is a number. For example, if p and q are the quadratic polynomials in the Opening Exercise, and we are considering the case when a is 5, then $p(5) = 56$ and $q(5) = 6$. Therefore, 5 cannot be a solution to the equation $p(x)q(x) = 0$.

MP.1

- Communicate to students that they can use the statement below to break problems into simpler parts:

  Given any two polynomial functions p and q, the set of solutions to the equation $p(x)q(x) = 0$ can be found by solving $p(x) = 0$, solving $q(x) = 0$, and combining the solutions into one set.

Ask your students to try the following exercise on their own.

Exercise 1 (2 minutes)

Exercise 1

1. **Find the solutions of $(x^2-9)(x^2-16)=0$.**

   The solutions to $(x^2-9)(x^2-16)$ are the solutions of $x^2-9=0$ combined with the solutions of of $x^2-16=0$. These solutions are -3, 3, -4, and 4.

In the next example, we look at a polynomial equation for which we already know a solution. The goal of this example and the discussion that follows is to use a solution to the equation $f(x)=0$ to further factor the polynomial f. In doing so, we end with a description of the zeros of a function, a concept first introduced in Algebra I, Module 4.

Example 1 (8 minutes)

Example 1

Suppose we know that the polynomial equation $4x^3-12x^2+3x+5=0$ has three real solutions and that one of the factors of $4x^3-12x^2+3x+5$ is $(x-1)$. How can we find all three solutions to the given equation?

Steer the discussion to help students conjecture that

$$4x^3-12x^2+3x+5=(x-1)(\text{some quadratic polynomial}).$$

Since $(x-1)$ is a factor and we know how to divide polynomials, we can find the quadratic polynomial by dividing:

$$\frac{4x^3-12x^2+3x+5}{x-1}=4x^2-8x-5.$$

Now we know that $4x^3-12x^2+3x+5=(x-1)(4x^2-8x-5)$, and we also know that $4x^2-8x-5$ is a quadratic polynomial that has linear factors $(2x+1)$ and $(2x-5)$.

Therefore, $4x^3-12x^2+3x+5=0$ has the same solutions as $(x-1)(4x^2-8x-5)=0$, which has the same solutions as

$$(x-1)(2x+1)(2x-5)=0.$$

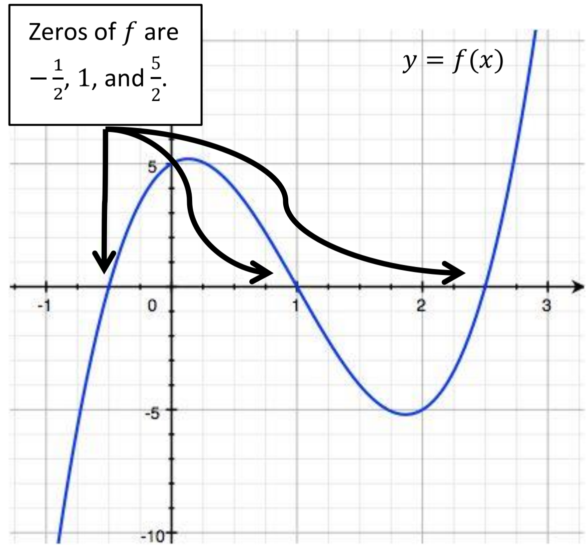
In this factored form, the solutions of $f(x)=0$ are readily apparent: $-\frac{1}{2}$, 1, and $\frac{5}{2}$.

Scaffolding:

Allow students to generate ideas about how the linear factors affect the behavior of the graph so that they can use the graph of a function to identify zeros.

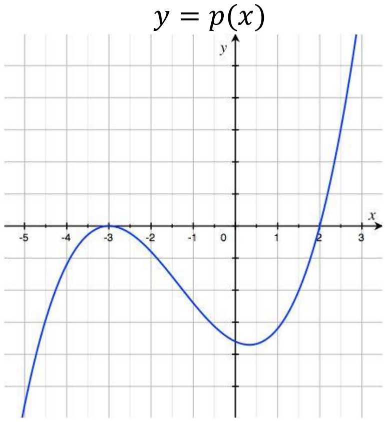
Discussion (8 minutes)

- In Example 1 above, we saw that factoring the polynomial into linear factors helped us to find solutions to the original polynomial equation $4x^3 - 12x^2 + 3x + 5 = 0$.
- There is a corresponding notion for the zeros of a function. Let f be a function whose domain is a subset of the real numbers and whose range is a subset of the real numbers. A *zero (or root) of the function* f is a number c such that $f(c) = 0$.
- The zeros of the function $f(x) = 4x^3 - 12x^2 + 3x + 5$ are the x-intercepts of the graph of f: these are $-\frac{1}{2}$, 1, and $\frac{5}{2}$.
- By definition, a zero of a polynomial function f is a solution to the equation $f(x) = 0$. If $(x - a)$ is a factor of a polynomial function f, then $f(a) = 0$ and a is a zero of f.

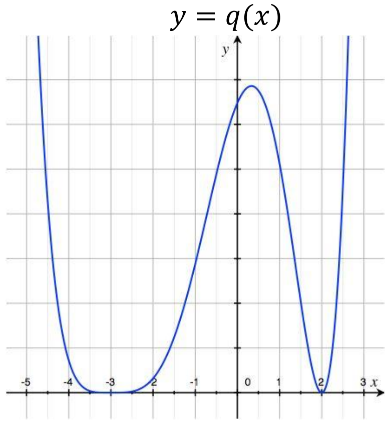


However, consider the polynomial functions $p(x) = (x-2)(x+3)^2$, $q(x) = (x-2)^2(x+3)^4$, and $r(x) = (x-2)^4(x-3)^5$. Because $p(2) = 0$, $q(2) = 0$, and $r(2) = 0$, the number 2 is a zero of p, q, and r. Likewise, -3 is also a zero of p, q, and r. Even though these polynomial functions have the same zeros, they are not the same function; they do not even have the same degree!

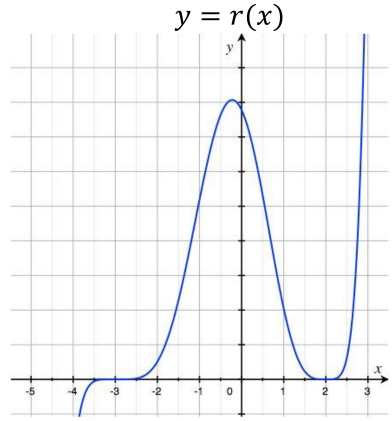
$y = p(x)$



$y = q(x)$



$y = r(x)$



We would like to be able to distinguish between the zeros of these two polynomial functions. If we write out all of the factors for p, q, and r, we see that

$$p(x) = (x-2)(x+3)(x+3)$$
$$q(x) = (x-2)(x-2)(x+3)(x+3)(x+3)(x+3)$$
$$r(x) = (x-2)(x-2)(x-2)(x-2)(x+3)(x+3)(x+3)(x+3)(x+3).$$

- We notice that $(x-2)$ is a factor of p once, and $(x+3)$ is a factor of p twice. Thus, we say that 2 is a zero of p of multiplicity 1, and -3 is a zero of p of multiplicity 2. Zeros of multiplicity 1 are usually just referred to as zeros, without mentioning the multiplicity.
- What are the zeros of q, with their multiplicities?
  - *For q, 2 is a zero of multiplicity 2, and -3 is a zero of multiplicity 4.*
- What are the zeros of r, with their multiplicities?
  - *For r, 2 is a zero of multiplicity 4, and -3 is a zero of multiplicity 5.*
- Can you look at the factored form of a polynomial equation and identify the zeros with their multiplicities? Explain how you know.
  - *Yes. Each linear factor $(ax-b)^m$ of the polynomial will produce a zero $\frac{b}{a}$ with multiplicity m.*
- Can multiplicity be negative? Can it be zero? Can it be a fraction?
  - *No. Multiplicity is the count of the number of times a factor appears in a factored polynomial expression. Polynomials can only have positive integer exponents, so a factor must have positive integer exponents. Thus, the multiplicity of a zero must be a positive integer.*

Note: In Lesson 14, students will use the zeros of a polynomial function together with their multiplicities to create a graph of the function, and in Lesson 19, students will use the zeros of a polynomial function with their multiplicities to construct the equation of the function.

Exercises 2–5 (8 minutes)

Exercises 2–5

2. Find the zeros of the following polynomial functions, with their multiplicities.

a. $f(x) = (x+1)(x-1)(x^2+1)$

-1 with multiplicity 1

1 with multiplicity 1

b. $g(x) = (x-4)^3(x-2)^8$

4 with multiplicity 3

2 with multiplicity 8

c. $h(x) = (2x-3)^5$

$\frac{3}{2}$ with multiplicity 5

d. $k(x) = (3x+4)^{100}(x-17)^4$

$-\frac{4}{3}$ with multiplicity 100

17 with multiplicity 4

3. **Find a polynomial function that has the following zeros and multiplicities. What is the degree of your polynomial?**

Zero	Multiplicity
2	3
-4	1
6	6
-8	10

$p(x) = (x-2)^3(x+4)(x-6)^6(x+8)^{10}$

The degree of p is 20.

4. **Is there more than one polynomial function that has the same zeros and multiplicities as the one you found in Exercise 3?**

Yes. Consider $(x) = (x^2+5)(x-2)^3(x+4)(x-6)^6(x+8)^{10}$. Since there are no real solutions to $x^2+5=0$, adding this factor does not produce a new zero. Thus p and q have the same zeros and multiplicities but are different functions.

5. **Can you find a rule that relates the multiplicities of the zeros to the degree of the polynomial function?**

Yes. If p is a polynomial function of degree n, then the sum of the multiplicities of all of the zeros is less than or equal to n. If p can be factored into linear terms, then the sum of the multiplicities of all of the zeros is exactly equal to n.

Closing (2 minutes)

Ask students to summarize the key ideas of the lesson, either in writing or with a neighbor. Consider posing the questions below.

- Part of the lesson today has been that given two polynomials, p and q, we can determine solutions to $p(x)q(x) = 0$ by solving both $p(x) = 0$ and $q(x) = 0$, even if they are high degree polynomials. If p and q are polynomial functions that do not have any real number zeros, do you think the equation $p(x)q(x) = 0$ still has real number solutions? Can you give an example of two such functions?
  - *If $p(x) \neq 0$ for all real numbers x and $q(x) \neq 0$ for all real numbers x, then there is no possible way to have $p(x)q(x) = 0$.*
  - *For example: If $p(x) = x^2 + 1$ and $q(x) = x^4 + 1$, then the equation $(x^2+1)(x^4+1) = 0$ has no real solutions.*

The following vocabulary was introduced in Algebra I (please see Module 3 and Module 4 in Algebra I). While you should not have to teach these terms explicitly, it may still be a good idea to go through them with your class.

Relevant Vocabulary Terms

In the definitions below, the symbol $\mathbb{R}$ stands for the set of real numbers.

Function: A *function* is a correspondence between two sets, X and Y, in which each element of X is assigned to one and only one element of Y.

The set X in the definition above is called the *domain of the function.* The *range (or image)* of the function is the subset of Y, denoted $f(X)$, that is defined by the following property: y is an element of $f(X)$ if and only if there is an x in X such that $f(x) = y$.

If $f(x) = x^2$ where x can be any real number, then the domain is all real numbers (denoted $\mathbb{R}$), and the range is the set of non-negative real numbers.

Polynomial Function: Given a polynomial expression in one variable, a *polynomial function in one variable* is a function $f: \mathbb{R} \to \mathbb{R}$ such that for each real number x in the domain, $f(x)$ is the value found by substituting the number x into all instances of the variable symbol in the polynomial expression and evaluating.

It can be shown that if a function $f: \mathbb{R} \to \mathbb{R}$ is a polynomial function, then there is some non-negative integer n and collection of real numbers $a_0, a_1, a_2, \dots, a_n$ with $a_n \neq 0$ such that the function satisfies the equation

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

for every real number x in the domain, which is called the *standard form of the polynomial function.* The function $f(x) = 3x^3 + 4x^2 + 4x + 7$, where x can be any real number, is an example of a function written in standard form.

Degree of a Polynomial Function: The *degree of a polynomial function* is the degree of the polynomial expression used to define the polynomial function.

The degree of $f(x) = 8x^3 + 4x^2 + 7x + 6$ is 3, but the degree of $g(x) = (x+1)^2 - (x-1)^2$ is 1 because when g is put into standard form, it is $g(x) = 4x$.

Constant Function: A *constant function* is a polynomial function of degree 0. A constant function is of the form $f(x) = c$, for a constant c.

Linear Function: A *linear function* is a polynomial function of degree 1. A linear function is of the form $f(x) = ax + b$, for constants a and b with $a \neq 0$.

Quadratic Function: A *quadratic function* is a polynomial function of degree 2. A quadratic function is in *standard form* if it is written in the form $f(x) = ax^2 + bx + c$, for constants a, b, c with $a \neq 0$ and any real number x.

Cubic Function: A *cubic function* is a polynomial function of degree 3. A cubic function is of the form $f(x) = ax^3 + bx^2 + cx + d$, for constants a, b, c, d with $a \neq 0$.

Zeros or Roots of a Function: A *zero* (or *root*) of a function $f: \mathbb{R} \to \mathbb{R}$ is a number x of the domain such that $f(x) = 0$. A zero of a function is an element in the solution set of the equation $f(x) = 0$.

Lesson Summary

Given any two polynomial functions p and q, the solution set of the equation $p(x)q(x) = 0$ can be quickly found by solving the two equations $p(x) = 0$ and $q(x) = 0$ and combining the solutions into one set.

The number a is zero of a polynomial function p with multiplicity m if the factored form of p contains $(x - a)^m$.

Exit Ticket (5 minutes)

Name ______________________________ Date ________________

Lesson 11: The Special Role of Zero in Factoring

Exit Ticket

A polynomial function p can be factored into seven factors: $(x-3)$, $(x+1)$, and 5 factors of $(x-2)$. What are its zeros with multiplicity, and what is the degree of the polynomial? Explain how you know.

Exit Ticket Sample Solutions

A polynomial function p can be factored into seven factors: $(x-3)$, $(x+1)$, and 5 factors of $(x-2)$. What are its zeros with multiplicity, and what is the degree of the polynomial? Explain how you know.

Zeros: 3 with multiplicity 1; -1 with multiplicity 1; 2 with multiplicity 5

The polynomial has degree seven. There are seven linear factors as given above, so $p(x) = (x-3)(x+1)(x-2)^5$. If the factors were multiplied out, the leading term would be x^7, so the degree of p is 7.

Problem Set Sample Solutions

For Problems 1–4, find all solutions to the given equations.

1. $(x-3)(x+2) = 0$

   $3, -2$

2. $(x-5)(x+2)(x+3) = 0$

   $5, -2, -3$

3. $(2x-4)(x+5) = 0$

   $2, -5$

4. $(2x-2)(3x+1)(x-1) = 0$

   $1, -\frac{1}{3}, 1$

5. **Find four solutions to the equation $(x^2 - 9)(x^4 - 16) = 0$.**

   $2, -2, 3, -3$

6. **Find the zeros with multiplicity for the function $p(x) = (x^3 - 8)(x^5 - 4x^3)$.**

   We can factor p to give $p(x) = x^3(x-2)(x^2+2x+4)(x-2)(x+2) = x^3(x-2)^2(x+2)(x^2+2x+4)$. Then 0 is a zero of multiplicity 3, -2 is a zero of multiplicity 1, and 2 is a zero of multiplicity 2.

7. **Find two different polynomial functions that have zeros at 1, 3, and 5 of multiplicity 1.**

   $p(x) = (x-1)(x-3)(x-5)$ and $q(x) = (x^2+1)(x-1)(x-3)(x-5)$

8. **Find two different polynomial functions that have a zero at 2 of multiplicity 5 and a zero at -4 of multiplicity 3.**

   $p(x) = (x-2)^5(x+4)^3$ and $q(x) = (x^2+1)(x-2)^5(x+4)^3$

9. **Find three solutions to the equation $(x^2 - 9)(x^3 - 8) = 0$.**

   From Lesson 6, we know that $(x-2)$ is a factor of (x^3-8), so three solutions are $3, -3$, and 2.

10. Find two solutions to the equation $(x^3 - 64)(x^5 - 1) = 0$.

From Lesson 6, we know that $(x-4)$ is a factor of $(x^3 - 64)$, and $(x-1)$ is a factor of $(x^5 - 1)$, so two solutions are 1 and 4.

11. If p, q, r, s are non-zero numbers, find the solutions to the equation $(px+q)(rx+s) = 0$ in terms of p, q, r, s.

Setting each factor equal to zero gives solutions $-\frac{q}{p}$ and $-\frac{s}{r}$.

Use the identity $a^2 - b^2 = (a-b)(a+b)$ to solve the equations given in Problems 12–13.

12. $(3x-2)^2 = (5x+1)^2$

Using algebra, we have $(3x-2)^2 - (5x+1)^2 = 0$. Applying the difference of squares formula, we have $\big((3x-2)-(5x+1)\big)\big((3x-2)+(5x+1)\big) = 0$. Combining like terms gives $(-2x-3)(8x-1) = 0$, so the solutions are $-\frac{3}{2}$ and $\frac{1}{8}$.

13. $(x+7)^2 = (2x+4)^2$

Using algebra, we have $(x+7)^2 - (2x+4)^2 = 0$. Then $\big((x+7)-(2x+4)\big)\big((x+7)+(2x+4)\big) = 0$, so we have $(-x+3)(3x+11) = 0$. Thus the solutions are $-\frac{11}{3}$ and 3.

14. Consider the polynomial function $P(x) = x^3 + 2x^2 + 2x - 5$.
    a. Divide P by the divisor $(x-1)$ and rewrite in the form $P(x) = (\text{divisor})(\text{quotient}) + \text{remainder}$.

    $P(x) = (x-1)(x^2+3x+5) + 0$

    b. Evaluate $P(1)$.

    $P(1) = 0$

15. Consider the polynomial function $Q(x) = x^6 - 3x^5 + 4x^3 - 12x^2 + x - 3$.
    a. Divide Q by the divisor $(x-3)$ and rewrite in the form $Q(x) = (\text{divisor})(\text{quotient}) + \text{remainder}$.

    $Q(x) = (x-3)(x^5 + 4x^2 + 1) + 0$

    b. Evaluate $Q(3)$.

    $Q(3) = 0$

16. Consider the polynomial function $R(x) = x^4 + 2x^3 - 2x^2 - 3x + 2$.
    a. Divide R by the divisor $(x+2)$ and rewrite in the form $R(x) = (\text{divisor})(\text{quotient}) + \text{remainder}$.

    $R(x) = (x+2)(x^3 - 2x + 1) + 0$

    b. Evaluate $R(-2)$.

    $R(-2) = 0$

17. Consider the polynomial function $S(x) = x^7 + x^6 - x^5 - x^4 + x^3 + x^2 - x - 1$.

a. Divide S by the divisor $(x+1)$ and rewrite in the form $S(x) = (\text{divisor})(\text{quotient}) + \text{remainder}$.

$S(x) = (x+1)(x^6 - x^4 + x^2 - 1) + 0$

b. Evaluate $S(-1)$.

$S(-1) = 0$

18. Make a conjecture based on the results of Questions 14–17.

It seems that the zeros a of a polynomial function correspond to factors $(x-a)$ in the equation of the polynomial.