Lesson 11: The Special Role of Zero in Factoring

Classwork

Opening Exercise

Find all solutions to the equation $\left(x^{2}+5x+6\right)\left(x^{2}-3x-4\right)=0$.

Exercise 1

1. Find the solutions of $\left(x^{2}-9\right)\left(x^{2}-16\right)=0$.

**Example 1**

Suppose we know that the polynomial equation $4x^{3}-12x^{2}+3x+5=0$ has three real solutions and that one of the factors of $4x^{3}-12x^{2}+3x+5$ is $(x-1)$. How can we find all three solutions to the given equation?

Exercises 2–5

1. Find the zeros of the following polynomial functions, with their multiplicities.
	1. $f\left(x\right)=(x+1)(x-1)(x^{2}+1)$
	2. $g\left(x\right)=\left(x-4\right)^{3}\left(x-2\right)^{8}$
	3. $h\left(x\right)=\left(2x-3\right)^{5}$
	4. $k\left(x\right)=\left(3x+4\right)^{100}\left(x-17\right)^{4}$
2. Find a polynomial function that has the following zeros and multiplicities. What is the degree of your polynomial?

|  |  |
| --- | --- |
| Zero | Multiplicity |
| $$2$$ | $$3$$ |
| $$-4$$ | $$1$$ |
| $$6$$ | $$6$$ |
| $$-8$$ | $$10$$ |

1. Is there more than one polynomial function that has the same zeros and multiplicities as the one you found in Exercise 3?
2. Can you find a rule that relates the multiplicities of the zeros to the degree of the polynomial function?

Relevant Vocabulary Terms

In the definitions below, the symbol $R$ stands for the set of real numbers.

**Function:**  A *function* is a correspondence between two sets, $X$ and $Y$*,* in which each element of $X$ is assigned to one and only one element of $Y$.

The set $X$ in the definition above is called the *domain of the function.* The *range (or image)* of the function is the subset of $Y$, denoted $f(X)$, that is defined by the following property: $y$ is an element of $f(X)$ if and only if there is an $x$ in $X$ such that $f(x)=y$.

If $f\left(x\right)=x^{2}$ where $x$ can be any real number, then the domain is all real numbers (denoted $R$), and the range is the set of non-negative real numbers.

**Polynomial Function:**  Given a polynomial expression in one variable, a *polynomial function in one variable* is a function $f: R\rightarrow R$ such that for each real number $x$ in the domain, $f(x)$ is the value found by substituting the number $x$ into all instances of the variable symbol in the polynomial expression and evaluating.

It can be shown that if a function $f: R\rightarrow R$ is a polynomial function, then there is some non-negative integer $n$ and collection of real numbers $a\_{0}$, $a\_{1}$, $a\_{2}$,$… $, $a\_{n}$ with $a\_{n}\ne 0$ such that the function satisfies the equation

$$f\left(x\right)=a\_{n}x^{n}+a\_{n-1}x^{n-1}+…+a\_{1}x+a\_{0},$$

for every real number $x$ in the domain, which is called the *standard form of the polynomial function.* The function
$f\left(x\right)=3x^{3}+4x^{2}+4x+7$, where $x$ can be any real number, is an example of a function written in standard form.

**Degree of a Polynomial Function:** The *degree of a polynomial function* is the degree of the polynomial expression used to define the polynomial function.

The degree of $f\left(x\right)=8x^{3}+4x^{2}+7x+6$ is 3, but the degree of $g\left(x\right)=\left(x+1\right)^{2}-\left(x-1\right)^{2}$ is $1$ because when $g$ is put into standard form, it is $g\left(x\right)=4x$.

**Constant Function:** A *constant function* is a polynomial function of degree 0. A constant function is of the form $f\left(x\right)=c$, for a constant $c$.

**Linear Function:**  A *linear function* is a polynomial function of degree $1$. A linear function is of the form $f\left(x\right)=ax+b$, for constants $a$ and $b$ with $a\ne 0$.

**Quadratic Function:** A *quadratic function* is a polynomial function of degree$ 2$. A quadratic function is in *standard form* if it is written in the form $f\left(x\right)=ax^{2}+bx+c$, for constants $a$, $b$,$ c$ with $a\ne 0$ and any real number$ x$.

**Cubic Function:**  A *cubic function* is a polynomial function of degree $3$. A cubic function is of the form $f\left(x\right)=ax^{3}+bx^{2}+cx+d,$ for constants $a, b, c,d$ with $a\ne 0$.

**Zeros or Roots of a Function:**  A *zero* (or *root*) of a function $f: R\rightarrow R$ is a number $x$ of the domain such that
$f(x)=0$. A zero of a function is an element in the solution set of the equation $f(x)=0$.

Problem Set

Lesson Summary

Given any two polynomial functions $p$ and $q$, the solution set of the equation $p\left(x\right)q\left(x\right)=0$ can be quickly found by solving the two equations $p\left(x\right)=0$ and $q\left(x\right)=0$ and combining the solutions into one set.

The number $a$ is zero of a polynomial function $p$ with multiplicity $m$ if the factored form of $p$ contains $\left(x-a\right)^{m}$.

For Problems 1–4, find all solutions to the given equations.

1. $\left(x-3\right)\left(x+2\right)=0$
2. $\left(x-5\right)\left(x+2\right)\left(x+3\right)=0$
3. $\left(2x-4\right)\left(x+5\right)=0$
4. $\left(2x-2\right)\left(3x+1\right)\left(x-1\right)=0$
5. Find four solutions to the equation $\left(x^{2}-9\right)\left(x^{4}-16\right)=0$.
6. Find the zeros with multiplicity for the function $p\left(x\right)=\left(x^{3}-8\right)\left(x^{5}-4x^{3}\right).$
7. Find two different polynomial functions that have zeros at $1$, $3$, and $5$ of multiplicity $1$.
8. Find two different polynomial functions that have a zero at $2$ of multiplicity $5$ and a zero at $-4$ of multiplicity $3$.
9. Find three solutions to the equation $\left(x^{2}-9\right)\left(x^{3}-8\right)=0$.
10. Find two solutions to the equation $\left(x^{3}-64\right)\left(x^{5}-1\right)=0$.
11. If $p$,$ q$,$ r$, $s$ are non-zero numbers, find the solutions to the equation $\left(px+q\right)\left(rx+s\right)=0$ in terms of $p$, $q$,$ r$,$ s$.

Use the identity $a^{2}-b^{2}=(a-b)(a+b)$ to solve the equations given in Problems 12–13.

1. $\left(3x-2\right)^{2}=\left(5x+1\right)^{2}$
2. $\left(x+7\right)^{2}=\left(2x+4\right)^{2}$
3. Consider the polynomial function $P\left(x\right)=x^{3}+2x^{2}+2x-5$.
	1. Divide $P$ by the divisor $(x-1)$ and rewrite in the form $P\left(x\right)=\left(divisor\right)\left(quotient\right)+remainder$.
	2. Evaluate $P(1)$.
4. Consider the polynomial function $Q\left(x\right)=x^{6}-3x^{5}+4x^{3}-12x^{2}+x-3$.
	1. Divide $Q$ by the divisor $(x-3)$ and rewrite in the form $Q\left(x\right)=\left(divisor\right)\left(quotient\right)+remainder$.
	2. Evaluate $Q(3)$.
5. Consider the polynomial function $R\left(x\right)=x^{4}+2x^{3}-2x^{2}-3x+2$.
	1. Divide $R$ by the divisor $(x+2)$ and rewrite in the form $R\left(x\right)=\left(divisor\right)\left(quotient\right)+remainder$.
	2. Evaluate $R(-2)$.
6. Consider the polynomial function $S\left(x\right)=x^{7}+x^{6}-x^{5}-x^{4}+x^{3}+x^{2}-x-1$.
	1. Divide $S$ by the divisor $(x+1)$ and rewrite in the form $S\left(x\right)=\left(divisor\right)\left(quotient\right)+remainder$.
	2. Evaluate $S(-1)$.
7. Make a conjecture based on the results of Questions 14—17.