Lesson 11: The Special Role of Zero in Factoring

Classwork

Opening Exercise

Find all solutions to the equation $(x^2 + 5x + 6)(x^2 - 3x - 4) = 0$.

Exercise 1

1. Find the solutions of $(x^2 - 9)(x^2 - 16) = 0$.

Example 1

Suppose we know that the polynomial equation $4x^3 - 12x^2 + 3x + 5 = 0$ has three real solutions and that one of the factors of $4x^3 - 12x^2 + 3x + 5$ is $(x - 1)$. How can we find all three solutions to the given equation?

Exercises 2–5

2. Find the zeros of the following polynomial functions, with their multiplicities.

   a. $f(x) = (x+1)(x-1)(x^2+1)$

   b. $g(x) = (x-4)^3(x-2)^8$

   c. $h(x) = (2x-3)^5$

   d. $k(x) = (3x+4)^{100}(x-17)^4$

3. Find a polynomial function that has the following zeros and multiplicities. What is the degree of your polynomial?

Zero	Multiplicity
2	3
-4	1
6	6
-8	10

4. Is there more than one polynomial function that has the same zeros and multiplicities as the one you found in Exercise 3?

5. Can you find a rule that relates the multiplicities of the zeros to the degree of the polynomial function?

Relevant Vocabulary Terms

In the definitions below, the symbol $\mathbb{R}$ stands for the set of real numbers.

<u>Function</u>: A *function* is a correspondence between two sets, X and Y, in which each element of X is assigned to one and only one element of Y.

The set X in the definition above is called the *domain of the function.* The *range (or image)* of the function is the subset of Y, denoted $f(X)$, that is defined by the following property: y is an element of $f(X)$ if and only if there is an x in X such that $f(x) = y$.

If $f(x) = x^2$ where x can be any real number, then the domain is all real numbers (denoted $\mathbb{R}$), and the range is the set of non-negative real numbers.

<u>Polynomial Function</u>: Given a polynomial expression in one variable, a *polynomial function in one variable* is a function $f: \mathbb{R} \to \mathbb{R}$ such that for each real number x in the domain, $f(x)$ is the value found by substituting the number x into all instances of the variable symbol in the polynomial expression and evaluating.

It can be shown that if a function $f: \mathbb{R} \to \mathbb{R}$ is a polynomial function, then there is some non-negative integer n and collection of real numbers $a_0, a_1, a_2, \dots, a_n$ with $a_n \neq 0$ such that the function satisfies the equation

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

for every real number x in the domain, which is called the *standard form of the polynomial function.* The function $f(x) = 3x^3 + 4x^2 + 4x + 7$, where x can be any real number, is an example of a function written in standard form.

<u>Degree of a Polynomial Function</u>: The *degree of a polynomial function* is the degree of the polynomial expression used to define the polynomial function.

The degree of $f(x) = 8x^3 + 4x^2 + 7x + 6$ is 3, but the degree of $g(x) = (x+1)^2 - (x-1)^2$ is 1 because when g is put into standard form, it is $g(x) = 4x$.

<u>Constant Function</u>: A *constant function* is a polynomial function of degree 0. A constant function is of the form $f(x) = c$, for a constant c.

<u>Linear Function</u>: A *linear function* is a polynomial function of degree 1. A linear function is of the form $f(x) = ax + b$, for constants a and b with $a \neq 0$.

<u>Quadratic Function</u>: A *quadratic function* is a polynomial function of degree 2. A quadratic function is in *standard form* if it is written in the form $f(x) = ax^2 + bx + c$, for constants a, b, c with $a \neq 0$ and any real number x.

<u>Cubic Function</u>: A *cubic function* is a polynomial function of degree 3. A cubic function is of the form $f(x) = ax^3 + bx^2 + cx + d$, for constants a, b, c, d with $a \neq 0$.

<u>Zeros or Roots of a Function</u>: A *zero* (or *root*) of a function $f: \mathbb{R} \to \mathbb{R}$ is a number x of the domain such that $f(x) = 0$. A zero of a function is an element in the solution set of the equation $f(x) = 0$.

Lesson Summary

Given any two polynomial functions p and q, the solution set of the equation $p(x)q(x) = 0$ can be quickly found by solving the two equations $p(x) = 0$ and $q(x) = 0$ and combining the solutions into one set.

The number a is zero of a polynomial function p with multiplicity m if the factored form of p contains $(x - a)^m$.

Problem Set

For Problems 1–4, find all solutions to the given equations.

1. $(x - 3)(x + 2) = 0$

2. $(x - 5)(x + 2)(x + 3) = 0$

3. $(2x - 4)(x + 5) = 0$

4. $(2x - 2)(3x + 1)(x - 1) = 0$

5. Find four solutions to the equation $(x^2 - 9)(x^4 - 16) = 0$.

6. Find the zeros with multiplicity for the function $p(x) = (x^3 - 8)(x^5 - 4x^3)$.

7. Find two different polynomial functions that have zeros at 1, 3, and 5 of multiplicity 1.

8. Find two different polynomial functions that have a zero at 2 of multiplicity 5 and a zero at -4 of multiplicity 3.

9. Find three solutions to the equation $(x^2 - 9)(x^3 - 8) = 0$.

10. Find two solutions to the equation $(x^3 - 64)(x^5 - 1) = 0$.

11. If p, q, r, s are non-zero numbers, find the solutions to the equation $(px + q)(rx + s) = 0$ in terms of p, q, r, s.

Use the identity $a^2 - b^2 = (a - b)(a + b)$ to solve the equations given in Problems 12–13.

12. $(3x - 2)^2 = (5x + 1)^2$

13. $(x + 7)^2 = (2x + 4)^2$

14. Consider the polynomial function $P(x) = x^3 + 2x^2 + 2x - 5$.
    a. Divide P by the divisor $(x - 1)$ and rewrite in the form $P(x) = (\text{divisor})(\text{quotient}) + \text{remainder}$.
    b. Evaluate $P(1)$.

15. Consider the polynomial function $Q(x) = x^6 - 3x^5 + 4x^3 - 12x^2 + x - 3$.
    a. Divide Q by the divisor $(x - 3)$ and rewrite in the form $Q(x) = (\text{divisor})(\text{quotient}) + \text{remainder}$.
    b. Evaluate $Q(3)$.

16. Consider the polynomial function $R(x) = x^4 + 2x^3 - 2x^2 - 3x + 2$.
    a. Divide R by the divisor $(x + 2)$ and rewrite in the form $R(x) = (\text{divisor})(\text{quotient}) + \text{remainder}$.
    b. Evaluate $R(-2)$.

17. Consider the polynomial function $S(x) = x^7 + x^6 - x^5 - x^4 + x^3 + x^2 - x - 1$.
    a. Divide S by the divisor $(x + 1)$ and rewrite in the form $S(x) = (\text{divisor})(\text{quotient}) + \text{remainder}$.
    b. Evaluate $S(-1)$.

18. Make a conjecture based on the results of Questions 14—17.