Lesson 9: Radicals and Conjugates

Classwork

Opening Exercise

Which of these statements are true for all $a, b > 0$? Explain your conjecture.

i. $2(a + b) = 2a + 2b$

ii. $\frac{a+b}{2} = \frac{a}{2} + \frac{b}{2}$

iii. $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$

Example 1

Express $\sqrt{50} - \sqrt{18} + \sqrt{8}$ in simplest radical form and combine like terms.

Exercises 1–5

1. $\sqrt{\frac{1}{4}} + \sqrt{\frac{9}{4}} - \sqrt{45}$

2. $\sqrt{2}\left(\sqrt{3} - \sqrt{2}\right)$

3. $\sqrt{\frac{3}{8}}$

4. $\sqrt[3]{\frac{5}{32}}$

5. $\sqrt[3]{16x^5}$

Example 2

Multiply and combine like terms. Then explain what you notice about the two different results.

$\left(\sqrt{3}+\sqrt{2}\right)\left(\sqrt{3}+\sqrt{2}\right)$

$\left(\sqrt{3}+\sqrt{2}\right)\left(\sqrt{3}-\sqrt{2}\right)$

Exercise 6

6. Find the product of the conjugate radicals.

$(\sqrt{5}+\sqrt{3})(\sqrt{5}-\sqrt{3})$

$(7+\sqrt{2})(7-\sqrt{2})$

$(\sqrt{5}+2)(\sqrt{5}-2)$

Example 3

Write $\frac{\sqrt{3}}{5-2\sqrt{3}}$ in simplest radical form.

Lesson Summary

- For real numbers $a \geq 0$ and $b \geq 0$, where $b \neq 0$ when b is a denominator,
$$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b} \text{ and } \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}.$$
- For real numbers $a \geq 0$ and $b \geq 0$, where $b \neq 0$ when b is a denominator,
$$\sqrt[3]{ab} = \sqrt[3]{a} \cdot \sqrt[3]{b} \text{ and } \sqrt[3]{\frac{a}{b}} = \frac{\sqrt[3]{a}}{\sqrt[3]{b}}.$$
- Two binomials of the form $\sqrt{a} + \sqrt{b}$ and $\sqrt{a} - \sqrt{b}$ are called conjugate radicals:
  $\sqrt{a} + \sqrt{b}$ is the conjugate of $\sqrt{a} - \sqrt{b}$, and
  $\sqrt{a} - \sqrt{b}$ is the conjugate of $\sqrt{a} + \sqrt{b}$.
  For example, the conjugate of $2 - \sqrt{3}$ is $2 + \sqrt{3}$.
- To express a numeric expression with a denominator of the form $\sqrt{a} + \sqrt{b}$ in simplest radical form, multiply the numerator and denominator by the conjugate $\sqrt{a} - \sqrt{b}$ and combine like terms.

Problem Set

1. Express each of the following as a rational number or in simplest radical form. Assume that the symbols a, b, and x represent positive numbers.
   a. $\sqrt{36}$
   b. $\sqrt{72}$
   c. $\sqrt{18}$
   d. $\sqrt{9x^3}$
   e. $\sqrt{27x^2}$
   f. $\sqrt[3]{16}$
   g. $\sqrt[3]{24a}$
   h. $\sqrt{9a^2 + 9b^2}$

2. Express each of the following in simplest radical form, combining terms where possible.
   a. $\sqrt{25} + \sqrt{45} - \sqrt{20}$
   b. $3\sqrt{3} - \sqrt{\frac{3}{4}} + \sqrt{\frac{1}{3}}$
   c. $\sqrt[3]{54} - \sqrt[3]{8} + 7\sqrt[3]{\frac{1}{4}}$
   d. $\sqrt[3]{\frac{5}{8}} + \sqrt[3]{40} - \sqrt[3]{\frac{8}{9}}$

3. Evaluate $\sqrt{x^2 - y^2}$ when $x = 33$ and $y = 15$.

4. Evaluate $\sqrt{x^2 + y^2}$ when $x = 20$ and $y = 10$.

5. Express each of the following as a rational expression or in simplest radical form. Assume that the symbols x and y represent positive numbers.
   a. $\sqrt{3}\left(\sqrt{7} - \sqrt{3}\right)$
   b. $\left(3 + \sqrt{2}\right)^2$
   c. $(2 + \sqrt{3})(2 - \sqrt{3})$
   d. $(2 + 2\sqrt{5})(2 - 2\sqrt{5})$
   e. $\left(\sqrt{7} - 3\right)\left(\sqrt{7} + 3\right)$
   f. $\left(3\sqrt{2} + \sqrt{7}\right)\left(3\sqrt{2} - \sqrt{7}\right)$
   g. $\left(x - \sqrt{3}\right)\left(x + \sqrt{3}\right)$
   h. $\left(2x\sqrt{2} + y\right)\left(2x\sqrt{2} - y\right)$

6. Simplify each of the following quotients as far as possible.
   a. $\left(\sqrt{21} - \sqrt{3}\right) \div \sqrt{3}$
   b. $\left(\sqrt{5} + 4\right) \div \left(\sqrt{5} + 1\right)$
   c. $\left(3 - \sqrt{2}\right) \div \left(3\sqrt{2} - 5\right)$
   d. $\left(2\sqrt{5} - \sqrt{3}\right) \div \left(3\sqrt{5} - 4\sqrt{2}\right)$

7. If $x = 2 + \sqrt{3}$, show that $x + \frac{1}{x}$ has a rational value.

8. Evaluate $5x^2 - 10x$ when the value of x is $\frac{2-\sqrt{5}}{2}$.

9. Write the factors of $a^4 - b^4$. Use the result to obtain the factored form of $\left(\sqrt{3} + \sqrt{2}\right)^4 - \left(\sqrt{3} - \sqrt{2}\right)^4$.

10. The converse of the Pythagorean Theorem is also a theorem: If the square of one side of a triangle is equal to the sum of the squares of the other two sides, then the triangle is a right triangle.

    Use the converse of the Pythagorean Theorem to show that for $A, B, C > 0$, if $A + B = C$, then $\sqrt{A} + \sqrt{B} > \sqrt{C}$, so that $\sqrt{A} + \sqrt{B} > \sqrt{A + B}$.