Lesson 6: Dividing by $x - a$ and by $x + a$

Student Outcomes

- Students work with polynomials with constant coefficients to prove polynomial identities.

Lesson Notes

Students extend their understanding of polynomial division to abstract situations that involve division by $x - a$ and by $x + a$. Through this work they derive the fundamental identities for the difference of two squares and the sum and difference of two cubes. Further, they connect this work back to divisibility of integers. This lesson bridges the first five lessons of Topic A of this module with the upcoming lessons in Topic B in which students extend work with division and polynomial identities to factoring. In those lessons, they will learn the usefulness of the factored form of a polynomial expression to solve polynomial equations and analyze the zeros of the graph of a polynomial function. This lesson addresses aspects of several standards (most notably **A.SSE.A.2** and **A.APR.C.4**) in a way that also emphasizes MP.7 and MP.8. Students recognize and then generalize patterns, and use them to fluently rewrite polynomial expressions and perform polynomial operations.

Classwork

Opening (1 minute)

Students may choose to solve the problems using either the reverse tabular method or long division. Encourage and model both approaches throughout this lesson. Have students work with a partner on these problems and then randomly select pairs to present each of the problems on the board. You can begin by paraphrasing the following statement as students start the Opening Exercise:

- Today we want to observe patterns when we divide certain types of polynomials and make some generalizations to help us quickly compute quotients without having to do the work involved with the reverse tabular method or the long division algorithm.

> *Scaffolding:*
>
> Provide support for struggling learners by asking them to recall the two methods for dividing polynomials: the reverse tabular method and the long division algorithm. Provide a table with the dividend and divisor in the proper location or a long division problem with the missing zero coefficient terms already in place.
>
> $$x-3\overline{\smash{)}x^2+0x-9}$$
>
> $$x-3\overline{\smash{)}x^3+0x^2+0x-27}$$
>
> Or using the tabular method as shown.
>
> | | x | $+3$ | |
> |---|---|---|---|
> | | x^2 | $3x$ | x |
> | x^2 | $-3x$ | -9 | -3 |
> | $0x$ | -9 | | |
>
> Ask advanced learners to generate a similar sequence of problems that have a quotient equal to $x - 4$ and to then explain how they determined their expressions.

Opening Exercise (4 minutes)

Opening Exercise

Find the following quotients, and write the quotient in standard form.

a. $\frac{x^2-9}{x-3}$

$x + 3$

b. $\frac{x^3-27}{x-3}$

x^2+3x+9

c. $\frac{x^4-81}{x-3}$

$x^3+3x^2+9x+27$

Discussion (5 minutes)

Have students come to the board to present their solutions. After students check and correct their work, discuss the patterns that they notice in these problems.

- What patterns do you notice in the Opening Exercise?
  - The expression $x-3$ *divides without a remainder into all three dividends, which means it is a factor of each dividend.*
  - *The dividends are differences of powers of* x *and powers of* 3. *For example,* $x^3-27=x^3-3^3$.
  - *The degree of the quotient is* 1 *less than the degree of the dividend. The terms of the quotient are products of powers of* x *and powers of* 3. *The exponents on* x *decrease by one and the exponents on* 3 *increase by one. Each term is positive.*
- Use the patterns you observed in the Opening Exercise to determine the quotient of $\frac{x^5-243}{x-3}$. Explain your reasoning.
  - *Since* $243=3^5$, *we should be able to apply the same pattern, and the quotient should be* $x^4+3x^3+9x^2+27x+81$.
- Test your conjecture by using long division or the reverse tabular method to compute the quotient.
  - *The result is the same.*

Exercise 1 (5 minutes)

Have students work in groups of two or three to complete these problems. Have them make and test conjectures about the quotient that results in each problem. Have the groups divide up the work so at least two students are working on each problem. Then have them share their results in their small groups.

Exercise 1

1. Use patterns to predict each quotient. Explain how you arrived at your prediction, and then test it by applying the reverse tabular method or long division.

a. $\frac{x^2-144}{x-12}$

$x+12$. *I arrived at this conclusion by noting that* $144=12^2$, *so I could apply the patterns in the previous problems to obtain the result.*

b. $\frac{x^3-8}{x-2}$

x^2+2x+4. The dividend is the difference of two perfect cubes, x and $2^3=8$. Based on the patterns in the Opening Exercise, the quotient will be a quadratic polynomial with coefficients that are ascending powers of 2 starting with 2^0.

c. $\frac{x^3-125}{x-5}$

$x^2+5x+25$

d. $\frac{x^6-1}{x-1}$

$x^5+x^4+x^3+x^2+1$

Once this exercise concludes and students have presented their work to the class, they should be ready to generalize a pattern for the quotient of $\frac{x^n-a^n}{x-a}$. The next example establishes this identity for the case where $n=2$.

Example 1 (4 minutes)

We can use the reverse tabular method to compute quotients like the ones in the Opening Exercise and Exercise 1 for any constant a. In this way, we can verify that the patterns you noticed work for any value of a. In this example and Exercises 1 and 2, students work with specific values for the exponents. An interesting extension for advanced students would be to show that $\frac{x^n-a^n}{x-a}=x^{n-1}+ax^{n-2}+a^2x^{n-3}+\cdots a^{n-2}x+a^{n-1}x^0$ using the reverse tabular method.

Example 1

What is the quotient of $\frac{x^2-a^2}{x-a}$? Use the reverse tabular method or long division.

	x	$+a$	
	x^2	ax	x
x^2	$-ax$	$-a^2$	$-a$
$0x$	$-a^2$		

The result of dividing x^2-a^2 by $x-a$ is $x+a$.

Scaffolding:

Provide additional support here by considering specific values of a for each part. For example, ask students to work with

$$\frac{x^3-8}{x-2}$$
$$\frac{x^3-27}{x-3}$$
$$\frac{x^3-64}{x-4},$$

and then ask them to solve the problem using the tabular method for the variables x and a.

Exercise 2 (7 minutes)

Exercises 2–4

2. Work with your group to find the following quotients.

a. $\frac{x^3-a^3}{x-a}$

x^2+ax+a^2

b. $\frac{x^4-a^4}{x-a}$

$x^3+ax^2+a^2x+a^3$

MP.7

Before moving on, discuss these results as a whole class. You may need to model a solution to the third question below if students are still struggling with connecting division back to multiplication.

- What patterns do you notice in the quotient?
  - *The terms are always added and each term is a product of a power of x and a power of a. As the powers of x decrease by 1 for each consecutive term, the powers of a increase by 1.*
- How do these patterns compare to the ones you observed in the opening exercises?
  - *They support the patterns we discovered earlier. This work shows that we can quickly compute the quotient for any problem that fits the pattern.*
- How can you rewrite these division problems as multiplication problems?
  - *The dividend is equal to the product of the quotient and the divisor. For example, $x^3-a^3=(x-a)(x^2+ax+a^2)$. The other problems would be $x^2-a^2=(x-a)(x+a)$ and $x^4-a^4=(x-a)(x^3+ax^2+a^2x+a^3)$.*

Exercise 3 (6 minutes)

The focus shifts to division by $x+a$. The expression $x+a$ divides into the difference of two squares x^2-a^2 without a remainder. Some students may be surprised that it does not divide without a remainder into the difference of two cubes. However, $x+a$ does divide without a remainder into the sum of two cubes but not into the sum of two squares. Prior to this point, students have not worked with polynomial division problems that result in a remainder. As you discuss these results as a class, you can draw a parallel to division of integers. Polynomial division with remainders will be addressed in later lessons in this same module. At this point, lead students to conclude that some of these quotients produce identities that may be helpful for quickly dividing polynomials and some do not.

Ask students to think about how these problems would be different if we were dividing by $x+a$. Have students discuss their ideas with a partner before starting this exercise. Students will most likely assume that there are similar patterns for dividing the sums of squares and cubes, but they may be surprised by their results to parts (b) and (c). For groups that finish early, have them guess and check the results of dividing x^4+a^4 and x^4-a^4 by $x+a$. You can provide additional concrete examples with numerical values of a such as $a=2, 3, 4,\ldots$ if needed to reinforce this concept.

The focus of this part of the lesson is to derive the three identities provided in the Lesson Summary and for students to realize that $x+a$ and $x-a$ do not divide into the sum of squares x^2+a^2 without a remainder.

3. **Predict without performing division whether or not the divisor will divide into the dividend without a remainder for the following problems. If so, find the quotient. Then check your answer.**

   a. $\frac{x^2-a^2}{x+a}$

   The quotient is $x-a$. This makes sense because we already showed that the result when dividing by $x-a$ is $x+a$.

   b. $\frac{x^3-a^3}{x+a}$

   This problem does not divide without a remainder; therefore, $x+a$ is not a factor of x^3-a^3.

   c. $\frac{x^2+a^2}{x+a}$

   This problem does not divide without a remainder; therefore, $x+a$ is not a factor of x^2+a^2.

   d. $\frac{x^3+a^3}{x+a}$

   The quotient is x^2-ax+a^2. This result is similar to our work in Exercise 2 except the middle term is $-ax$ instead of ax.

Exercise 4 (5 minutes)

Students consider the special case when $a=1$ for different values of n. They should be able to quickly generalize a pattern. In part (b), you can introduce the use of the ellipsis (…) to indicate the missing powers of x when displaying the general result since you cannot write all of the terms. In this exercise, we ask students to look for patterns in the quotient $\frac{x^n-1}{x-1}$ for integer exponents n greater than 1.

MP.8

4. **Find the quotient $\frac{x^n-1}{x-1}$ for $n=2, 3, 4$ and 8.**

   For $n=2$, the quotient is $x+1$.

   For $n=3$, the quotient is x^2+x+1.

   For $n=4$, the quotient is x^3+x^2+x+1.

   For $n=8$, the quotient is $x^7+x^6+x^5+\cdots+x+1$.

   a. **What patterns do you notice?**

   The degree of the quotient is 1 less than the degree of the dividend. The degree of each term is 1 less than the degree of the previous term. The last term is 1. The number of terms will be equal to the degree of the dividend.

   b. **Use your work in this problem to write an expression equivalent to $\frac{x^n-1}{x-1}$ for any integer $n>1$.**

   $x^{n-1}+x^{n-2}+x^{n-3}+\cdots+x^1+1$

Closing (5 minutes)

The summary details the identities derived in this lesson. Ask students to summarize the important results of this lesson either in writing, to a partner, or as a class. Take the opportunity to informally assess their understanding of this lesson before moving on to the Exit Ticket. Note that we have not formally derived the last identity, merely used inductive reasoning to generalize the pattern based on the work done in Exercise 4.

Lesson Summary

Based on the work in this lesson, we can conclude the following statements are true for all real values of x and a:

$$\begin{aligned} x^2 - a^2 &= (x-a)(x+a) \\ x^3 - a^3 &= (x-a)(x^2+ax+a^2) \\ x^3 + a^3 &= (x+a)(x^2-ax+a^2), \end{aligned}$$

and it seems that the following statement is also an identity for all real values of x and a:

$x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + x^{n-3} + \cdots + x^1 + 1)$**, for integers** $n > 1$**.**

Exit Ticket (3 minutes)

In this Exit Ticket, students actually apply the identities they worked with to determine quotients. This Exit Ticket will allow you to test their fluency in working with these new relationships. The Lesson Summary is reinforced by having them rewrite each quotient as a product.

Name ______________________________ Date ______________

Lesson 6: Dividing by $x - a$ and by $x + a$

Exit Ticket

Compute each quotient using the identities you discovered in this lesson. Then, write the problem as a product. An example is shown.

QUOTIENT	PRODUCT
$\frac{x^2-9}{x+3} = x - 3$	$x^2 - 9 = (x+3)(x-3)$

1. $\frac{x^4-16}{x-2}$

2. $\frac{x^3+1000}{x+10}$

3. $\frac{x^5-1}{x-1}$

Exit Ticket Sample Solutions

Compute each quotient using the identities you discovered in this lesson. Then, write the problem as a product. An example is shown.

	QUOTIENT	PRODUCT
	$\frac{x^2-9}{x+3} = x-3$	$x^2-9=(x+3)(x-3)$

1. $\frac{x^4-16}{x-2}$

   QUOTIENT: $\frac{x^4-16}{x-2} = x^3+2x^2+4x+8$ ***PRODUCT:*** $x^4-16=(x-2)(x^3+2x^2+4x+8)$

2. $\frac{x^3+1000}{x+10}$

   QUOTIENT: $\frac{x^3+1000}{x+10} = x^2-10x+100$ ***PRODUCT:*** $x^3+1000=(x+10)(x^2-10x+100)$

3. $\frac{x^5-1}{x-1}$

   QUOTIENT: $\frac{x^5-1}{x-1} = x^4+x^3+x^2+x+1$ ***PRODUCT:*** $x^5-1=(x-1)(x^4+x^3+x^2+x+1)$

Problem Set Sample Solutions

1. **Compute each quotient.**

   a. $\frac{x^2-625}{x-25}$

   $x+25$

   b. $\frac{x^3+1}{x+1}$

   x^2-x+1

   c. $\frac{x^3-\frac{1}{8}}{x-\frac{1}{2}}$

   $x^2+\frac{1}{2}x+\frac{1}{4}$

   d. $\frac{x^2-0.01}{x-0.1}$

   $x+0.1$

2. In the next exercises, you can use the same identities you applied in the previous problem. Fill in the blanks in the problems below to help you get started. Check your work by using the reverse tabular method or long division to make sure you are applying the identities correctly.

a. $\frac{16x^2-121}{4x-11} = \frac{(\_\_)^2-(\_\_)^2}{4x-11} = (\_\_) + 11$

$4x, 11, 4x$

b. $\frac{25x^2-49}{5x+7} = \frac{(\_\_)^2-(\_\_)^2}{5x+7} = (\_\_) - (\_\_) =$ ____________

$5x, 7, 5x, 7, 5x - 7$

c. $\frac{8x^3-27}{2x-3} = \frac{(\_\_)^3-(\_\_)^3}{2x-3} = (\_\_)^2 + (\_\_)(\_\_) + (\_\_)^2 =$ ____________________

$2x, 3, 2x, 2x, 3, 3, 4x^2 + 6x + 9$

3. Show how the patterns and relationships learned in this lesson could be applied to solve the following arithmetic problems by filling in the blanks.

a. $\frac{625-81}{16} = \frac{(\_\_)^2-(9)^2}{25-(\_\_)} = (\_\_) + (\_\_) = 34$

$\frac{25^2 - 9^2}{25 - 9} = 25 + 9 = 34$

b. $\frac{1000-27}{7} = \frac{(\_\_)^3-(\_\_)^3}{(\_\_)-3} = (\_\_)^2 + (10)(\_\_) + (\_\_)^2 =$ _______

$\frac{10^3 - 3^3}{10 - 3} = 10^2 + 10(3) + 3^2 = 139$

c. $\frac{100-9}{7} = \frac{(\_\_)^2-(\_\_)^2}{(\_\_)-3} =$ _______

$\frac{10^2 - 3^2}{10 - 3} = 10 + 3 = 13$

d. $\frac{1000+64}{14} = \frac{(\_\_)^3+(\_\_)^3}{(\_\_)+(\_\_)} = (\_\_)^2 - (\_\_)(\_\_) + (\_\_)^2 =$ _______

$\frac{10^3 + 4^3}{10 + 4} = 10^2 - 10(4) + 4^2 = 76$

4. Apply the identities from this lesson to compute each quotient. Check your work using the reverse tabular method or long division.

a. $\frac{16x^2-9}{4x+3}$

$4x - 3$

b. $\frac{81x^2-25}{18x-10}$

$\frac{9}{2}x + \frac{5}{2}$

c. $\frac{27x^3-8}{3x-2}$

$9x^2+6x+4$

5. Extend the patterns and relationships you learned in this lesson to compute the following quotients. Explain your reasoning, and then check your answer by using long division or the tabular method.

a. $\frac{8+x^3}{2+x}$

The quotient is $4+2x+x^2$. This problem just has the variable and constant terms re-written using the commutative property, so it is the same as computing $(x^3+8)\div(x+2)$.

b. $\frac{x^4-y^4}{x-y}$

The quotient is $x^3+x^2y+xy^2+y^3$. This problem is similar to Opening Exercise part (c), except that instead of 81 and 3 in the dividend and quotient, we have a power of y. You can also extend the patterns for $\frac{x^3-a^3}{x-a}=x^2+ax^2+a^2$ using the variable y instead of the variable a.

c. $\frac{27x^3+8y^3}{3x+2y}$

The quotient is $9x^2-6xy+4y^2$. In this example, $3x$ is in the x position and $2y$ is in the a position. Then, the divisor fits the pattern of x^3+a^3.

d. $\frac{x^7-y^7}{x-y}$

The quotient is $x^6+x^5y+x^4y^2+x^3y^3+x^2y^4+xy^5+y^6$. In this problem, replace 1 with y and extend the powers of y pattern using the identities in the Lesson Summary.