Lesson 5: Putting It All Together

Student Outcomes

- Students perform arithmetic operations on polynomials and write them in standard form.
- Students understand the structure of polynomial expressions by quickly determining the first and last terms if the polynomial were to be written in standard form.

Lesson Notes

In this lesson, students work with all four polynomial operations. The first part of the lesson is a relay exercise designed to build fluency, and the second part of the lesson includes combining two or more operations to write a polynomial in standard form. You will need to prepare a set of notecards as described below for Exercises 1–15.

The Algebra Progressions name three different forms for a quadratic expression: standard, factored, and vertex. Examples of these forms are shown below.

- Standard Form: $x^2 + 4x - 5$
- Factored Form: $(x + 5)(x - 1)$
- Vertex Form: $(x + 2)^2 - 9$

We can define a standard and factored form of a degree n polynomial expression as well. The final lessons in this module introduce the Fundamental Theorem of Algebra which states that a degree n polynomial with real coefficients can be written as the product of n linear factors. Precise definitions of the following terms are provided for teacher reference at the end of this lesson: monomial, polynomial expression, coefficient of a monomial, degree of a monomial, terms of a polynomial, standard form of a polynomial in one variable, and degree of a polynomial in one variable. This lesson concludes by challenging students to quickly determine the first and last term of a polynomial expression if it were to be written in standard form. Students who have this capacity can quickly analyze the end behavior of the graph of a polynomial function or compute key features such as a y-intercept without having to fully re-write the expression in standard form. This allows for quick analysis when applying properties of polynomials in future mathematics courses such as Grade 12 mathematics or Calculus.

> *Scaffolding:*
>
> Before beginning these exercises it may be necessary to review the definition and/or post an example of a polynomial in standard form. You can also remind students of vertex and factored form of quadratic expressions using the examples in the Lesson Notes if needed.
>
> For example,
>
> - Standard Form: $x^2 + 4x - 5$
> - Factored Form: $(x + 5)(x - 1)$
> - Vertex Form: $(x + 2)^2 - 9$

Classwork

Exercises 1–15 (20 minutes): Polynomial Pass

Prior to the lesson, write each Exercise 1 to 15 on an index card, one exercise per card. On the back of each card, write the solution to the previous exercise. For example, on the back of the card for Exercise 2, write the answer to Exercise 1. On the back of the index card for Exercise 1, write the answer to Exercise 15. Students should be seated in a circle if possible. If more than 15 different cards are needed, create additional exercises that require addition, subtraction, multiplication, or division without remainder of linear, quadratic, or cubic polyomials.

To complete the exercise, have students work in pairs. Pass out cards to pairs in numerical order so that when the cards are passed, each pair gets a new exercise that contains the answer to the one they just worked on the back of the card. (Alternatively, have students work individually and make two sets of cards so that cards 16–30 are duplicates of 1–15.) Allow one minute for students to work each problem on a separate sheet of paper, and then have them pass the card to the next pair and receive a card from the previous pair. They should check their answers and begin the next exercise. After all the rounds are completed, have students complete the graphic organizer on their student handouts to model each operation.

Exercises 1–15: Polynomial Pass

Perform the indicated operation to write each polynomial in standard form.

1. $(x^2-3)(x^2+3x-1)$ $\quad$ $x^4+3x^3-4x^2-9x+3$
2. $(5x^2-3x-7)-(x^2+2x-5)$ $\quad$ $4x^2-5x-2$
3. $(x^3-8)\div(x-2)$ $\quad$ x^2+2x+4
4. $(x+1)(x-2)(x+3)$ $\quad$ x^3+2x^2-5x-6
5. $(x+1)-(x-2)-(x+3)$ $\quad$ $-x$
6. $(x+2)(2x^2-5x+7)$ $\quad$ $2x^3-x^2-3x+14$
7. $\frac{x^3-2x^2-65x+18}{x-9}$ $\quad$ x^2+7x-2
8. $(x^2-3x+2)-(2-x+2x^2)$ $\quad$ $-x^2-2x$
9. $(x^2-3x+2)(2-x+2x^2)$ $\quad$ $2x^4-7x^3+9x^2-8x+4$
10. $\frac{x^3-x^2-5x-3}{x-3}$ $\quad$ x^2+2x+1
11. $(x^2+7x-12)(x^2-9x+1)$ $\quad$ $x^4-2x^3-74x^2+115x-12$
12. $(2x^3-6x^2-7x-2)+(x^3+x^2+6x-12)$ $\quad$ $3x^3-5x^2-x-14$
13. $(x^3-8)(x^2-4x+4)$ $\quad$ $x^5-4x^4+4x^3-8x^2+32x-32$
14. $(x^3-2x^2-5x+6)\div(x+2)$ $\quad$ x^2-4x+3
15. $(x^3+2x^2-3x-1)+(4-x-x^3)$ $\quad$ $2x^2-4x+3$

Exercise 16 (5 minutes)

Use this exercise to help students see the structure of the expressions and to create a graphic organizer of how to perform the four polynomial operations for future reference. They should complete this exercise either with a partner or in groups of four.

Exercises 16–22

16. Review Exercises 1–15 and then select one exercise for each category and record the steps in the operation below as an example. Be sure to show all your work.

Addition Exercise	**Multiplication Exercise**
Subtraction Exercise	**Division Exercise**

Exercises 17–20 (5 minutes)

Before starting the next exercises, lead a short discussion transitioning into the problems below that combine polynomial operations. Say (or paraphrase) the following:

- In the previous exercises you applied one operation to two or three polynomials, but many times we work with expressions that contain more than one operation. In the next exercises, more than one operation is indicated. How do you determine which operation to perform first?
  - *The parentheses and the use of the fraction bar for division tell us which operations to perform first. The order of operations also can be applied to polynomials. For example, you have to multiply before you can add or subtract.*

After this discussion, have students work with the same partner (or group) that they worked with on Exercise 16. Have different groups present their work to the class. Students could use whiteboards or chart paper to present their solutions to the class.

For Exercises 17–20, re-write each polynomial in standard form by applying the operations in the appropriate order.

17. $\dfrac{(x^2+5x+20)+(x^2+6x-6)}{x+2}$

$\mathbf{2x+7}$

18. $(x^2-4)(x+3)-(x^2+2x-5)$

$\mathbf{x^3+2x^2-6x-7}$

19. $\frac{(x-3)^3}{x^2-6x+9}$

$x-3$

20. $(x+7)(2x-3)-(x^3-2x^2+x-2)\div(x-2)$

$x^2+11x-22$

Exercise 21 (4 minutes)

MP.7

This exercise along with Exercise 22 helps students understand that they can learn quite a bit about the nature of a polynomial expression without performing all the operations required to write it in standard form.

- Sometimes we do not need to perform the entire operation to understand the structure of an expression. Can you think of a situation where we might only need to know the first term or the last term of a polynomial expression?
  - *If we wanted to understand the shape of the graph of a polynomial, like $p(x)=x^2+2x+3$, we would only need to know the coefficient and degree of the first term to get a general idea of its behavior.*
  - *The constant term of a polynomial expression indicates the y-intercept of the corresponding graph.*

21. What would be the first and last terms of the polynomial if it was re-written in standard form? Answer these quickly without performing all of the indicated operations.

a. $(2x^3-x^2-9x+7)+(11x^2-6x^3+2x-9)$

First term: $-4x^3$, *Last term:* -2

b. $(x-3)(2x+3)(x-1)$

First term: $2x^3$, *Last term:* 9

c. $(2x-3)(3x+5)-(x+1)(2x^2-6x+3)$

First term: $-2x^3$, *Last term:* -18

d. $(x+5)(3x-1)-(x-4)^2$

First term: $2x^2$, *Last term:* -21

After students complete this exercise, lead a short discussion.

- How did you determine your answer quickly?
  - *I knew that the first term would be the one with the highest degree, so I focused on that operation. I knew that the last term would be the constant term unless it turned out to equal 0.*
- Did any of the solutions surprise you?
  - *In parts (a) and (c), I had to pay attention to the order of the terms and which operation would produce the largest degree term.*

Exercise 22 (4 minutes)

22. What would the first and last terms of the polynomial be if it was re-written in standard form?

a. $(n+1)(n+2)(n+3)\dots(n+9)(n+10)$ *First term:* n^{10}, *Last term:* $10!$

b. $(x-2)^{10}$ *First term:* x^{10}, *Last term:* $(-2)^{10}$

c. $\frac{(x-2)^{10}}{(x-2)}$ *First term:* x^{9}, *Last term:* $(-2)^{9}$

d. $\frac{n(n+1)(2n+1)}{6}$ *First term:* $\frac{1}{3}n^3$, *Last term:* $\frac{n}{6}$

> *Scaffolding:*
>
> For advanced learners or early finishers, you can increase the complexity of these exercises by posing the following problem (or one similar to it):
>
> - Generate three different polynomial expressions NOT already expressed in standard form that would have a first term of $-3x^3$ and a last term of $\frac{1}{2}$ if the polynomial expression was written in standard form.

Closing (2 minutes)

Consider having students record their answers to these questions in writing or by sharing their thoughts with a partner.

- How is polynomial arithmetic similar to integer arithmetic?
  - *The four operations produce a new polynomial. The four operations can be combined by following the order of operations conventions.*
- How can you quickly determine the first and last terms of a polynomial without performing all of the operations needed to rewrite it in standard form?
  - *Analyze the problem to identify the highest degree terms and perform the indicated operation on only those terms. Do the same with the lowest degree terms to determine the last term.*

Precise definitions of terms related to polynomials are presented here. These definitions were first introduced in Algebra I, Modules 1 and 4. These definitions are for teacher reference and can be shared with students at your discretion. Following these definitions are discussion questions for closing this lesson.

Relevent Vocabulary

Polynomial Expression: A *polynomial expression* is either a numerical expression or a variable symbol or the result of placing two previously generated polynomial expressions into the blanks of the addition operator (__+__) or the multiplication operator (__×__).

The definition of polynomial expression includes subtraction ($a-b = a+(-1\cdot b)$), exponentiation by a non-negative integer ($x^3 = (x\cdot x)\cdot x$), and dividing by a non-zero number (multiplying by the multiplicative inverse). Because subtraction, exponentiation, and division still apply, we will continue to use the regular notation for these operations. In other words, we will continue to write polynomials simply as $\frac{(x^3-3x)}{2}$ instead of

$$\frac{1}{2}\cdot\left(\left((x\cdot x)\cdot x\right)+(-1\cdot(3\cdot x))\right)$$

Note, however, that the definition excludes dividing by x or dividing by any polynomial in x.

All polynomial expressions are algebraic expressions.

Monomial: A *monomial* is a polynomial expression generated using only the multiplication operator (__×__).

Monomials are products whose factors are numerical expressions or variable symbols.

Coefficient of a Monomial: The *coefficient of a monomial* is the value of the numerical expression found by substituting the number 1 into all the variable symbols in the monomial.

Sometimes the coefficient is considered as a constant (like the constant a in ax^2) instead of an actual number. In those cases, when a monomial is expressed as a product of a constant and variables, then the constant is called a *constant coefficient.*

Degree of a Monomial: The *degree of a* non-zero *monomial* is the sum of the exponents of the variable symbols that appear in the monomial.

For example, the degree of $7x^2y^4$ is 6, the degree of $8x$ is 1, the degree of 8 is 0, and the degree of $9x^2y^3x^{10}$ is 15.

Terms of a Polynomial: When a polynomial is expressed as a monomial or a sum of monomials, each monomial in the sum is called a *term of the polynomial*.

A monomial can now be described as a "polynomial with only one term." But please realize that the word "term" means many, seemingly disparate things in mathematics (e.g., $\sin x$ is often called a term of the expression $\sin x + \cos x$), whereas monomial is a specific object. This is why we have chosen to define monomial first and then use monomial to define a *term of a polynomial*.

Standard Form of a Polynomial in One Variable: A polynomial expression with one variable symbol x is in *standard form* if it is expressed as,

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where n is a non-negative integer, and a_0, a_1, a_2,…, a_n are constant coefficients with $a_n \neq 0$.

A polynomial expression in x that is in standard form is often called a *polynomial in x*.

The *degree of the polynomial in standard form* is the highest degree of the terms in the polynomial, namely n. The term $a_n x^n$ is called the *leading term* and a_n is called the *leading coefficient.* The *constant term* is the value of the numerical expression found by substituting 0 into all the variable symbols of the polynomial, namely a_0.

In general, one has to be careful about determining the degree of polynomials that are not in standard form: For example, the degree of the polynomial,

$$(x+1)^2 - (x-1)^2,$$

is not 2 since its standard form is $4x$.

Degree of a Polynomial in One Variable: The *degree of a polynomial expression in one variable* is the degree of the polynomial in standard form that is equivalent to it.

Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 5: Putting It All Together

Exit Ticket

Jenny thinks that the expression below is equal to $x^2 - 4$. If you agree, show that she is correct. If you disagree, show that she is wrong by re-writing this expression as a polynomial in standard form.

$$\frac{(x-2)^3}{x-2}$$

Exit Ticket Sample Solutions

MP.3

Jenny thinks that the expression below is equal to $x^2 - 4$. If you agree, show that she is correct. If you disagree, show that she is wrong by re-writing this expression as a polynomial in standard form.

$$\frac{(x-2)^3}{x-2}$$

Multiple approaches are possible to justify why Jenny is incorrect. One possible solution is shown below.

Jenny is incorrect. To perform this operation, you can first divide by $x-2$ then expand the quotient.

$$\begin{aligned}\frac{(x-2)^3}{x-2} &= (x-2)^2\\ &= x^2-4x+4\end{aligned}$$

Problem Set Sample Solutions

For Problems 1–7, rewrite each expression as a polynomial is standard form.

1. $(3x-4)^3$

   $27x^3 - 108x^2 + 144x - 64$

2. $(2x^2 - x^3 - 9x + 1) - (x^3 + 7x - 3x^2 + 1)$

   $-2x^3 + 5x^2 - 16x$

3. $(x^2 - 5x + 2)(x - 3)$

   $x^3 - 8x^2 + 17x - 6$

4. $\frac{x^4 - x^3 - 6x^2 - 9x + 27}{x-3}$

   $x^3 + 2x^2 - 9$

5. $(x+3)(x-3) - (x+4)(x-4)$

   7

6. $(x+3)^2 - (x+4)^2$

   $-2x - 7$

7. $\frac{x^2 - 5x + 6}{x-3} + \frac{x^3 - 1}{x-1}$

   $x^2 + 2x - 1$

For Problems 8–9: Quick, what would be the first and last terms of the polynomial if it was written in standard form?

8. $2(x^2 - 5x + 4) - (x + 3)(x + 2)$

The first and last term are x^2 and 2.

9. $\frac{(x-2)^5}{x-2}$

The first and last terms are x^4 and 16.

10. The profit a business earns by selling x items is given by the polynomial function

$$p(x) = x(160 - x) - (100x + 500).$$

What is the last term in the standard form of this polynomial? What does it mean in this situation?

The last term is -500, so that $p(0) = -500$. This means that if no items are sold the company would lose $\$500$.

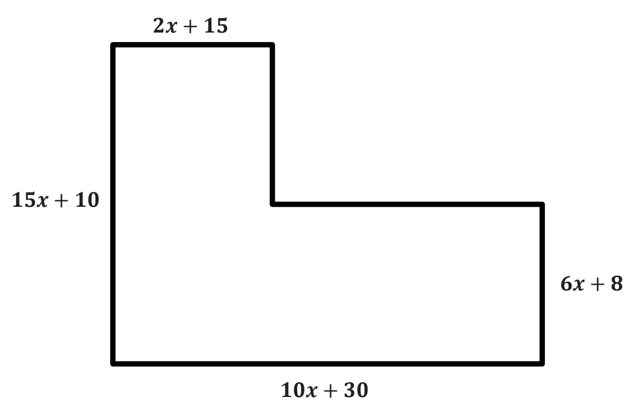
11. Explain why these two quotients are different. Compute each one. What do they have in common? Why?

$$\frac{(x-2)^4}{x-2} \text{ and } \frac{x^4-16}{x-2}$$

The quotients are $x^3 - 6x^2 + 12x - 8$ and $x^3 + 2x^2 + 4x + 8$.

They are different because the dividends are not equivalent expressions. The quotients have the first and last terms in common because division is going to reduce the degree by the difference of the degrees of the numerator and denominator, and their leading coefficients were both one. When multiplying, the last term of a polynomial in standard form is the product of the lowest degree terms in each factor. Therefore, when dividing, the last term of the quotient will be the quotient of the last term of the dividend and divisor.

12. What are the area and perimeter of the figure? Assume a right angle at each vertex.



The missing horizontal side length is $8x + 15$. The missing vertical side length is $9x + 2$. I determined these lengths by subtracting the vertical lengths and by subtracting the horizontal lengths. The perimeter is $50x + 80$. I got this by adding the lengths of all of the sides together. The area can be found by splitting the shape either horizontally or vertically into two rectangles. If split vertically, the areas of the rectangles are $(15x + 10)(2x + 15)$ and $(6x + 8)(8x + 15)$. The total area of the figure is the sum of these two products, $78x^2 + 399x + 270$.