Lesson 1: Successive Differences in Polynomials

Student Outcomes

- Students write explicit polynomial expressions for sequences by investigating successive differences of those sequences.

Lesson Notes

This first lesson of the year tells students that this course is about thinking and reasoning with mathematics. It reintroduces the study of polynomials in a surprising new way involving sequences. This gives you a chance to evaluate how much your students recall from Grade 9. The lesson starts with discussions of expressions, polynomials, sequences, and equations. In this lesson, students continue the theme that began in Grade 6 of evaluating and building expressions. Explore ways to test your students' recall of the vocabulary terms listed at the end of this lesson.

Throughout this lesson, listen carefully to your students' discussions. Their reactions will inform you how to best approach the rest of the module. The homework set to this lesson should also give you insight into how much they remember from previous grades and how well they can read instructions. In particular, if they have trouble with evaluating or simplifying expressions or solving equations, then you might want to revisit Lessons 6–9 in Grade 9, Module 1, and Lesson 2 in Grade 9, Module 4. If they are having trouble solving equations, use Lessons 10–12, 15–16, and 19 in Grade 9, Module 1 to give them extra practice.

Finally, the use of the term *constant* may need a bit of extra discussion. It is used throughout this PK–12 curriculum in two ways: either as a constant number (e.g., the a in $ax^2 + bx + c$ is a number chosen once-and-for-all at the beginning of a problem) or as a constant rate (e.g., a copier that reproduces at a constant rate of 40 copies/minute). We offer both uses in this lesson.

Classwork

Opening Exercise (7 minutes)

This exercise provides an opportunity to think about and generalize the main concept of today's lesson: that the second differences of a quadratic polynomial are constant. This generalizes to the n^{th} differences of a degree n polynomial. Your goal is to help students investigate, discuss, and generalize the second and higher differences in this exercise.

MP.1 & MP.8

Present the exercise to your students and ask them (in groups of two) to study the table and explain to their partner how to calculate each line in the table. If they get stuck, help them find entry points into this question, possibly by drawing segments connecting the successive differences on their papers (e.g., connect 5.76 and 11.56 to 5.8 and ask, "How are these three numbers related?"). This initial problem of the school year is designed to encourage students to persevere and look for and express regularity in repeated reasoning.

Teachers may also use the Opening Exercise to informally assess students' pattern-finding abilities and fluency with rational numbers.

Scaffolding:

Before presenting the problem below, you might start by displaying the first two rows of the table on the board and asking students to investigate the relationship between them, including making a conjecture about the nature of the relationship.

Opening Exercise

John noticed patterns in the arrangement of numbers in the table below.

Number	2.4		3.4		4.4		5.4		6.4
Square	5.76		11.56		19.36		29.16		40.96
First Differences		5.8		7.8		9.8		11.8	
Second Differences			2		2		2		

MP.8

Assuming that the pattern would continue, he used it to find the value of 7.4^2. Explain how he used the pattern to find 7.4^2, and then use the pattern to find 8.4^2.

To find 7.4^2, John assumed the next term in the first differences would have to be 13.8 since 13.8 is 2 more than 11.8. Therefore, the next term in the square numbers would have to be $40.96 + 13.8$, which is 54.76. Checking with a calculator, we also find $7.4^2 = 54.76$.

To find 8.4^2, we follow the same process: The next term in the first differences would have to be 15.8, so the next term in the square numbers would be $54.76 + 15.8$, which is 70.56. Check: $8.4^2 = 70.56$.

How would you label each row of numbers in the table?

Number, Square, First Differences, Second Differences

Scaffolding:

For students working below grade level, consider using positive integers $\{1, 2, 3, \ldots\}$ and corresponding squares $\{1, 4, 9, \ldots\}$ instead of using $\{2.4, 3.4, 4.4, \ldots\}$.

Discuss with students the relationship between each row and the row above it and how to label the rows based upon that relationship. Feel free to have this discussion before or after they find 7.4^2 and 8.4^2. They are likely to come up with labels such as "subtract" or "difference" for the third and fourth row. However, guide them to call the third and fourth rows "First Differences" and "Second Differences," respectively.

Discussion (3 minutes)

The pattern illustrated in the Opening Exercise is a particular case of a general phenomenon about polynomials. In Algebra I, Module 3, we saw how to recognize linear functions and exponential functions by recognizing similar growth patterns; that is, linear functions grow by a "constant difference over successive intervals of equal length," and exponential functions grow by a "constant factor over successive intervals of equal length." In this lesson, we generalize the linear growth pattern to polynomials of second degree (quadratic expressions) and third degree (cubic expressions).

Discussion

Let the sequence $\{a_0, a_1, a_2, a_3, \ldots\}$ be generated by evaluating a polynomial expression at the values $0, 1, 2, 3, \ldots$. The numbers found by evaluating $a_1 - a_0$, $a_2 - a_1$, $a_3 - a_2, \ldots$ form a new sequence which we will call the *first differences* of the polynomial. The differences between successive terms of the first differences sequence are called the *second differences* and so on.

It is a good idea to use an actual sequence of numbers such as the square numbers $\{1, 4, 9, 16, \ldots\}$ to help explain the meaning of the terms "first differences" and "second differences."

Example 1 (4 minutes)

Although you may be tempted to work through Example 1 using numbers instead of a and b, using symbols a and b actually makes the *structure* of the first differences sequence obvious, whereas numbers could hide that structure. Also, working with constant coefficients gives the generalization all at once.

Note: Consider using Example 1 to informally assess students' fluency with algebraic manipulations.

Example 1

What is the sequence of first differences for the linear polynomial given by $ax+b$, where a and b are constant coefficients?

The terms of the first differences sequence are found by subtracting consecutive terms in the sequence generated by the polynomial expression $ax+b$, namely, $\{b, a+b, 2a+b, 3a+b, 4a+b, \ldots\}$.

1^{st} term: $(a+b)-b=a$,

2^{nd} term: $(2a+b)-(a+b)=a$,

3^{rd} term: $(3a+b)-(2a+b)=a$,

4^{th} term: $(4a+b)-(3a+b)=a$.

The first differences sequence is $\{a, a, a, a, \ldots\}$. For first degree polynomial expressions, the first differences are constant and equal to a.

What is the sequence of second differences for $ax+b$?

Since $a-a=0$, the second differences are all 0. Thus the sequence of second differences is $\{0, 0, 0, 0, \ldots\}$.

Scaffolding:

Try starting the example by first asking students to generate sequences of first differences for $2x+3$, $3x-1$, and $4x+2$. For example, the sequence generated by $2x+3$ is $\{3, 5, 7, 9, \ldots\}$ and its sequence of first differences is $\{2, 2, 2, 2 \ldots\}$.

You can then use these three sequences as a source of examples from which to make

- How is this calculation similar to the arithmetic sequences you studied in Algebra I, Module 3?
  - *The constant derived from the first differences of a linear polynomial is the same constant addend used to define the arithmetic sequence generated by the polynomial. That is, the a in $A(n)=an+b$ for $n \geq 0$. Written recursively this is $A(0)=b$ and $A(n+1)=A(n)+a$ for $n \geq 0$.*

For Examples 2 and 3, let students work in groups of two to fill in the blanks of the tables (3 min. max for each table). Walk around the room, checking student work for understanding. Afterward, discuss the paragraphs below each table as a whole class.

Example 2 (5 minutes)

Example 2

Find the first, second, and third differences of the polynomial $ax^2 + bx + c$ by filling in the blanks in the following table.

x	$ax^2 + bx + c$	First Differences	Second Differences	Third Differences
0	c			
		$\underline{a+b}$		
1	$a+b+c$		$\underline{2a}$	
		$\underline{3a+b}$		$\underline{0}$
2	$4a+2b+c$		$\underline{2a}$	
		$\underline{5a+b}$		$\underline{0}$
3	$9a+3b+c$		$\underline{2a}$	
		$\underline{7a+b}$		$\underline{0}$
4	$16a+4b+c$		$\underline{2a}$	
		$\underline{9a+b}$		
5	$25a+5b+c$			

MP.7 & MP.8

The table shows that the second differences of the polynomial $ax^2 + bx + c$ all have the constant value $2a$. The second differences hold for any sequence of values of x where the values in the sequence differ by 1, as the Opening Exercise shows. For example, if we studied the second differences for x-values $\pi, \pi+1, \pi+2, \pi+3, \ldots$, we would find that the second differences would also be $2a$. In your homework, you will show that this fact is indeed true by finding the second differences for the values $n+0, n+1, n+2, n+3, n+4$.

Ask students to describe what they notice in the sequences of first, second, and third differences. Have them make a conjecture about the third and fourth differences of a sequence generated by a third degree polynomial.

Students are likely to say that the third differences have the constant value $3a$ (which is incorrect). Have them work through the next example to help them discover what the third differences really are. This is a good example of why we need to follow up conjecture based on observation with proof.

Example 3 (7 minutes)

Example 3

Find the second, third, and fourth differences of the polynomial $ax^3 + bx^2 + cx + d$ by filling in the blanks in the following table.

x	$ax^3 + bx^2 + cx + d$	First Differences	Second Differences	Third Differences	Fourth Differences
0	d				
		$a+b+c$			
1	$a+b+c+d$		$\underline{6a+2b}$		
		$7a+3b+c$		$\underline{6a}$	
2	$8a+4b+2c+d$		$\underline{12a+2b}$		$\underline{0}$
		$19a+5b+c$		$\underline{6a}$	
3	$27a+9b+3c+d$		$\underline{18a+2b}$		$\underline{0}$
		$37a+7b+c$		$\underline{6a}$	
4	$64a+16b+4c+d$		$\underline{24a+2b}$		
		$61a+9b+c$			
5	$125a+25b+5c+d$				

The third differences of $ax^3 + bx^2 + cx + d$ all have the constant value $6a$. Also, if a different sequence of values for x that differed by 1 was used instead, the third differences would still have the value $6a$.

- Ask students to make a conjecture about the fourth differences of a sequence generated by degree 4 polynomial. Students who were paying attention to their (likely wrong) conjecture of the third differences before doing this example may guess that the fourth differences are constant and equal to $(1 \cdot 2 \cdot 3 \cdot 4)a$, which is $24a$. This pattern continues: the n^{th} differences of any sequence generated by an n^{th} degree polynomial with leading coefficient a will be constant and have the value $a \cdot (n!)$.
- Ask students to make a conjecture about the $(n+1)^{\text{st}}$ differences of a degree n polynomial, for example, the 5^{th} differences of a fourth degree polynomial.

Students are now ready to tackle the main goal of this lesson—using differences to recognize polynomial relationships and build polynomial expressions.

Example 4 (7 minutes)

When collecting bivariate data on an event or experiment, the data does not announce, "I satisfy a quadratic relationship," or "I satisfy an exponential relationship." We need ways to recognize these relationships in order to model them with functions. In Algebra I, Module 3, students studied the conditions upon which they could conclude that the data satisfied a linear or exponential relationship. Either the first differences were constant or "first factors" were constant. By checking that the second or third differences of the data are constant, students now have a way to recognize a quadratic or cubic relationship and can write an equation to describe that relationship (**A-CED.A.3**, **F-BF.A.1a**).

Give students an opportunity to attempt this problem in groups of two. Walk around the room helping them find the leading coefficient.

Example 4

What type of relationship does the set of ordered pairs (x, y) satisfy? How do you know? Fill in the blanks in the table below to help you decide. (The first differences have already been computed for you.)

x	y	First Differences	Second Differences	Third Differences
0	**2**			
		-1		
1	**1**		6	
		5		6
2	**6**		12	
		17		6
3	**23**		18	
		35		6
4	**58**		24	
		59		
5	**117**			

Since the third differences are constant, the pairs could represent a cubic relationship between x and y.

Find the equation of the form $y = ax^3 + bx^2 + cx + d$ that all ordered pairs (x, y) above satisfy. Give evidence that your equation is correct.

Since third differences of a cubic polynomial are equal to $6a$, using the table above, we get $6a = 6$, so that $a = 1$. Also, since $(0, 2)$ satisfies the equation, we see that $d = 2$. Thus, we need only find b and c. Substituting $(1, 1)$ and $(2, 6)$ into the equation, we get

$$\begin{aligned} 1 &= 1 + b + c + 2 \\ 6 &= 8 + 4b + 2c + 2. \end{aligned}$$

Subtracting two times the first equation from the second, we get $4 = 6 + 2b - 2$, so that $b = 0$. Substituting 0 in for b in the first equation gives $c = -2$. Thus, the equation is $y = x^3 - 2x + 2$.

- After finding the equation, have students check that the pairs $(3, 23)$ and $(4, 58)$ satisfy the equation.

Help students to persevere in finding the coefficients (MP.1). They will most likely try to plug three ordered pairs into the equation, which gives a 3×3 system of linear equations in a, b, and c after they find that $d = 2$. Using the fact that the third differences of a cubic polynomial are $6a$ will greatly simplify the problem. (It implies $a = 1$ immediately, which reduces the system to the easy 2×2 system above.) Walk around the room as they work, and ask questions that lead them to realize that they can use the third differences fact if they get too stuck. Alternatively, find a student who used the fact, and then have the class discuss and understand his or her approach.

Closing (7 minutes)

- What are some of the key ideas that we learned today?
  - *Sequences whose second differences are constant satisfy a quadratic relationship.*
  - *Sequences whose third differences are constant satisfy a cubic relationship.*

The following terms were introduced and taught in Module 1 of Algebra I. The terms are listed here for completeness and reference.

Relevant Vocabulary

Numerical Symbol: A *numerical symbol* is a symbol that represents a specific number. Examples: $1, 2, 3, 4, \pi, -3.2$.

Variable Symbol: A *variable symbol* is a symbol that is a placeholder for a number from a specified set of numbers. The set of numbers is called the *domain of the variable.* Examples: x, y, z.

Algebraic Expression: An *algebraic expression* is either

1. **a numerical symbol or a variable symbol or**
2. **the result of placing previously generated algebraic expressions into the two blanks of one of the four operators ((__)+(__), (__)−(__), (__)×(__), (__)÷(__)) or into the base blank of an exponentiation with an exponent that is a rational number.**

Following the definition above, $\left(\left((x) \times (x)\right) \times (x)\right) + \left((3) \times (x)\right)$ is an algebraic expression, but it is generally written more simply as $x^3 + 3x$.

Numerical Expression: A *numerical expression* is an algebraic expression that contains only numerical symbols (no variable symbols) that evaluates to a single number. Example: The numerical expression $\frac{(3 \cdot 2)^2}{12}$ evaluates to 3.

Monomial: A *monomial* is an algebraic expression generated using only the multiplication operator (__×__). The expressions x^3 and $3x$ are both monomials.

Binomial: A *binomial* is the sum of two monomials. The expression $x^3 + 3x$ is a binomial.

<u>Polynomial Expression</u>: A *polynomial expression* is a monomial or sum of two or more monomials.

<u>Sequence</u>: A *sequence* can be thought of as an ordered list of elements. The elements of the list are called the *terms of the sequence.*

<u>Arithmetic Sequence</u>: A sequence is called *arithmetic* if there is a real number d such that each term in the sequence is the sum of the previous term and d.

Exit Ticket (5 minutes)

Name ______________________________ Date ________________

Lesson 1: Successive Differences in Polynomials

Exit Ticket

1. What type of relationship is indicated by the following set of ordered pairs? Explain how you know.

x	y
0	0
1	2
2	10
3	24
4	44

2. Find an equation that all ordered pairs above satisfy.

Exit Ticket Sample Solutions

1. **What type of relationship is indicated by the following set of ordered pairs? Explain how you know.**

x	y	*First Differences*	*Second Differences*
0	0		
		2	
1	2		6
		8	
2	10		6
		14	
3	24		6
		20	
4	44		

Since the second differences are constant, there is a quadratic relationship between x and y.

2. **Find an equation that all ordered pairs above satisfy.**

Since $(0, 0)$ satisfies an equation of the form $y = ax^2 + bx + c$, we have that $c = 0$. Using the points $(1, 2)$ and $(2, 10)$, we have

$$2 = a + b$$
$$10 = 4a + 2b$$

Subtracting twice the first equation from the second gives $6 = 2a$, which means $a = 3$. Substituting 3 into the first equation gives $b = -1$. Thus, $y = 3x^2 - x$ is the equation.

OR

Since the pairs satisfy a quadratic relationship, the second differences must be equal to $2a$. Therefore, $6 = 2a$, so $a = 3$. Since $(0, 0)$ satisfies the equation, $c = 0$. Using the point $(1, 2)$, we have that $2 = 3 + b + 0$, so $b = -1$. Thus, $y = 3x^2 - x$ is the equation that is satisfied by these points.

Problem Set Sample Solutions

1. **Create a table to find the second differences for the polynomial $36 - 16t^2$ for integer values of t from 0 to 5.**

t	$36 - 16t^2$	*First Differences*	*Second Differences*
0	36		
		-16	
1	20		-32
		-48	
2	-28		-32
		-80	
3	-108		-32
		-112	
4	-220		-32
		-144	
5	-364		

2. Create a table to find the third differences for the polynomial $s^3 - s^2 + s$ for integer values of s from -3 to 3.

s	$s^3 - s^2 + s$	*First Differences*	*Second Differences*	*Third Differences*
-3	-39			
		25		
-2	-14		-14	
		11		6
-1	-3		-8	
		3		6
0	0		-2	
		1		6
1	1		4	
		5		6
2	6		10	
		15		
3	21			

3. Create a table of values for the polynomial x^2, using $n, n+1, n+2, n+3, n+4$ as values of x. Show that the second differences are all equal to 2.

x	x^2	*First Differences*	*Second Differences*
n	n^2		
		$2n+1$	
$n+1$	n^2+2n+1		2
		$2n+3$	
$n+2$	n^2+4n+4		2
		$2n+5$	
$n+3$	n^2+6n+9		2
		$2n+7$	
$n+4$	$n^2+8n+16$		

4. Show that the set of ordered pairs (x, y) in the table below satisfies a quadratic relationship. (Hint: Find second differences.) Find the equation of the form $y = ax^2 + bx + c$ that all of the ordered pairs satisfy.

x	0	1	2	3	4	5
y	5	4	-1	-10	-23	-40

Students show that second differences are constant and equal to -4. The equation: $y = -2x^2 + x + 5$.

5. Show that the set of ordered pairs (x, y) in the table below satisfies a cubic relationship. (Hint: Find third differences.) Find the equation of the form $y = ax^3 + bx^2 + cx + d$ that all of the ordered pairs satisfy.

x	0	1	2	3	4	5
y	20	4	0	20	76	180

Students show that third differences are constant and equal to 12. The equation: $y = 2x^3 - 18x + 20$.

6. The distance d ft. required to stop a car traveling at $10v$ mph under dry asphalt conditions is given by the following table.

v	0	1	2	3	4	5
d	0	5	19.5	43.5	77	120

a. What type of relationship is indicated by the set of ordered pairs?

Students show that second differences are constant and equal to 9.5. Therefore, the relationship is quadratic.

b. Assuming that the relationship continues to hold, find the distance required to stop the car when the speed reaches 60 mph, when $v = 6$.

172.5 *ft.*

c. (Challenge) Find an equation that describes the relationship between the speed of the car v and its stopping distance d.

$\mathrm{d} = 4.75\mathrm{v}^2 + 0.25\mathrm{v}$ *(Note: Students do not need to find the equation to answer part (b).)*

7. Use the polynomial expressions $5x^2 + x + 1$ and $2x + 3$ to answer the questions below.

a. Create a table of second differences for the polynomial $5x^2 + x + 1$ for the integer values of x from 0 to 5.

x	$5x^2 + x + 1$	*First Differences*	*Second Differences*
0	1		
		6	
1	7		10
		16	
2	23		10
		26	
3	49		10
		36	
4	85		10
		46	
5	131		

b. Justin claims that for $n \geq 2$, the n^{th} differences of the sum of a degree n polynomial and a linear polynomial are the same as the n^{th} differences of just the degree n polynomial. Find the second differences for the sum $(5x^2 + x + 1) + (2x + 3)$ of a degree 2 and a degree 1 polynomial and use the calculation to explain why Justin might be correct in general.

Students compute that the second differences are constant and equal to 10, just as in part (a). Justin is correct because the differences of the sum are the sum of the differences. Since the second (and all other higher) differences of the degree 1 polynomial are constant and equal to zero, only the n^{th} differences of the degree n polynomial contribute to the n^{th} difference of the sum.

c. Jason thinks he can generalize Justin's claim to the product of two polynomials. He claims that for $n \geq 2$, the $(n+1)^{th}$ differences of the product of a degree n polynomial and a linear polynomial are the same as the n^{th} differences of the degree n polynomial. Use what you know about second and third differences (from Examples 2 and 3) and the polynomial $(5x^2 + x + 1)(2x + 3)$ to show that Jason's generalization is incorrect.

The second differences of a quadratic polynomial are $2a$, so the second differences of $5x^2 + x + 1$ are always 10. Since $(5x^2 + x + 1)(2x + 3) = 10x^3 + 17x^2 + 5x + 3$ and third differences are equal to $6a$, we have that the third differences of $(5x^2 + x + 1)(2x + 3)$ are always 60, which is not 10.