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# Extending to Three Dimensions



[1] Each lesson is ONE day, and ONE day is considered a 45-minute period.

Geometry • Module 3

# Extending to Three Dimensions

## OVERVIEW

Module 3, Extending to Three Dimensions, builds on students' understanding of congruence in Module 1 and similarity in Module 2 to prove volume formulas for solids.

Topic A studies informal limit arguments to find the area of a rectangle with an irrational side length and of a disk (**G-GMD.A.1**). It also focuses on properties of area that arise from unions, intersections, and scaling. These topics prepare for understanding limit arguments for volumes of solids.

Topic B is introduced by a lesson where students experimentally discover properties of three-dimensional space that are necessary to describe three-dimensional solids such as cylinders and prisms, cones and pyramids, and spheres. Cross-sections of these solids are studied and are classified as similar or congruent (**G-GMD.B.4**). A dissection is used to show the volume formula for a right triangular prism after which limit arguments give the volume formula for a general right cylinder (**G-GMD.A.1**).

In Lesson 10, two-dimensional cross-sections of solids are used to approximate solids by general right cylindrical slices and leads to an understanding of Cavalieri's principle (**G-GMD.A.1**). Congruent cross-sections for a general (skew) cylinder and Cavalieri's principle lead to the volume formula for a general cylinder.

To find the volume formula of a pyramid, a cube is dissected into six congruent pyramids to find the volume of each one. Scaling the given pyramids, according to a scaling principle analogous to the one introduced in Topic A, gives the volume formula for a right rectangular pyramid. The cone cross-section theorem and Cavalieri's principle are then used to find the volume formula of a general cone (**G-GMD.A.1, G-GMD.A.3**).

Cavalieri's principle is used to show that the volume of a right circular cylinder with radius $R$ and height $R$ is the sum of the volume of hemisphere of radius $R$ and the volume of a right circular cone with radius $R$ and height $R$. This information leads to the volume formula of a sphere (**G-GMD.A.2, G-GMD.A.3**).

## Focus Standards

### Explain volume formulas and use them to solve problems.[2]

**G-GMD.A.1** Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. *Use dissection arguments, Cavalieri's principle, and informal limit arguments.*

**G-GMD.A.3** Use volume formulas for cylinders, pyramids, cones and spheres to solve problems.★

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[2] The (+) standard on the volume of the sphere is an extension of G-GMD.A.1. It is explained by the teacher in this grade and used by students in G-GMD.A.3. Note: Students are not assessed on proving the volume of a sphere formula until Precalculus.

### Visualize relationships between two-dimensional and three-dimensional objects.

**G-GMD.B.4** Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.

### Apply geometric concepts in modeling situations.

**G-MG.A.1** Use geometric shapes, their measures, and their properties to describe objects (e.g. modeling a tree trunk or a human torso as a cylinder).★

**G-MG.A.2** Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot).★

**G-MG.A.3** Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios).★

## Extension Standards

### Explain volume formulas and use them to solve problems.

**G-GMD.A.2** (+) Give an informal argument using Cavalieri's principle for the formulas for the volume of a sphere and other solid figures.

## Foundational Standards

### Draw, construct, and describe geometrical figures and describe the relationships between them.

**7.G.A.3** Describe the two-dimensional figures that result from slicing three-dimensional figures, as in plane sections of right rectangular prisms and right rectangular pyramids.

### Solve real-life and mathematical problems involving angle measure, area, surface area, and volume.

**7.G.B.4** Know the formulas for the area and circumference of a circle and use them to solve problems; give an informal derivation of the relationship between the circumference and the area of a circle.

### Understand and apply the Pythagorean Theorem.

**8.G.B.7** Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions.

**Solve real-life and mathematical problems involving volume of cylinders, cones, and spheres.**

**8.G.C.9** Know the formulas for the volumes of cones, cylinders, and spheres and use them to solve real-world and mathematical problems.

# Focus Standards for Mathematical Practice

**MP.6** **Attend to precision**. Students will formalize definitions, using explicit language to define terms such as *right rectangular prism* that have been informal and more descriptive in earlier grade levels.

**MP.7** **Look for and make use of structure**. The theme of approximation in Module 3 is an interpretation of structure. Students approximate both area and volume (curved two-dimensional shapes and cylinders and cones with curved bases) polyhedral regions. They must understand how and why it is possible to create upper and lower approximations of a figure's area or volume. The derivation of the volume formulas for cylinders, cones, and spheres, and the use of Cavalieri's principle is also based entirely on understanding the structure and sub-structures of these figures.

# Terminology

## New or Recently Introduced Terms

- **Cavalieri's Principle** (Given two solids that are included between two parallel planes, if every plane parallel to the two planes intersects both solids in cross-sections of equal area, then the volumes of the two solids are equal.)
- **Cone** (Let $B$ be a region in a plane $E$, and $V$ be a point not in $E$. The *cone with base $B$ and vertex $V$* is the union of all segments $\overline{VP}$ for all points $P$ in $B$. If the base is a polygonal region, then the cone is usually called a *pyramid*.)
- **General Cylinder** (Let $E$ and $E'$ be two parallel planes, let $B$ be a region in the plane $E$, and let $L$ be a line which intersects $E$ and $E'$ but not $B$. At each point $P$ of $B$, consider the segment $\overline{PP'}$ parallel to $L$, joining $P$ to a point $P'$ of the plane $E'$. The union of all these segments is called a *cylinder with base $B$*.)
- **Inscribed Polygon** (A polygon is *inscribed in* a circle if all of the vertices of the polygon lie on the circle.)
- **Intersection** (The *intersection* of $A$ and $B$ is the set of all objects that are elements of $A$ and also elements of $B$. The intersection is denoted $A \cap B$.)
- **Rectangular Pyramid** (Given a rectangular region $B$ in a plane $E$, and a point $V$ not in $E$, the *rectangular pyramid with base $B$ and vertex $V$* is the union of all segments $\overline{VP}$ for points $P$ in $B$.)
- **Right Rectangular Prism** (Let $E$ and $E'$ be two parallel planes. Let $B$ be a rectangular region in the plane $E$. At each point $P$ of $B$, consider the segment $\overline{PP'}$ perpendicular to $E$, joining $P$ to a point $P'$ of the plane $E'$. The union of all these segments is called a *right rectangular prism*.)

- **Solid Sphere or Ball** (Given a point $C$ in the three-dimensional space and a number $r > 0$, the *solid sphere (or ball) with center $C$ and radius $r$* is the set of all points in space whose distance from point $C$ is less than or equal to $r$.)
- **Sphere** (Given a point $C$ in the three-dimensional space and a number $r > 0$, the *sphere with center $C$ and radius $r$* is the set of all points in space that are distance $r$ from the point $C$.)
- **Subset** (A set $A$ is a *subset* of a set $B$ if every element of $A$ is also an element of $B$.)
- **Tangent to a Circle** (A *tangent line to a circle* is a line that intersects a circle in one and only one point.)
- **Union** (The *union* of $A$ and $B$ is the set of all objects that are either elements of $A$ or of $B$ or of both. The union is denoted $A \cup B$.)

### Familiar Terms and Symbols[3]

- Disk
- Lateral Edge and Face of a Prism

## Suggested Tools and Representations

- Three-dimensional models of rectangular prisms, right circular cylinders, right pyramids
- Deck of cards
- Stack of coins
- Images of "sliced" figures, suchas a loaf of bread or a stack of deli cheese

## Assessment Summary

| Assessment Type | Administered | Format | Standards Addressed |
|---|---|---|---|
| Assessment Task | After Topic B | Constructed response with rubric | G-GMD.A.1, G-GMD.A.3, G-GMD.B.4, G-MG.A.1, G-MG.A.2, G-MG.A.3 |

[3] These are terms and symbols students have seen previously.

# Topic A:

# Area

**G-GMD.A.1**

| | | |
|---|---|---|
| **Focus Standard:** | G-GMD.A.1 | Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. *Use dissection arguments, Cavalieri's principle, and informal limit arguments.* |
| **Instructional Days:** | 4 | |
| **Lesson 1:** | What Is Area? (E)[1] | |
| **Lesson 2:** | Properties of Area (E) | |
| **Lesson 3:** | The Scaling Principle for Area (E) | |
| **Lesson 4:** | Proving the Area of a Disk (S) | |

The topic of area must be revisited in order to have a conversation about figures in three dimensions; we first have the necessary discussion around area. Area is introduced in Grade 3, but only figures that are easy to "fill" with units are considered. In Grade 5, the need to use parts of unit squares for some figures is understood, and is applied in Grade 6. In Grade 7, students realize that a figure may have an area even if, like the disk, it cannot be decomposed into a finite number of unit squares, but (appropriately) this is treated at the intuitive level. As the grades progress, the link between area as a measurable geometric quantity and area represented by numbers for calculations is elaborated. The culmination is a universal method for measuring areas, even when they are not finite unions of unit squares or simple parts of unit squares. Mathematicians refer to this as Jordan measure. While this may seem like an intimidating idea to introduce at this level, it can be thought of simply as the well-known properties of area that students are already familiar with. This is not only an intuitively acceptable approach to area, but it is completely rigorous at the university level. The concept forms an important bridge to calculus, as Jordan measure is the idea employed in defining the Riemann integral.

In this topic, Lesson 1 shows how finding the area of a curved figure can be approximated by rectangles and triangles. By refining the size of the rectangles and triangles, the approximation of the area becomes closer to the actual area. Students experience a similar process of approximation in Grade 8 (Module 7, Lesson 14) in order to estimate $\pi$. The informal limit argument prepares students for the development of volume

[1] Lesson Structure Key: **P**-Problem Set Lesson, **M**-Modeling Cycle Lesson, **E**-Exploration Lesson, **S-**Socratic Lesson

formulas for cylinders and cones and foreshadows ideas that students will formally explore in calculus. This process of approximation is important to developing the volume formula of cylinders and cones. In Lesson 2, students study the basic properties of area using set notation; in Topic B they will see how the properties are analogous to those of volume. In Lesson 3, students study the scaling principle, which states that if a planar region is scaled by factors $a$ and $b$ in two perpendicular directions, then its area is multiplied by a factor of $a \times b$. Again, we study this in two dimensions to set the stage for three dimensions when we scale solids. Finally, in Lesson 4, students develop the formula for the area of a disk, and just as in Lesson 1, incorporate an approximation process. Students approximate the area of the disk, or circle, by inscribing a polygon within the circle, and consider how the area of the polygonal region changes as the number of sides increases and the polygon looks more and more like the disk it is inscribed within.

# Lesson 1: What Is Area?

## Student Outcomes

- Students review the area formula for rectangles with rational side lengths and prove the area formula for an arbitrary rectangle.
- Students use a square grid to estimate the area of a curved region using lower approximations, upper approximations, and average approximations.

## Lesson Notes

In Grade 8, Module 7, Lesson 11, we approximated the value of an irrational number, and in Lesson 14 of the same module, we approximated the decimal expansion of $\pi$; both lessons used the idea of upper and lower approximations. We use the strategy of upper and lower approximations in this lesson. In fact, we see this theme repeatedly throughout the module; for the area of a region we do not normally have a method to compute, we can approximate the area by using areas of regions larger and smaller than the target region that we *do* know how to compute. These larger and smaller areas act as upper and lower approximations, respectively. This work provides the foundation for using informal limits to describe the area of a circle (**G-GMD.A.1**) and foreshadows what students will see in integral calculus.

Exploratory Challenge 1 reminds students of what area is and why we can use the formula $\text{length} \times \text{width}$ to compute the area of a rectangle. Students use what they know about determining the area of a rectangle with whole number and rational side lengths to justify why the $\text{length} \times \text{width}$ formula holds when the side lengths are irrational. The Discussion extends what students have learned about area in Grade 3, Module 4 regarding multiplication and area in general as well as in Grade 5, Module 5 where students use area models to multiply fractions and interpret the work as multiplying side lengths of a rectangle to find area.

## Classwork

### Exploratory Challenge 1 (4 minutes)

Use the Exploratory Challenge to get students thinking about how and why they calculate the area of a rectangular region.

**Exploratory Challenge 1**

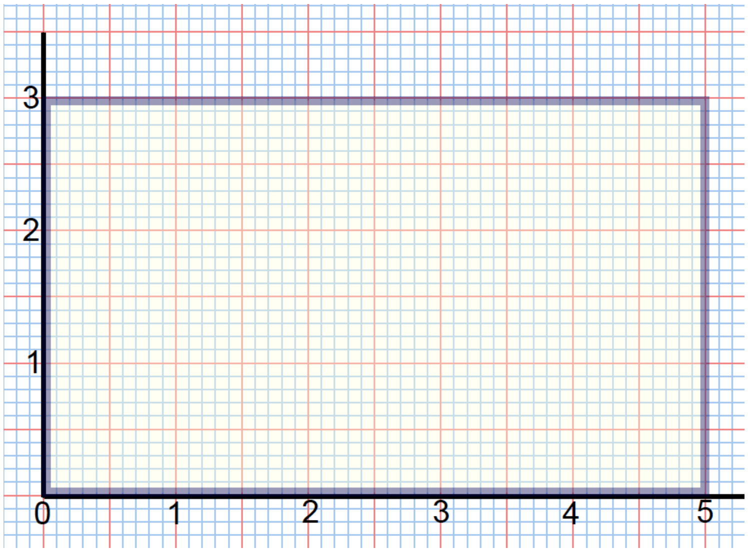
**a. What is area?**

***Responses will vary. Expect a variety of descriptions of area. For example, "it is a space, like a picnic area" or "it is $\text{length} \times \text{width}$." Take a few responses and move on, as this question is more precisely answered in the Discussion following Exploratory Challenge 1.***

Consider prompting students further with the following questions:

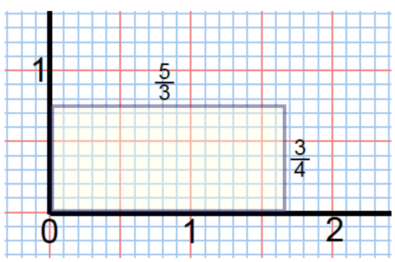
- Give an example of when you would need to know area.
- Use the word area in a sentence.
- What are the units associated with area?

b. **What is the area of the rectangle below whose side lengths measure $3$ units by $5$ units?**



*The area is $15$ square units.*

c. **What is the area of the $\frac{3}{4} \times \frac{5}{3}$ rectangle below?**



*The area is $\frac{15}{12} = \frac{5}{4}$ square units.*

Once students have answered and shared out responses for parts (b) and (c), pose the question below as a starting point to the Discussion. Allow them to express ideas, but confirm the concept in the Discussion.

- Why can we use the formula $\text{length} \times \text{width}$ to calculate area of a rectangular region with whole number and rational side lengths?

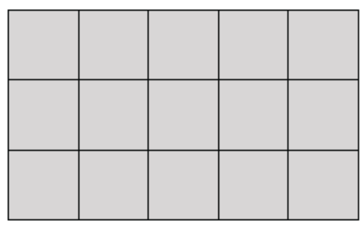
It should be noted that when we discuss area of a region, the area is a quantity that measures the region. It is common practice to accept the phrase *area of a rectangle* to be synonymous with area of a rectangular region, but students should be aware of the distinction.

Students begin their study of area with whole numbers in Grade 3 and progress to the use of rational side lengths in Grade 5. Here in this lesson, they will see why the area formula for a rectangle with *irrational* side lengths is the same as that of the formula for rectangles with whole number and rational side lengths.

### Discussion (5 minutes)

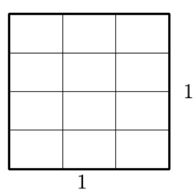
The purpose of the following discussion is to prepare students for working with rectangles with irrational side lengths and ultimately finding the area under a curve.

- Area is a way of associating to any region a quantity that reflects our intuitive sense of how big the region is, without reference to the shape of the region.
- Recall that we measure area by fixing a unit square and defining the area of a region as the number of unit squares (and partial unit squares) needed to fill or tile the region.
- Whether we can see the grid lines of such unit squares, we know that if we were to place a rectangular region on a grid of unit squares, we could count the number of unit squares, or use multiplication since it is simply an expedited means of counting (repeated addition). Therefore, we know the area of a rectangle can be determined by the formula $\text{length} \times \text{width}$.
- Thus, the area of the $3 \times 5$ rectangle in part (b), filled with three rows of $5$ unit squares each, is $15$.

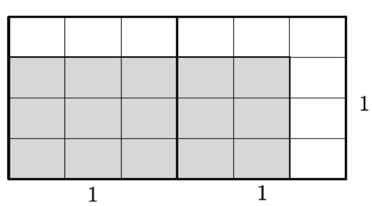


Area $= 3 \times 5 = 15$

- The formula $\text{length} \times \text{width}$ can be extended to rectangles with rational side lengths. For example, consider a rectangle that is $\frac{3}{4} \times \frac{5}{3}$. The area of this rectangle is the same as $(3 \times 5)$ copies of smaller rectangles, each of which has area $\frac{1}{4\times3}$ square units. To find the area of the rectangle we can divide one unit square into $4 \times 3$, or $12$ smaller rectangles:




- What is the area of each smaller rectangular region?
  - $\frac{1}{12}$ *square units.*
- Now we can define a rectangle that is $\frac{3}{4} \times \frac{5}{3}$ or $3 \times 5$ twelfths:




- Notice that the calculation process that yields an area $3 \times 5$ twelfths, or $\frac{15}{12}$, is relative to $1$ square unit.

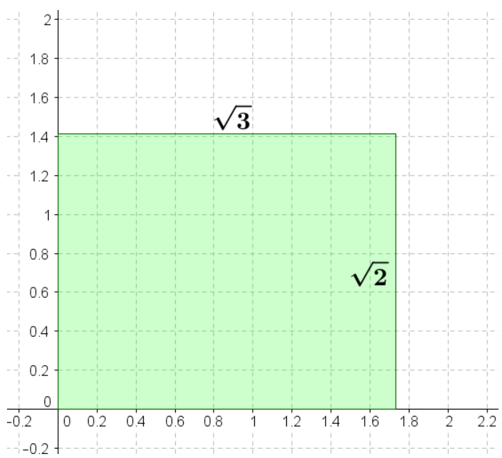
For more on how this content has been addressed in earlier grade levels, please review Grade 5, Module 4 (the content is spread over several topics) and Topic C of Grade 5, Module 4. Module 4 develops the rules of fraction multiplication using area models but does not explicitly connect the arithmetic with finding the area of the figure in square units. The content in Module 5 makes the area connection explicit.

### Exploratory Challenge 2 (5 minutes)

In Exploratory Challenge 2, students are prompted to consider irrational side lengths of a rectangular region and how that impacts the calculation of its area. As students approximate the area in part (a), prompt them to justify their answer by counting how many of the small squares are in 1 square unit and predicting how many small squares should then be in $2.3$ square units. Have them verify this by counting squares in the rectangle.

**Exploratory Challenge 2**

a. **What is the area of the rectangle below whose side lengths measure $\sqrt{3}$ units by $\sqrt{2}$ units? Use the unit squares on the graph to guide your approximation. Explain how you determined your answer.**




*Scaffolding:*

- Use scaffolded questions with the whole class or a small group.
- Count the boxes to come up with an approximation that you know is too big. Count the boxes to come up with an approximation that you know is too small. Use your predictions to check if your final answer makes sense.

***Answers will vary; some students may respond with a decimal approximation, such as approximately $2.3$ square units. Others might respond with the answer $\sqrt{6}$ square units; ask students how they know in this case.***

b. **Is your answer precise?**

***Students who offered decimal answers should recognize that their results are not precise because their side length estimations are just that: estimations of irrational values. A precise answer would be $\sqrt{6}$ square units, but again, press students for a justification for this answer.***

### Discussion (15 minutes)

We use Exploratory Challenge 2 to continue the discussion on area but now with rectangles with irrational side lengths. The overarching goal is to show that even with irrational side lengths the area of a rectangle can be calculated using the length $\times$ width formula.

There are two additional sub-goals: (1) The Discussion is an opportunity to understand the process of multiplying two irrational numbers in the context of approximating the area of a rectangle with two irrational side lengths. Teachers should note to their students that this process works in general and is one way to visualize the meaning of products like $\sqrt{2} \times \pi$ or $\sqrt{2} \times \sqrt{5}$. (2) The refinement of the underestimates and overestimates of area we derive in this example will be a theme that we will keep coming back to throughout this entire module.

- Do you think we can give the same explanation for a rectangle with irrational side lengths like the rectangle in Exploratory Challenge 1, part (c)?
  - *Student responses will vary. Students should reason that we cannot use the same explanation because we typically approximate the value of irrational numbers using finite decimals in order to compute with them. Therefore, the area of any rectangle with irrational side lengths, given in decimal or fractional form, can only be approximated.*
- Now, we will use a strategy that allows us to approximate the area of a rectangle with irrational side lengths more accurately. We start by trying to show that the area of a rectangle with irrational side lengths can be found using the formula $\text{length} \times \text{width}$.

MP.1

Allow students the opportunity to wrestle with the following question:

- Consider the rectangle that is $\sqrt{2} \times \sqrt{3}$. How can we approximate its area, $A$?
  - *Have students share their responses to Exploratory Challenge 2, part (a).*
- Since $\sqrt{2} = 1.41421356\ldots$ and $\sqrt{3} = 1.73205080\ldots$ then the area, $A$, of the rectangle (shown below in green) is greater than the area of a rectangle that is $1 \times 1$ (shown below in blue) but less than the area of a rectangle that is $2 \times 2$ (shown below in red). This is our first, very rough, upper and lower approximations of the area, $A$.

*Scaffolding:*

- For grade-level students, ask them to find fraction representations of $1.8$ and $1.5$. The idea is to remind students that rational values can be written as both fractions (as in the Exploratory Challenges) and as decimals.
- Provide struggling learners with the fractions $\frac{9}{5}$ and $\frac{3}{2}$, and ask them how they are related to the upper approximation of the area, $A$, in this step.

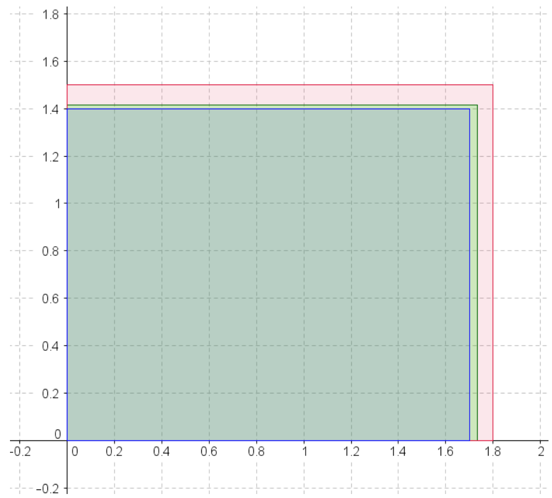
**Discussion**

**Use Figures 1, 2, and 3 to find upper and lower approximations of the given rectangle.**

**Figure 1**

- We can make a more precise estimate of the area, $A$, by looking at side lengths that are more precise. Since we know how to find the area of rectangles with rational side lengths, let us pick more precise side lengths that are closer in value to $\sqrt{3} \times \sqrt{2}$; we will select arbitrary, but appropriate, rational values. Let us create an upper approximation of the area, $A$, by using a rectangle with dimensions $1.8 \times 1.5$ (shown in red) and a lower approximation by using a rectangle with dimensions $1.7 \times 1.4$ (shown in blue).

**Figure 2**

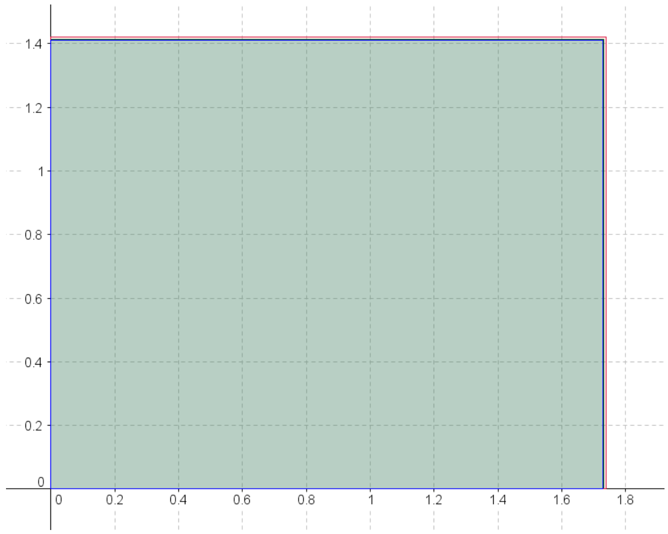



- Can we make a more precise approximation? Describe dimensions of a possible lower approximation for $A$ that is more precise than $1.7 \times 1.4$.

Allow students to share out possible values. Remind them of the values of the dimensions they are targeting and that the goal is to use approximate dimensions that are closer to the true values: $\sqrt{3} = 1.73205080\ldots$ by $\sqrt{2} = 1.41421356$. Guide students toward the idea that the way to refine the dimensions is to rely on the digits directly from the decimal values of $\sqrt{3}$ and $\sqrt{2}$. For the lower approximation dimensions, the exact digits are selected but truncated from the expansion. With the upper approximation dimensions, for each successive dimension, and each additional place value, one digit greater than the digit in place of the actual expansion is taken.

- The area, $A$, of the rectangle is greater than the area of a rectangle that is $1.73 \times 1.41$ (shown in blue) but less than the area of a rectangle that is $1.74 \times 1.42$ (shown in red).

Figure 3




- We can continue to refine the approximations. The area, $A$ (shown in green), of the rectangle is greater than the area of a rectangle that is $1.732 \times 1.414$ (shown below in blue) but less than the area of a rectangle that is $1.733 \times 1.415$ (shown below in red). Notice now that, even as we zoom in, it is difficult to distinguish between the red, black, and blue rectangles because they are so close in size.

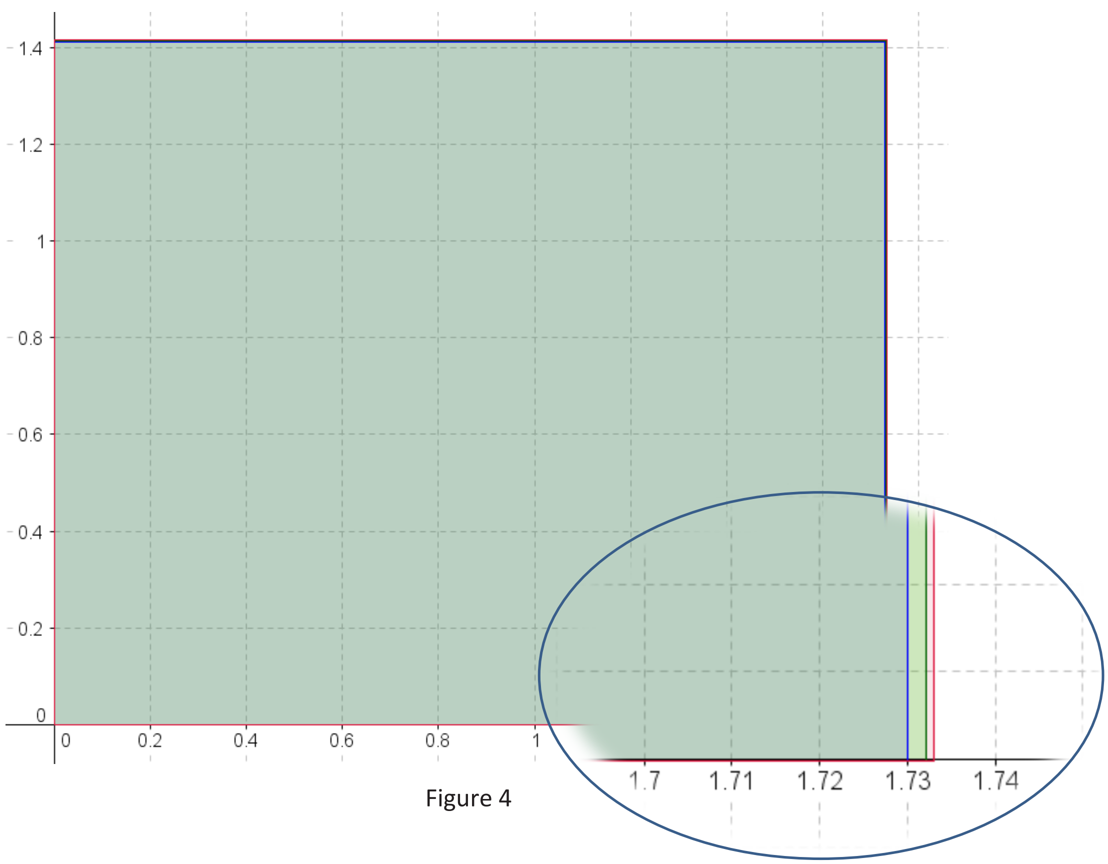



Figure 4

- The following tables summarize our observations so far and include a few additional values. The following table shows lower approximations first (blue rectangles).

| Lower Approximations | | |
|---|---|---|
| Less than $\sqrt{2}$ | Less than $\sqrt{3}$ | Less than or equal to $A$ |
| $1$ | $1$ | $1 \times 1 = \quad 1$ |
| $1.4$ | $1.7$ | $1.4 \times 1.7 = \quad 2.38$ |
| $1.41$ | $1.73$ | $1.41 \times 1.73 = \quad 2.4393$ |
| $1.414$ | $1.732$ | $1.414 \times 1.732 = \quad 2.449048$ |
| $1.4142$ | $1.7320$ | $1.4142 \times 1.7320 = \quad 2.449344$ |
| $1.41421$ | $1.73205$ | $1.41421 \times 1.73205 = \quad 2.4494824305$ |
| $1.414213$ | $1.732050$ | $1.414213 \times 1.732050 = \quad 2.44948762665$ |

Provide time for students to make sense of the data in the table. Ask the following questions to ensure students understand the data table:

- What do you notice happening in the first column as you go down the rows?
  - *We are using approximations of $\sqrt{2}$ that are less than the actual value but becoming closer and closer to the actual value.*
- What do you notice happening in the second column as you go down the rows?
  - *We are using approximations of $\sqrt{3}$ that are less than the actual value but becoming closer and closer to the actual value.*
- What impact do these approximations have on the area, $A$, as you go down the rows in the third column?
  - *The more precise of an estimate we use for $\sqrt{2}$ and $\sqrt{3}$, the closer we get to the precise area of the rectangle.*
- What do you notice about decimal expansions of the product?
  - *More and more of the digits in the expansion of the product are the same.*
- Why are all of these calculations less than $A$? When will it be equal to $A$?
  - *Students may not have precise answers to these questions. They are prompts to help shape their number sense.*
- Find the decimal value of $\sqrt{6}$. How do the decimal expansions of the product relate to $\sqrt{6}$?
  - *They are getting closer to the decimal expansion of $\sqrt{6}$.*
- What does $\sqrt{6}$ have to do with $\sqrt{2}$ *and* $\sqrt{3}$?
  - *$\sqrt{6}$ is the product of $\sqrt{2}$ and $\sqrt{3}$.*
- In the table above, we underestimate the side lengths so the area estimate is less than the actual area of the rectangle. Similarly, in the table below, we overestimate the side lengths (red rectangles) so our area estimate is greater than the actual area. Again, a few additional values beyond those we discussed have been added.

| Upper Approximations | | |
|---|---|---|
| Greater than $\sqrt{2}$ | Greater than $\sqrt{3}$ | Greater than or equal to $A$ |
| $2$ | $2$ | $2 \times 2 = 4$ |
| $1.5$ | $1.8$ | $1.5 \times 1.8 = 2.7$ |
| $1.42$ | $1.74$ | $1.42 \times 1.74 = 2.4708$ |
| $1.415$ | $1.733$ | $1.415 \times 1.733 = 2.452195$ |
| $1.4143$ | $1.7321$ | $1.4143 \times 1.7321 = 2.44970903$ |
| $1.41422$ | $1.73206$ | $1.41422 \times 1.73206 = 2.4495138932$ |
| $1.414214$ | $1.732051$ | $1.414214 \times 1.732051 = 2.449490772914$ |

- Does the relationship with $\sqrt{6}$ still seem to hold relative to the upper approximation of the area as it did with the lower approximation?
  - *Yes, each successive upper approximation of $A$ is closer and closer to $\sqrt{6}$.*
- Putting these two tables together we can get two estimates for the area $A$: one that is less than $A$ and one that is greater than $A$.
- Recall that if we know that the true value of the area is a number between $2.38$ and $2.7$, then we know that the absolute value of the error is less than $2.7 - 2.38$ and that $A > 2.38$. Then the upper bound on the error using the numbers $2.38$ and $2.7$ as an approximation $X$ for $A$ is found by $\frac{|X-A|}{A} < \frac{0.32}{2.38} \approx 0.134453$, and the percent error is approximately $13.45\%$.
- If we continue the lower and upper estimates for the area $A$, we would find that there is only one number that is greater than or equal to all of the lower estimates and less than or equal to all of the upper estimates: $\sqrt{6}$.
- Thus, we say that the area of the rectangle is $\sqrt{6}$ because this number is the only number that is always between the lower and upper estimates using rectangles with rational side lengths. Using this reasoning, we can say that the area of any rectangle is given by $\text{length} \times \text{width}$.

### Discussion (10 minutes)

In the last example, we approximated the area of a rectangle with irrational length sides to get better and better approximations of its area. We extend this idea in the next example to show that the same procedure can also be done with regions that are bounded by curves. Present the question to students, and then allow them time to develop strategies in partners or small groups.

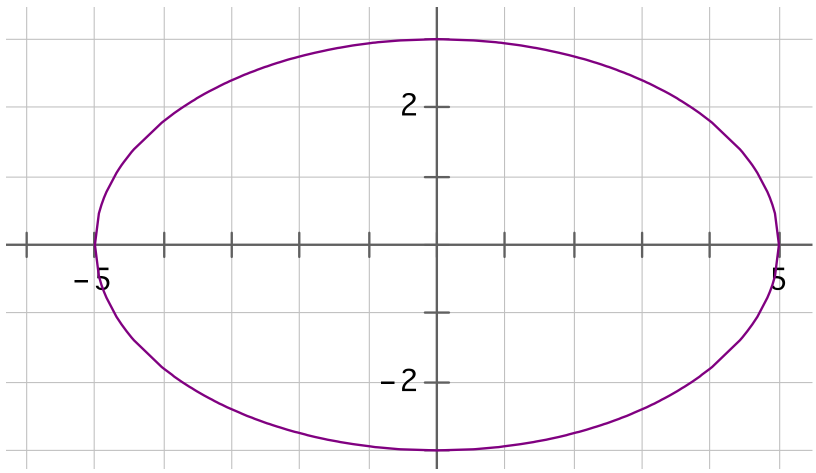
*Scaffolding:*

- Teachers may choose to use a circle instead of an ellipse for this Discussion.

MP.1 & MP.3

**Discussion**

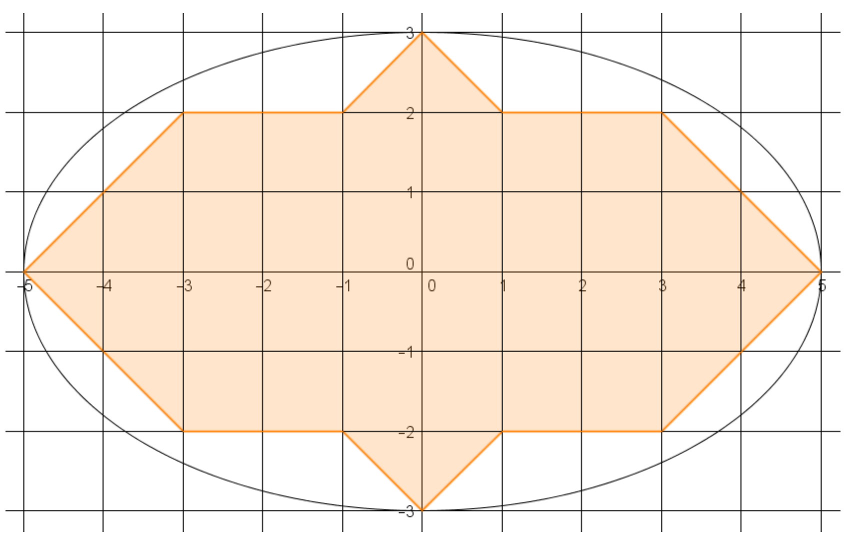
**If it takes one can of paint to cover a unit square in the coordinate plane, how many cans of paint are needed to paint the region within the curved figure?**




Provide time for students to discuss the problem with partners or in small groups. Then have students share their strategies with the whole class. If necessary, the bullet points below can be used to guide students' thinking.

Strategies developed by students will vary. Encourage those strategies that approximate the area of the curved region by making approximations first with whole squares, then partial squares, and finally comparing the area of regions outside the curve and inside the curve, as close to the actual area as possible.

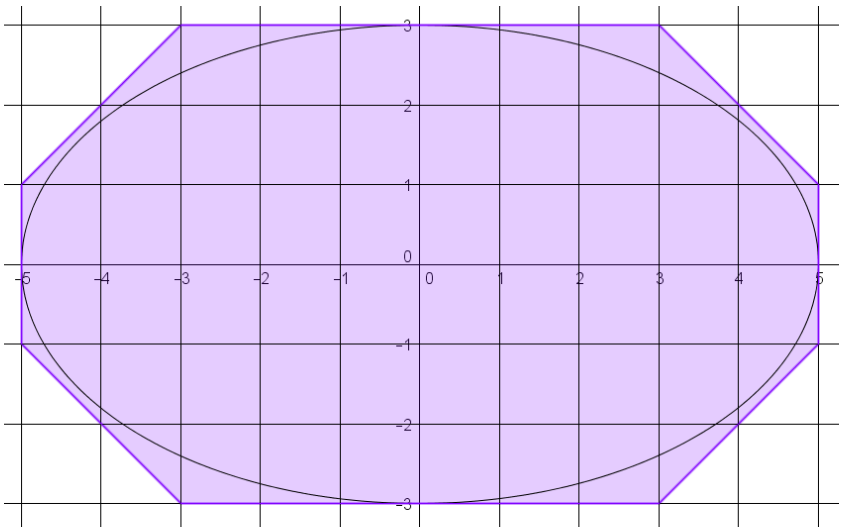
- First, find how many squares and half squares (since we know how to find the area of a half square region) fit inside the region.



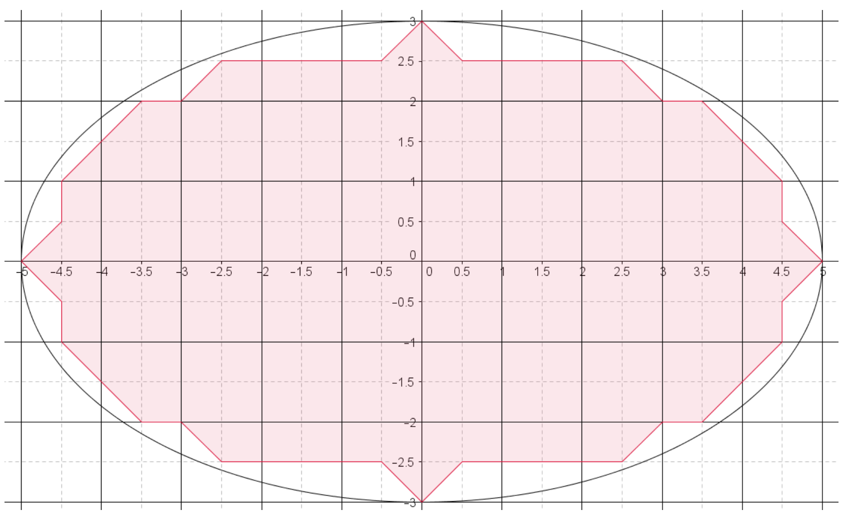

  - *This strategy gives us the answer that it will take more than* $34$ *cans of paint.*

- Now consider the area of the region just outside the curve. How many cans of paint will it take if we count the squares and half squares?

Point out to students that the shaded area is the minimum area needed to fill the ellipse, comprised of whole and half squares, completely.



  - *This strategy gives us the answer that it will take less than* $52$ *cans of paint.*
- An estimate of the area is the average of $34$ and $52$, or $43$ cans of paint. How can we improve the accuracy of our estimate?
  - *If we used smaller square units, we could make a more accurate estimate of the area.*
- If we divide each unit square into four smaller squares and count all of the squares and half squares, how many cans of paint will it take to cover the region inside the curve?

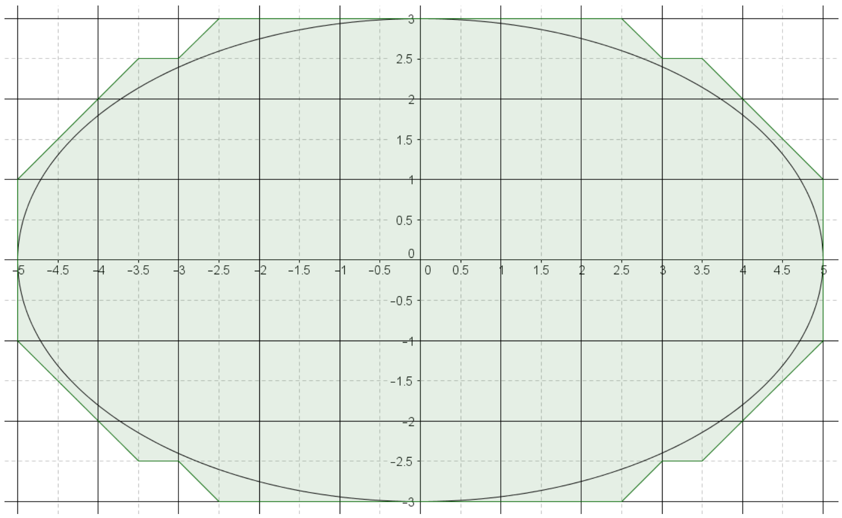


*Scaffolding:*

- For struggling learners, teachers may choose to tile the figure with manipulative units so that students can easily rearrange them into whole unit squares.
- Ask students who may be above grade level to make and justify an estimate of the area independently.

  - *There are* $164$ *squares and half squares in the region inside the curve. Since it takes* $1$ *can of paint to cover every* $4$ *of these squares, then it will take more than* $41$ *cans of paint to cover this region.*

- Now consider the area of the region just outside the curve. How many cans of paint will it take if we count the squares and half squares?



> *Scaffolding:*
> - If time is an issue, arrange for some groups to work on the over approximation using $11$ squares, while some work on the under approximation; do the same for the $0.25 \times 0.25$ squares.

   - *There are $204$ squares and half squares covered, so it will take less than $51$ cans of paint to cover the region inside the curve.*

MP.3

- What is a good estimate for the area of the region? Explain how you know.
   - *An estimate of the area is the average of $41$ and $51$, or about $46$ cans of paint.*
- There are ways to determine the precise area of the curved region. We have not learned them yet so I will just tell you that the precise area of the region is $15\pi \approx 47.12$. If the area is known to be $15\pi$, which estimate is better? How do you know?
- This is the unique number that is greater than every lower approximation and less than every upper approximation. The first estimate of $43$ cans of paint had an error of about $8.75\%$. The second estimate, $46$, had an error of about $2.4\%$. We could continue to improve the accuracy of our estimate by using even smaller squares in the grid.

### Closing (1 minute)

Ask students the following questions. You may choose to have students respond in writing, to a partner, or to the whole class.

- Explain a method for approximating the interior region bounded by a curve.
   - *We can make a rough estimate by underestimating the region with a polygonal region that lies within the region and by overestimating the region with a polygonal region that contains the region.*
- How can we improve the accuracy of our estimation of area?
   - *The accuracy of our estimate can be improved by approximating the region with better and better polygonal regions (those which we have the means to calculate the area of). For example, if using a grid, we can look at smaller and smaller squares. If using the decimal approximation of an irrational number, we can improve the accuracy of the estimate of area by using values with more decimal digits.*

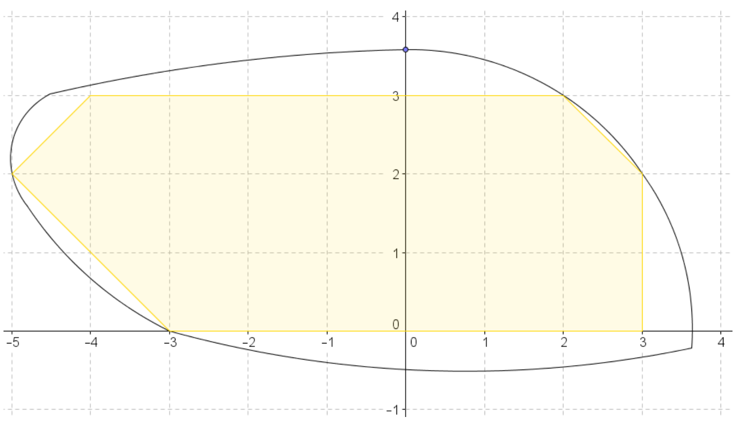
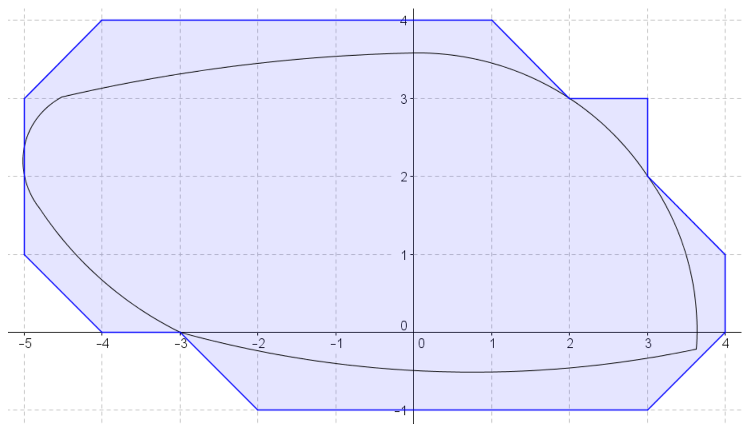
### Exit Ticket (5 minutes)

Name ______________________________ Date ________________

# Lesson 1: What Is Area?

## Exit Ticket

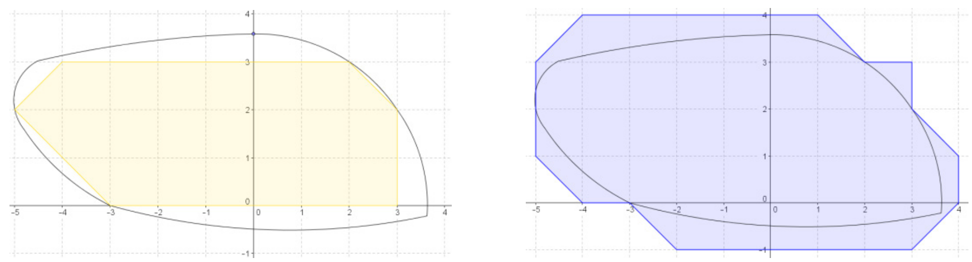
1. Explain how to use the shaded polygonal regions shown to estimate the area $A$ inside the curve.

2. Use Problem 1 to find an average estimate for the area inside the curve.

## Exit Ticket Sample Solutions

1. Explain how to use the shaded polygonal regions shown to estimate the area $A$ inside the curve.



*I can use whole and half squares to determine a lower estimate of* $21$ *square units and an upper estimate of* $37$ *square units.*

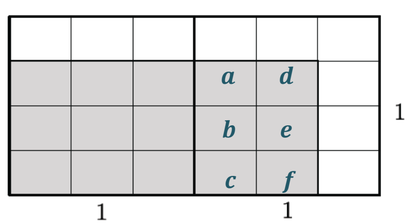
2. Use Problem 1 to find an average estimate for the area inside the curve.

*Solution:* $\frac{1}{2}(21+37)=29$

*An average estimate for the area inside the curve is* $29$ *square units.*

## Problem Set Sample Solutions

1. Use the following picture to explain why $\frac{15}{12}$ is the same as $1\frac{1}{4}$.




*Rectangles* $a$, $b$, *and* $c$ *can be used to fill in the first unit square. That leaves rectangles* $d$, $e$, *and* $f$, *which make up one quarter of a unit square. Thus,* $\frac{15}{12}$ *is the same as* $1\frac{1}{4}$.

2. **Figures 1 and 2 below show two polygonal regions used to approximate the area of the region inside an ellipse and above the $x$-axis.**

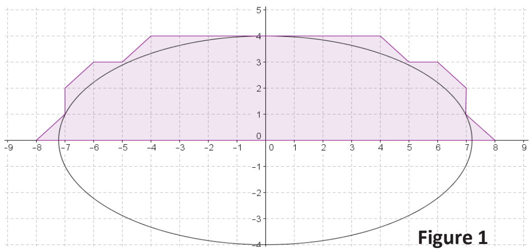


**Figure 1**

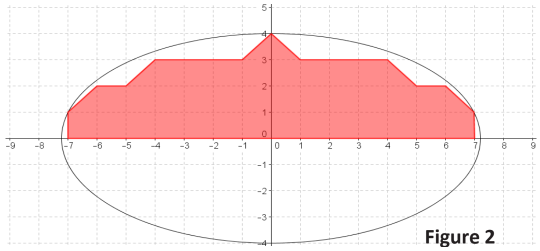


**Figure 2**

a. **Which polygonal region has a greater area? Explain your reasoning.**

*The area of the polygon in Figure 1 has a greater area because it includes all of the area inside the ellipse (above the $x$-axis) plus some area outside of the ellipse. The polygon in Figure 2 does not even fill the region inside the ellipse (above the $x$-axis).*

b. **Which, if either, of the polygonal regions do you believe is closer in area to the region inside the ellipse and above the $x$-axis?**

*Answers will vary.*

3. **Figures 1 and 2 below show two polygonal regions used to approximate the area of the region inside a parabola and above the $x$-axis.**

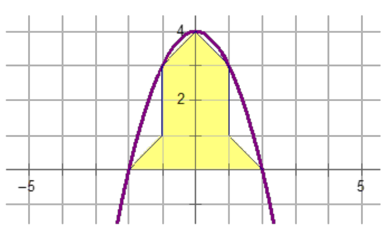


Figure 1

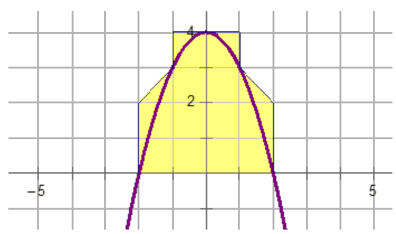


Figure 2

a. **Use the shaded polygonal region in Figure 1 to give a lower estimate of the area $a$ under the curve and above the $x$-axis.**

*Lower estimate:* $8$ *square units.*

b. **Use the shaded polygonal region to give an upper estimate of the area $a$ under the curve and above the $x$-axis.**

*Upper estimate:* $13$ *square units.*

c. **Use (a) and (b) to give an average estimate of the area $a$.**

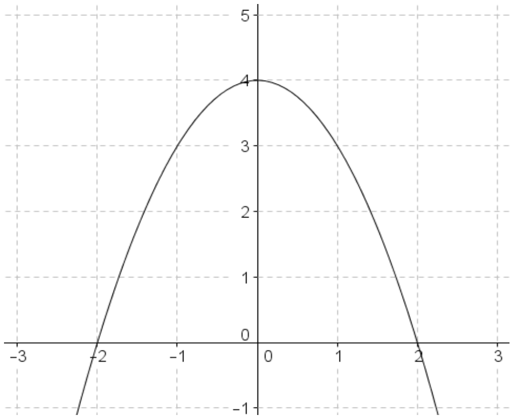
*Average estimate* $= \frac{1}{2}$(*lower estimate* $+$ *upper estimate*)

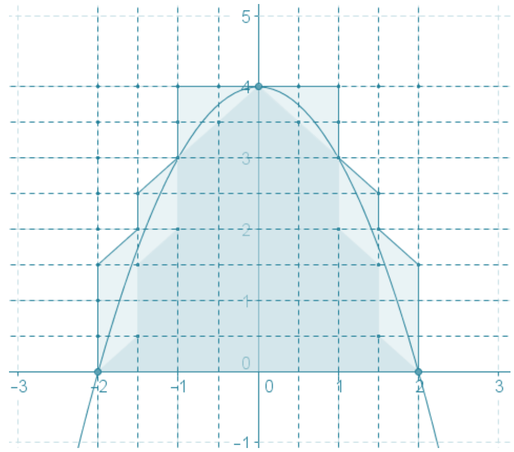
*Average estimate* $= \frac{1}{2}(8 + 13)$

*Average estimate* $= 10.5$

*An average estimate of the area under the given curve above the $x$-axis is* $10.5$ *square units.*

4. Problem 4 is an extension of Problem 3. Using the diagram, draw grid lines to represent each $\frac{1}{2}$ unit.





a. What do the new grid lines divide each unit square into?

*Each unit square is divided into $4$ quarter-unit squares.*

b. Use the squares described in part (a) to determine a lower estimate of area $a$ in the diagram.

*Lower estimate: $36$ quarter-square units or $9$ square units.*

c. Use the squares described in part (a) to determine an upper estimate of area $a$ in the diagram.

*Upper estimate: $50$ quarter-square units or $12.5$ square units.*

d. Calculate an average estimate of the area under the curve and above the $x$-axis based on your upper and lower estimates in parts (b) and (c).

*Average estimate* $= \frac{1}{2}$(*lower estimate* $+$ *upper estimate*)

*Average estimate* $= \frac{1}{2}(9 + 12.5)$

*Average estimate* $= 10.75$

*An average estimate of the area under the curve and above the $x$-axis is $10.75$ square units.*

e. Do you think your average estimate in Problem 4 is more or less precise than your estimate from Problem 3? Explain.

*Student answers will vary. Possible student answer: The areas of the lower and upper estimates in Problem 2 are closer to one another than the estimates found in Problem 1, so I think that the average estimate for the area under the curve in Problem 2 is more precise than the average estimate in Problem 1.*

# Lesson 2: Properties of Area

## Student Outcomes

- Students understand properties of area:
  1. Students understand that the area of a set in the plane is a number greater than or equal to zero that measures the size of the set and not the shape.
  2. The area of a rectangle is given by the formula $\text{length} \times \text{width}$. The area of a triangle is given by the formula $\frac{1}{2} \times \text{base} \times \text{height}$. A polygonal region is the union of finitely many non-overlapping triangular regions; its area is the sum of the areas of the triangles.
  3. Congruent regions have the same area.
  4. The area of the union of two regions is the sum of the areas minus the area of the intersection.
  5. The area of the difference of two regions, where one is contained in the other, is the difference of the areas.

## Lesson Notes

In this lesson, we make precise what we mean about area and the properties of area. We already know the area formulas for rectangles and triangles; this will be our starting point. In fact, the basic definition of area and most of the area properties listed in the student outcomes above were first explored by students in third grade (**3.MD.C.3** **3.MD.C.6**, **3.MD.C.7**). Since their introduction, students have had continuous exposure to these properties in a variety of situations involving triangles, circles, etc. (**4.MD.A.2**, **5.NF.B.4b**, **6.G.A.1**, **6.G.A.4**). It is the goal of this lesson to state the properties learned in earlier grades explicitly. In that sense, this lesson is a summative experience for students rather than an introductory experience. Furthermore, the review is preparatory to the exploration of volume, which will come later. The examination will allow us to show the parallels between area and volume more explicitly, which helps set the stage for understanding why volume formulas for cylinders, pyramids, and cones work and, consequently, the application of each of those formulas (**G-GMD.A.1**, **G-GMD.A.3**).

If these facts seem brand new to students, please refer to the following lessons:

- Property 1, Grade 3, Module 4, Lessons 1–4
- Property 2, Grade 3, Module 4, Topic B focuses on rectangles. Triangles are not studied in Grade 3, Module 4, but there is practice decomposing regions into rectangles and adding up areas of smaller rectangles in Lesson 13. Introduction of $\text{length} \times \text{width}$ happens in Grade 4, Module 3, Lesson 1.
- Property 3, though not addressed explicitly, is observed in Grade 3, Module 4, Lesson 5, Problem 1(a) and 1(c).
- Property 4, Grade 3, Module 4, Lesson 13
- Property 5, Grade 3, Module 4, Lesson 13, Problem 3

This lesson does cover some new material. We introduce some notation for set operations for regions in the plane: specific notation related to a union of two regions, $\cup$; an intersection of two regions, $\cap$; and a subset of a region $\subseteq$. Students begin by exploring the properties of area, which are then solidified in a whole-class discussion.

This treatment of area and area properties in this lesson is usually referred to as Jordan measure by mathematicians. It is part of the underlying *theory of area* needed by students for learning integral calculus. While it is not necessary to bring up this term with your students, we encourage you to read up on it by searching for it on the Internet (the brave of heart might find Terence Tao's online book on *measure theory* fascinating, even if it is just to ponder the questions he poses in the opening to Chapter 1).

If more time is needed for students to relate their previous experience working with area with these explicit properties, consider splitting the lesson over two days. This means a readjustment of pacing so as not to address all five properties in one day.

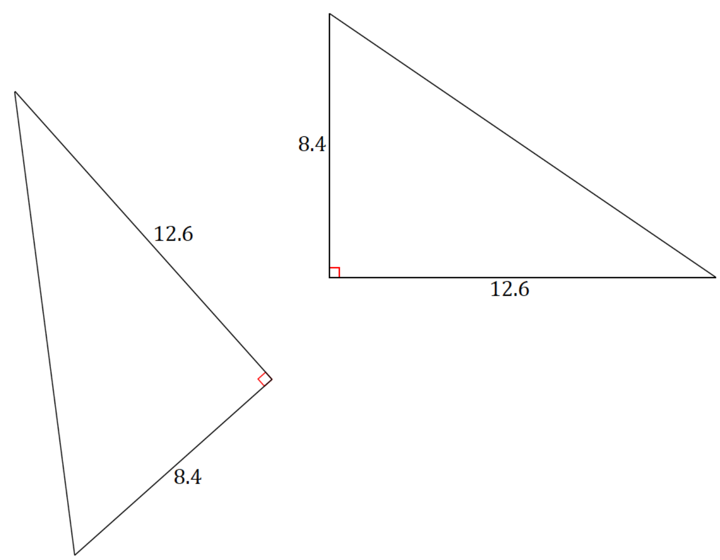
## Classwork

### Exploratory Challenge/Exercises 1–4 (15 minutes)

The exercises below relate to the properties of area that students know; these exercises facilitate the conversation around the formal language of the properties in the following Discussion. The exercises are meant to be quick; divide the class into groups so that each group works on a separate problem. Then have each group present their work.

**Exploratory Challenge/Exercises 1–4**

1. **Two congruent triangles are shown below.**




   a. **Calculate the area of each triangle.**

   $\frac{1}{2}(12.6)(8.4) = 52.92$

   b. **Circle the transformations that, if applied to the first triangle, would always result in a new triangle with the same area:**

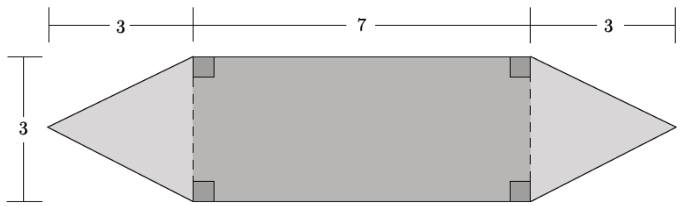
   **Translation** **Rotation** **Dilation** **Reflection**

c. **Explain your answer to part (b).**

*Two congruent figures have equal area. If two figures are congruent, it means that there exists a transformation that maps one figure onto the other completely. For this reason, it makes sense that they would have equal area because the figures would cover the exact same region.*

**2.**

a. **Calculate the area of the shaded figure below.**




$2\left(\frac{1}{2}\right)(3)(3) = 9$

$7(3) = 21$

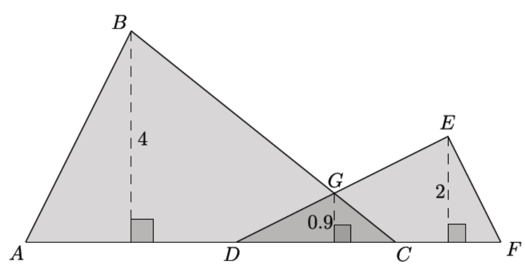
*The area of the figure is* $9 + 21 = 30$.

b. **Explain how you determined the area of the figure.**

*First, I realized that the two shapes at the ends of the figure were triangles with a base of* $3$ *and a height of* $3$ *and the shape in the middle was a rectangle with dimensions* $3 \times 7$*. To find the area of the shaded figure, I found the sum of all three shapes.*

MP.7

**3.** **Two triangles $\triangle ABC$ and $\triangle DEF$ are shown below. The two triangles overlap forming $\triangle DGC$.**




a. **The base of figure $ABGEF$ is comprised of segments of the following lengths: $AD = 4$, $DC = 3$, and $CF = 2$. Calculate the area of the figure $ABGEF$.**

*The area of* $\triangle ABC$: $\frac{1}{2}(4)(7) = 14$

*The area of* $\triangle DEF$: $\frac{1}{2}(2)(5) = 5$

*The area of* $\triangle DGC$: $\frac{1}{2}(0.9)(3) = 1.35$

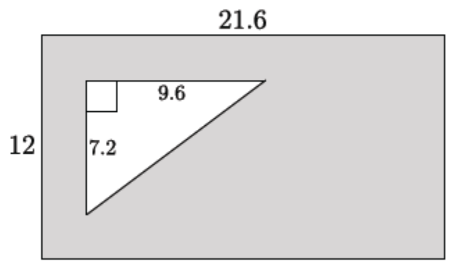
*The area of figure* $ABGEF$ *is* $14 + 5 - 1.35 = 17.65$.

MP.7

b. **Explain how you determined the area of the figure.**

*Since the area of* $\triangle\ DGC$ *is counted twice, it needs to be subtracted from the sum of the overlapping triangles.*

4. **A rectangle with dimensions $21.6 \times 12$ has a right triangle with a base $9.6$ and a height of $7.2$ cut out of the rectangle.**




a. **Find the area of the shaded region.**

*The area of the rectangle:* $(12)(21.6) = 259.2$

*The area of the triangle:* $\frac{1}{2}(7.2)(9.6) = 34.56$

*The area of the shaded region is* $259.2 - 34.56 = 224.64$.

b. **Explain how you determined the area of the shaded region.**

*I subtracted the area of the triangle from the area of the rectangle to determine the shaded region.*

## Discussion (20 minutes)

The Discussion formalizes the properties of area that students have been studying since Grade 3, debriefs the exercises in the Exploratory Challenge, and introduces set notation appropriate for discussing area. Have the properties displayed in a central location as you refer to each one; and have students complete a foldable organizer on an $8.5 \times 11$ in. sheet of paper like the example here, where they record each property, and new set notation and definitions, as it is discussed.

| Property | Example | In my own words |
|---|---|---|
| | | |

- (**Property 1**) We describe area of a set in the plane as a number, greater than or equal to zero, that measures the size of the set and not the shape. How would you describe area in your own words?

Prompt students to think back to Lesson 1, where this question was asked and reviewed as part of the Discussion.

  - *Area is a way of quantifying the size of a region without any reference to the shape of the region.*

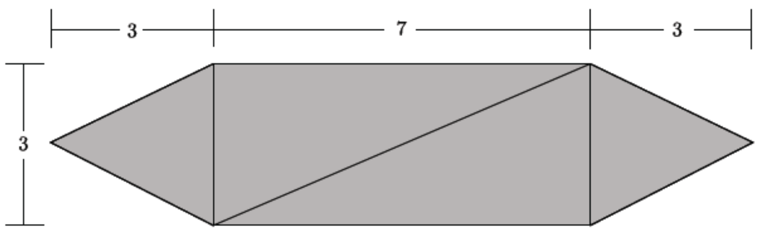
- (**First half of Property 2**) The area of a rectangle is given by the formula length $\times$ width. The area of a triangle is given by the formula $\frac{1}{2} \times$ base $\times$ height. How can we use Exercise 1 to support this?
  - *The two congruent right triangles can be fitted together to form a rectangle with dimensions* $12.6 \times 8.4$. *These dimensions are the length and width, or base and height, of the rectangle.*
  - *Since the two right triangles are congruent, each must have half the area of the entire rectangle or an area described by the formula* $\frac{1}{2} \times$ base $\times$ height.

- (**Property 3**) Notice that the congruent triangles in the Exercise 1 each have the same area as the other.
- (**Second half of Property 2**) We describe a polygonal region as the union of finitely many non-overlapping triangular regions. The area of the polygonal region is the sum of the areas of the triangles. Let's see what this property has to do with Exercise 2.
- Explain how you calculated the area for the figure in Exercise 2.

Select students to share their strategies for calculating the area of the figure. Sample responses are noted in the exercise. Most students will likely find the sum of the area of the rectangle and the area of the two triangles.

- Would your answer be any different if we divided the rectangle into two congruent triangles?

Show the figure below. Provide time for students to check via calculation or discuss in pairs.




  - *No, the area of the figure is the same whether we consider the middle portion of the figure as a rectangle or two congruent triangles.*

- Triangular regions *overlap* if there is a point that lies in the interior of each. One way to find the area of such a region is to split it into non-overlapping triangular regions and add the areas of the resulting triangular regions as shown below. Figure 2 is the same region as Figure 1 but is one (of many) possible decomposition into non-overlapping triangles.

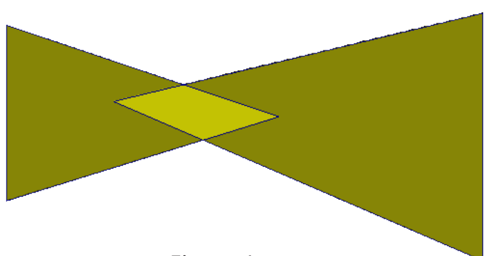


Figure 1

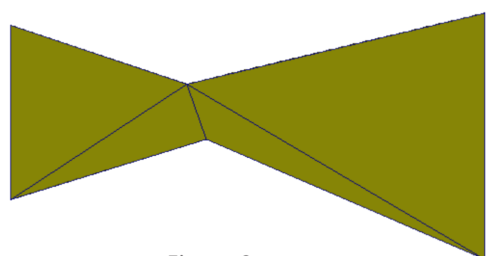


Figure 2

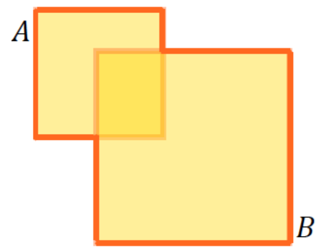
- This mode of determining area can be done for *any* polygonal region.

Provide time for students to informally verify this fact by drawing quadrilaterals and pentagons (ones that are not regular) and showing that each is the union of triangles.

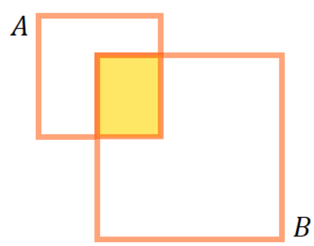
- (**Property 4**) The area of the union of two regions is the sum of the areas minus the area of the intersection. Exercise 3 can be used to break down this property, particularly the terms *union* and *intersection*. How would you describe these terms?

Take several responses from students; some may explain their calculation by identifying the overlapping region of the two triangles as the intersection and the union as the region defined by the boundary of the two shapes. Then use the points below to explicitly demonstrate each union and intersection and what they have to do with determining the area of a region.

- If $A$ and $B$ are regions in the plane, then $A \cup B$ denotes the union of the two regions; that is, all points that lie in $A$ or in $B$, including points that lie in both $A$ and $B$.




- Notice that the area of the union of the two regions is not the sum of the areas of the two regions, which would count the overlapping portion of the figure twice.
- If $A$ and $B$ are regions in the plane, then $A \cap B$ denotes the intersection of the two regions; that is, all points that lie in both $A$ and $B$.




*Scaffolding:*

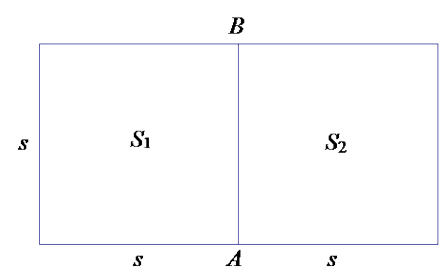
- As notation is introduced, have students make a chart that includes the name, symbol, and simple drawing that represents each term.
- Consider posting a chart in the classroom for students to reference throughout the module.

- (**Property 4**) Use this notation to show that the area of the union of two regions is the sum of the areas minus the area of the intersection.
  - $\text{Area}(A \cup B) = \text{Area}(A) + \text{Area}(B) - \text{Area}(A \cap B)$

Discuss Property 4 in terms of two regions that coincide at a vertex or an edge. This elicits the idea that the area of a segment must be 0, since there is nothing to subtract when regions overlap at a vertex or an edge.

Here is an optional proof:

Two squares $S_1$ and $S_2$ meet along a common edge $\overline{AB}$, or $S_1 \cap S_2 = \overline{AB}$.




Since $S_1 \cup S_2$ is a $2s \times s$ rectangle, its area is $2s^2$. The area of each of the squares is $s^2$.

Since

$$\text{Area}(S_1 \cup S_2) = \text{Area}(S_1) + \text{Area}(S_2) - \text{Area}(S_1 \cap S_2)$$

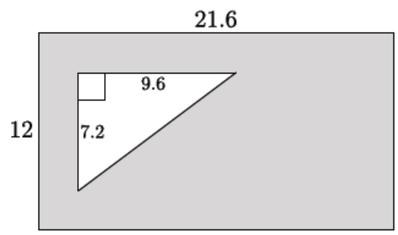
we get

$$2s^2 = s^2 + s^2 - \text{Area}(\overline{AB}).$$

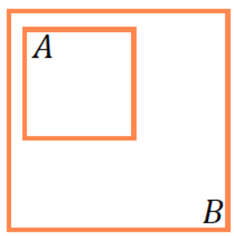
Solving for $\text{Area}(\overline{AB})$ shows that $\text{Area}(\overline{AB}) = 0$.

It is further worth mentioning that when we discuss the area of a triangle or rectangle (or mention the area of any polygon as such), we really mean the area of the triangular region because the area of the triangle itself is zero since it is the area of three line segments, each being zero.

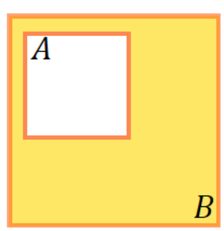
- (**Property 5**) The area of the difference of two regions where one is contained in the other is the difference of the areas. In which exercise was one area *contained* in the other?
  - *Exercise 4*




- If $A$ *is contained in* $B$, or in other words is *a subset of* $B$, denoted as $A \subseteq B$, it means that all of the points in $A$ are also points in $B$.




- What does Property 5 have to do with Exercise 4?
  - *In Exercise 4, all the points of the triangle are also points in the rectangle. That is why the area of the shaded region is the difference of the areas.*
- (**Property 5**) Use set notation to state Property 5: The area of the difference of two regions where one is contained in the other is the difference of the areas.
  - *When* $A \subseteq B$, *then* $\text{Area}(B - A) = \text{Area}(B) - \text{Area}(A)$ *is the difference of the areas*.



## Closing (5 minutes)

Have students review the properties in partners. Ask them to paraphrase each one and to review the relevant set notation and possibly create a simple drawing that helps illustrate each property. Select students to share their paraphrased versions of the properties with the whole class.

1. Students understand that the area of a set in the plane is a number, greater than or equal to zero, that measures the size of the set and not the shape.
2. The area of a rectangle is given by the formula $\text{length} \times \text{width}$. The area of a triangle is given by the formula $\frac{1}{2} \times \text{base} \times \text{height}$. A polygonal region is the union of finitely many non-overlapping triangular regions and has area the sum of the areas of the triangles.
3. Congruent regions have the same area.
4. The area of the union of two regions is the sum of the areas minus the area of the intersection.
5. The area of the difference of two regions where one is contained in the other is the difference of the areas.

**Lesson Summary**

**SET (description): A *set* is a well-defined collection of objects. These objects are called *elements* or *members* of the set.**

**SUBSET: A set $A$ is a *subset* of a set $B$ if every element of $A$ is also an element of $B$. The notation $A \subseteq B$ indicates that the set $A$ is a subset of set $B$.**

**UNION: The *union* of $A$ and $B$ is the set of all objects that are either elements of $A$ or of $B$, or of both. The union is denoted $A \cup B$.**

**INTERSECTION: The *intersection* of $A$ and $B$ is the set of all objects that are elements of $A$ and also elements of $B$. The intersection is denoted $A \cap B$.**

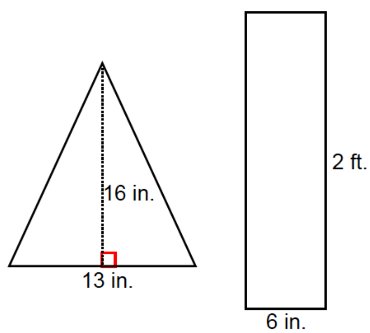
## Exit Ticket (5 minutes)

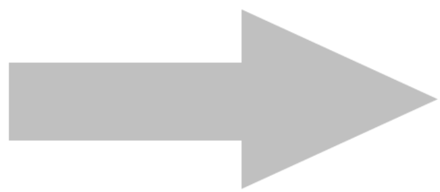
Name ___________________________________________ Date ____________________

# Lesson 2: Properties of Area

## Exit Ticket

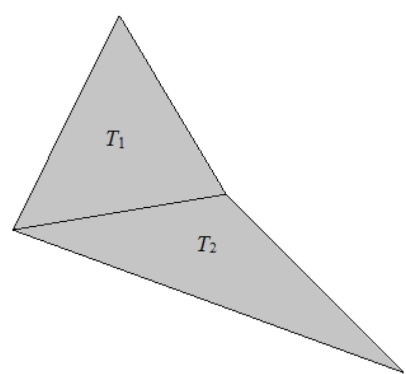
1. Wood pieces in the following shapes and sizes are nailed together in order to create a sign in the shape of an arrow. The pieces are nailed together so that the rectangular piece overlaps with the triangular piece by $4$ in. What is the area of the region in the shape of the arrow?



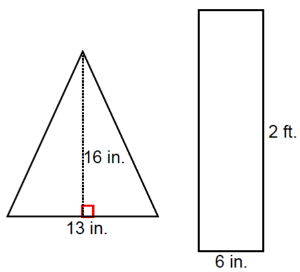
arrow-shaped sign

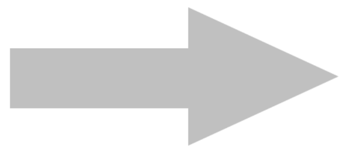
2. A quadrilateral $Q$ is the union of two triangles $T_1$ and $T_2$ that meet along a common side as shown in the diagram. Explain why $\text{Area}(Q) = \text{Area}(T_1) + \text{Area}(T_2)$.



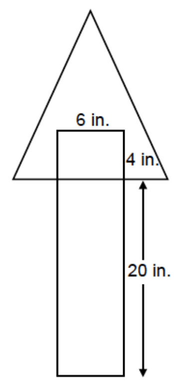
## Exit Ticket Sample Solutions

1. Wooden pieces in the following shapes and sizes are nailed together in order to create a sign in the shape of an arrow. The pieces are nailed together so that the rectangular piece overlaps with the triangular piece by $4$ in. What is the area of the region in the shape of the arrow?



arrow-shaped sign



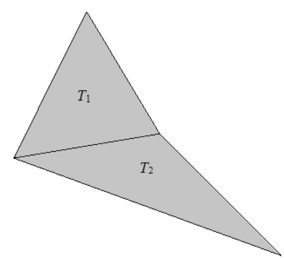

$\text{Area(Arrow)} = \text{Area(Rectangle)} + \text{Area(Triangle)} - \text{Area(Overlap)}$

$\text{Area(Arrow)} = 144 + 104 - 24$

$\text{Area(Arrow)} = 224$

*The area of the region in the shape of the arrow is* $224\ \text{in}^2$.

2. A quadrilateral $Q$ is the union of two triangles $T_1$ and $T_2$ that meet along a common side as shown in the diagram. Explain why $\text{Area}(Q) = \text{Area}(T_1) + \text{Area}(T_2)$.




$Q = T_1 \cup T_2$*, so* $\text{Area}(Q) = \text{Area}(T_1) + \text{Area}(T_2) - \text{Area}(T_1 \cap T_2)$.

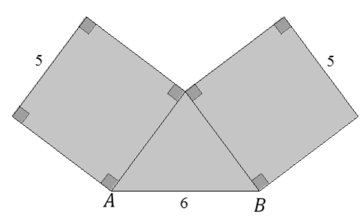
*Since* $T_1 \cap T_2$ *is a line segment, the area of* $T_1 \cap T_2$ *is* $0$.

$\text{Area}(T_1) + \text{Area}(T_2) - \text{Area}(T_1 \cap T_2) = \text{Area}(T_1) + \text{Area}(T_2) - 0$

$\text{Area}(T_1) + \text{Area}(T_2) - \text{Area}(T_1 \cap T_2) = \text{Area}(T_1) + \text{Area}(T_2)$

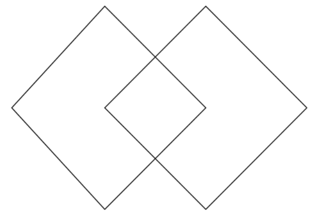
## Problem Set Sample Solutions

1. Two squares with side length $5$ meet at a vertex and together with segment $AB$ form a triangle with base $6$ as shown. Find the area of the shaded region.




*The altitude of the isosceles triangle splits it into two right triangles, each having a base of* $3$ *units in length and hypotenuse of* $5$ *units in length. By the Pythagorean theorem, the height of the triangles must be* $4$ *units in length. The area of the isosceles triangle is* $12$ *square units. Since the squares and the triangle share sides only, the sum of their areas is the area of the total figure. The areas of the square regions are each* $25$ *square units, making the total area of the shaded region* $62$ *square units.*

2. If two $2 \times 2$ square regions $S_1$ and $S_2$ meet at midpoints of sides as shown, find the area of the square region, $S_1 \cup S_2$.

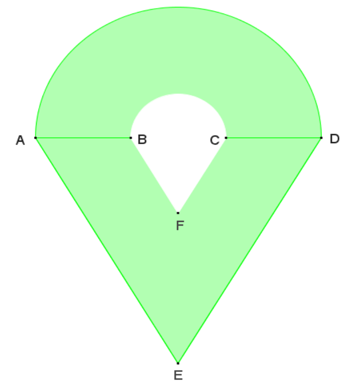


*The area of $S_1 \cap S_2 = 1$ because it is a $1 \times 1$ square region.*

$\text{Area}(S_1) = \text{Area}(S_2) = 4$.

*By Property 3, the area of $S_1 \cup S_2 = 4 + 4 - 1 = 7$.*

3. The figure shown is composed of a semicircle and a non-overlapping equilateral triangle, and contains a hole that is also composed of a semicircle and a non-overlapping equilateral triangle. If the radius of the larger semicircle is $8$, and the radius of the smaller semicircle is $\frac{1}{3}$ that of the larger semicircle, find the area of the figure.




*The area of the large semicircle:* $\text{Area} = \frac{1}{2}\pi \cdot 8^2 = 32$

*The area of the smaller semicircle:* $\text{Area} = \frac{1}{2}\pi\left(\frac{8}{3}\right)^2 = \frac{32}{9}\pi$

*The area of the large equilateral triangle:* $\text{Area} = \frac{1}{2} \cdot 8 \cdot 4\sqrt{3} = 16\sqrt{3}$

*The area of the smaller equilateral triangle:* $\text{Area} = \frac{1}{2} \cdot \frac{8}{3} \cdot \frac{4}{3}\sqrt{3} = \frac{16}{9}\sqrt{3}$

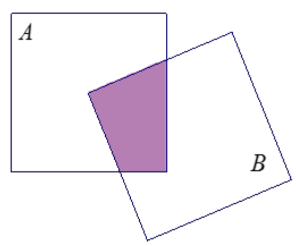
*Total Area:*

$$\text{Total area} = 32\pi - \frac{32}{9}\pi + 16\sqrt{3} - \frac{16}{9}\sqrt{3}$$

$$\text{Total area} = \frac{256}{9}\pi + \frac{128}{9}\sqrt{3} \approx 114$$

*The area of the figure is approximately $142$.*

4. Two square regions $A$ and $B$ each have $\text{Area}(8)$. One vertex of square $B$ is the center point of square $A$. Can you find the area of $A \cup B$ and $A \cap B$ without any further information? What are the possible areas?




*Rotating the shaded area about the center point of square $A$ by a quarter turn three times gives four congruent non-overlapping regions. Each region must have area one-fourth the area of the square. So, the shaded region has* $\text{Area}(2)$.

$\text{Area}(A \cup B) = 8 + 8 - 2 = 14$

$\text{Area}(A \cap B) = 2$

5. **Four congruent right triangles with leg lengths $a$ and $b$ and hypotenuse length $c$ are used to enclose the green region in Figure 1 with a square and then are rearranged inside the square leaving the green region in Figure 2.**

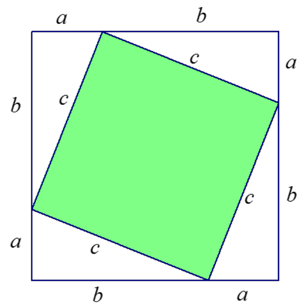



Figure 1

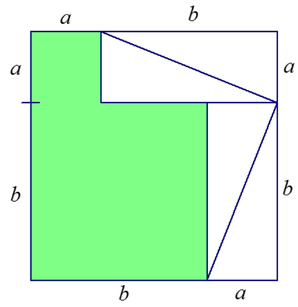



Figure 2

a. **Use Property 4 to explain why the green region in Figure 1 has the same area as the green region in Figure 2.**

*The white polygonal regions in each figure have the same area, so the green region (difference of the big square and the four triangles) has the same area in each figure.*

b. **Show that the green region in Figure 1 is a square and compute its area.**

*Each vertex of the green region is adjacent to the two acute angles in the congruent right triangles whose sum is $90°$. The adjacent angles also lie along a line (segment), so their sum must be $180°$. By addition, it follows that each vertex of the green region in Figure 1 has a degree measure of $90°$. This shows that the green region is at least a rectangle.*

*The green region was given as having side lengths of $c$, so together with having four right angles, the green region must be a square.*

c. **Show that the green region in Figure 2 is the union of two non-overlapping squares and compute its area.**

*The congruent right triangles are rearranged such that their acute angles are adjacent, forming a right angle. The angles are adjacent to an angle at a vertex of an $a \times a$ green region, and since the angles are all adjacent along a line (segment), the angle in the green region must then be $90°$. If the green region has four sides of length $a$, and one angle is $90°$, the remaining angles must also be $90°$, and the region a square.*

*A similar argument shows that the green $b \times b$ region is also a square. Therefore, the green region in Figure 2 is the union of two non-overlapping squares. The area of the green region is then $a^2 + b^2$.*

d. **How does this prove the *Pythagorean theorem*?**

*Because we showed the green regions in Figures 1 and 2 to be equal in area, the sum of the areas in Figure 2 being $a^2 + b^2$, therefore, must be equal to the area of the green square region in Figure 1, $c^2$. The lengths $a$, $b$, and $c$ were given as the two legs and hypotenuse of a right triangle, respectively, so the above line of questions shows that the sum of the squares of the legs of a right triangle is equal to the square of the hypotenuse.*

# Lesson 3: The Scaling Principle for Area

## Student Outcomes

- Students understand that a similarity transformation with scale factor $r$ multiplies the area of a planar region by a factor of $r^2$.
- Students understand that if a planar region is scaled by factors of $a$ and $b$ in two perpendicular directions, then its area is multiplied by a factor of $ab$.

## Lesson Notes

In Lesson 3, students experiment with figures that have been dilated by different scale factors and observe the effect that the dilation has on the area of the figure (or pre-image) as compared to its image. In Topic B, the move will be made from the *scaling principle for area* to the *scaling principle for volume.* This shows up in the use of the formula $V = Bh$; more importantly, it is the way we establish the volume formula for pyramids and cones. The scaling principle for area helps us to develop the scaling principle for volume, which in turn helps us develop the volume formula for general cones (**G-GMD.A.1**).

## Classwork

### Exploratory Challenge (10 minutes)

In the Exploratory Challenge, students determine the area of similar triangles and similar parallelograms and then compare the scale factor of the similarity transformation to the value of the ratio of the area of the image to the area of the pre-image. The goal is for students to see that the areas of similar figures are related by the square of the scale factor. It may not be necessary for students to complete all of the exercises in order to see this relationship. As you monitor the class, if most students understand it, move into the Discussion that follows.

**Exploratory Challenge**

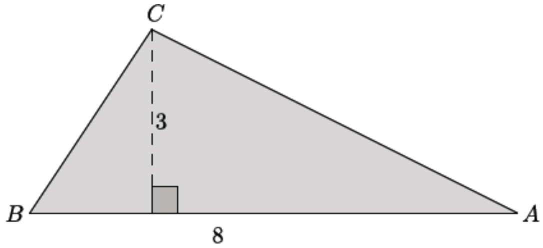
**Complete parts (i)–(iii) of the table for each of the figures in questions (a)–(d): (i) Determine the area of the figure (pre-image), (ii) determine the scaled dimensions of the figure based on the provided scale factor, and (iii) determine the area of the dilated figure. Then, answer the question that follows.**

**In the final column of the table, find the value of the ratio of the area of the similar figure to the area of the original figure.**

| (i) Area of Original Figure | Scale Factor | (ii) Dimensions of Similar Figure | (iii) Area of Similar Figure | Ratio of Areas $\text{Area}_{\text{similar}}:\text{Area}_{\text{original}}$ |
|---|---|---|---|---|
| $12$ | $3$ | $24 \times 9$ | $108$ | $\frac{108}{12} = 9$ |
| $7.5$ | $2$ | $10 \times 6$ | $30$ | $\frac{30}{7.5} = 4$ |
| $20$ | $\frac{1}{2}$ | $2.5 \times 2$ | $5$ | $\frac{5}{20} = \frac{1}{4}$ |
| $6$ | $\frac{3}{2}$ | $4.5 \times 3$ | $13.5$ | $\frac{13.5}{6} = \frac{27}{12} = \frac{9}{4}$ |

*Scaffolding:*
- Consider dividing the class and having students complete two of the four problems, and then share their results before making the conjecture in Exploratory Challenge, part (e).
- Model the process of determining the dimensions of the similar figure for the whole class or a small group.

**a.**

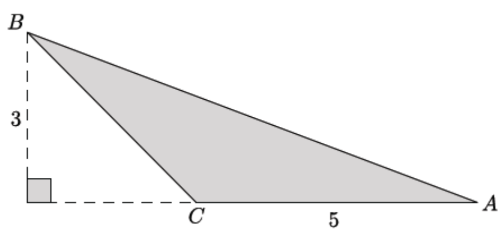



*i.* $\frac{1}{2}(8)(3) = 12$

*ii.* ***The base of the similar triangle is*** $8(3) = 24$***, and the height of the similar triangle is*** $3(3) = 9$.

*iii.* $\frac{1}{2}(24)(9) = 108$

**b.**

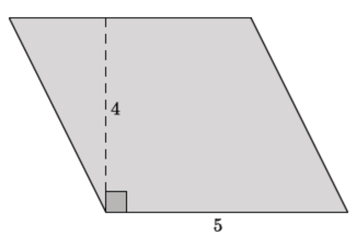



*i.* $\frac{1}{2}(5)(3) = 7.5$

*ii.* ***The base of the similar triangle is*** $5(2) = 10$***, and the height of the similar triangle is*** $3(2) = 6$.

*iii.* $\frac{1}{2}(10)(6) = 30$

**c.**

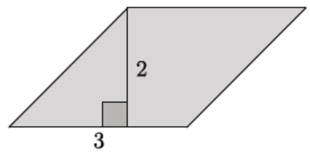



*i.* $(5)(4) = 20$

*ii.* ***The base of the similar parallelogram is*** $5\left(\frac{1}{2}\right) = 2.5$***, and the height of the similar parallelogram is*** $4\left(\frac{1}{2}\right) = 2$.

*iii.* $2.5(2) = 5$

**d.**




*i.* $(3)(2) = 6$

*ii.* ***The base of the similar parallelogram is*** $3\left(\frac{3}{2}\right) = 4.5$***, and the height of the similar parallelogram is*** $2\left(\frac{3}{2}\right) = 3$.

*iii.* $4.5(3) = 13.5$.

MP.3 & MP.8

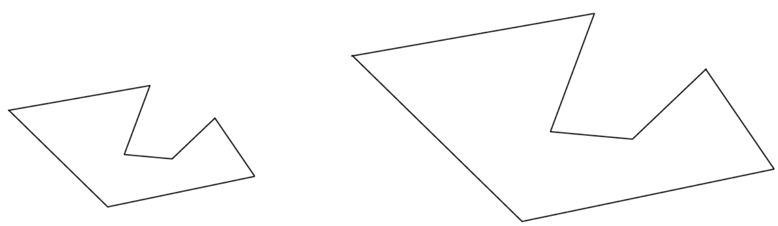
**e.** **Make a conjecture about the relationship between the areas of the original figure and the similar figure with respect to the scale factor between the figures.**

***It seems as though the value of the ratio of the area of the similar figure to the area of the original figure is the square of the scale factor of dilation.***

## Discussion (13 minutes)

Select students to share their conjecture from Exploratory Challenge, part (e). Then formalize their observations with the Discussion below about the scaling principle of area.

- We have conjectured that the relationship between the area of a figure and the area of a figure similar to it is the square of the scale factor.
- Polygon $Q$ is the image of Polygon $P$ under a similarity transformation with scale factor $r$. How can we show that our conjecture holds for a polygon such as this?



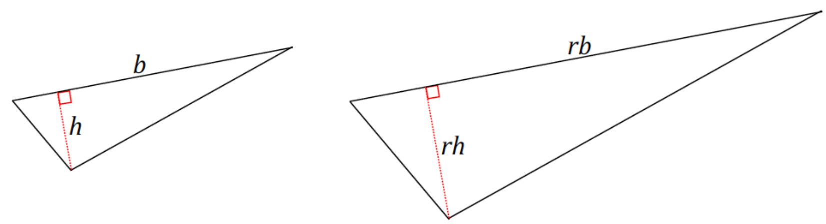
Polygon $P$ Polygon $Q$

- Polygon $Q$ is the image of Polygon $P$ under a similarity transformation with scale factor $r$. How can we show that our conjecture holds for a polygon such as this?
  - *We can find the area of each and compare the areas of the two figures.*
- How can we compute the area of a polygon like this?
  - *We can break it up into triangles.*
- Can any polygon be decomposed into non-overlapping triangles?
  - *Yes.*
- If we can prove that the relationship holds for any triangle, then we can extend the relationship to any polygon.

**THE SCALING PRINCIPLE FOR TRIANGLES:**

***If similar triangles $S$ and $T$ are related by a scale factor of $r$, then the respective areas are related by a factor of $r^2$.***

- To prove the *scaling principle for triangles,* consider a triangle $S$ with base and height, $b$ and $h$, respectively. Then the base and height of the image of $T$ are $rb$ and $rh$, respectively.




Triangle $S$

Triangle $T$

  - *The area of $S$ is* $\text{Area}(S) = \frac{1}{2}bh$*, and the area of $T$ is* $\text{Area}(T) = \frac{1}{2}rbrh = \left(\frac{1}{2}bh\right)r^2$.
- How could we show that the ratio of the areas of $T$ and $S$ is equal to $r^2$?
  - $\frac{\text{Area}(T)}{\text{Area}(S)} = \frac{\left(\frac{1}{2}bh\right)r^2}{\frac{1}{2}bh} = r^2$

    *Therefore, we have proved the scaling principle for triangles.*

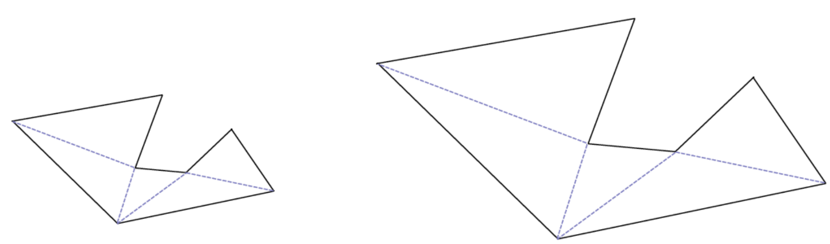
MP.3

- Given the scaling principle for triangles, can we use that to come up with a *scaling principle for any polygon*?
  - *Any polygon can be subdivided into non-overlapping triangles. Since each area of a scaled triangle is $r^2$ times the area of its original triangle, then the sum of all the individual, scaled areas of triangles should be the area of the scaled polygon.*

**THE SCALING PRINCIPLE FOR POLYGONS:**

***If similar polygons $P$ and $Q$ are related by a scale factor of $r$, then their respective areas are related by a factor of $r^2$.***

- Imagine subdividing similar polygons $P$ and $Q$ into non-overlapping triangles.



Polygon $P$

Polygon $Q$

- Each of the lengths in polygon $Q$ is $r$ times the corresponding lengths in polygon $P$.

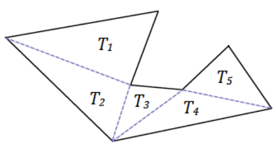
- The area of polygon $P$ is,

$$\text{Area}(P) = T_1 + T_2 + T_3 + T_4 + T_5\,,$$

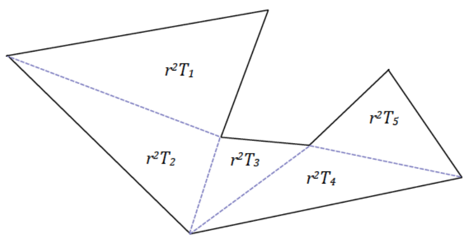
where $T_i$ is the area of the $i^{\text{th}}$ triangle, as shown.

- By the scaling principle of triangles, the areas of each of the triangles in $T$ is $r^2$ times the areas of the corresponding triangles in $Q$.
- Then the area of polygon $Q$ is,

$$\text{Area}(Q) = r^2T_1 + r^2T_2 + r^2T_3 + r^2T_4 + r^2T_5$$
$$\text{Area}(Q) = r^2(T_1 + T_2 + T_3 + T_4 + T_5)$$
$$\text{Area}(Q) = r^2\big(\text{Area}(P)\big)$$




Polygon $P$




Polygon $Q$

- Since the same reasoning will apply to any polygon, we have proven the scaling principle for polygons.

### Exercises 1–2 (8 minutes)

Students apply the scaling principle for polygons to determine unknown areas.

**Exercises 1–2**

1. **Rectangles $A$ and $B$ are similar and are drawn to scale. If the area of rectangle $A$ is $88\text{ mm}^2$, what is the area of rectangle $B$?**

*Length scale factor:* $\frac{30}{16} = \frac{15}{8} = 1.875$

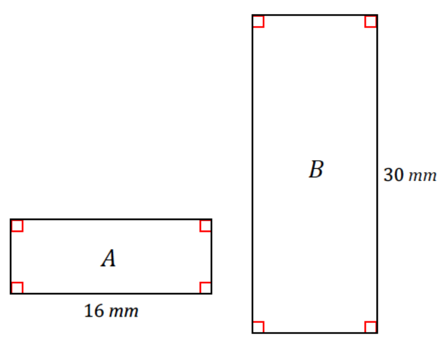
*Area scale factor:* $(1.875)^2$

$\text{Area}(B) = (1.875)^2 \times \text{Area}(A)$

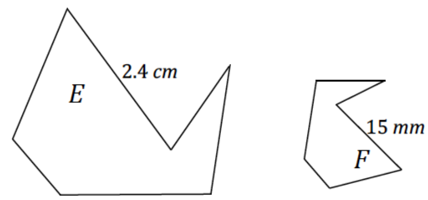
$\text{Area}(B) = (1.875)^2 \times 88$

$\text{Area}(B) = 309.375$

*The area of rectangle $B$ is* $309.375\text{ mm}^2$.



2. Figures $E$ and $F$ are similar and are drawn to scale. If the area of figure $E$ is $120\ \text{mm}^2$, what is the area of figure $F$?




$2.4\ \text{cm} = 24\ \text{mm}$

*Length scale factor:* $\frac{15}{24} = \frac{5}{8} = 0.625$

*Area scale factor:* $(0.625)^2$

$\text{Area}(F) = (0.625)^2 \times \text{Area}(E)$

$\text{Area}(F) = (0.625)^2 \times 120$

$\text{Area}(F) = 46.875$

*The area of figure $F$ is $46.875\ \text{mm}^2$.*

## Discussion (7 minutes)

- How can you describe the scaling principle for area?

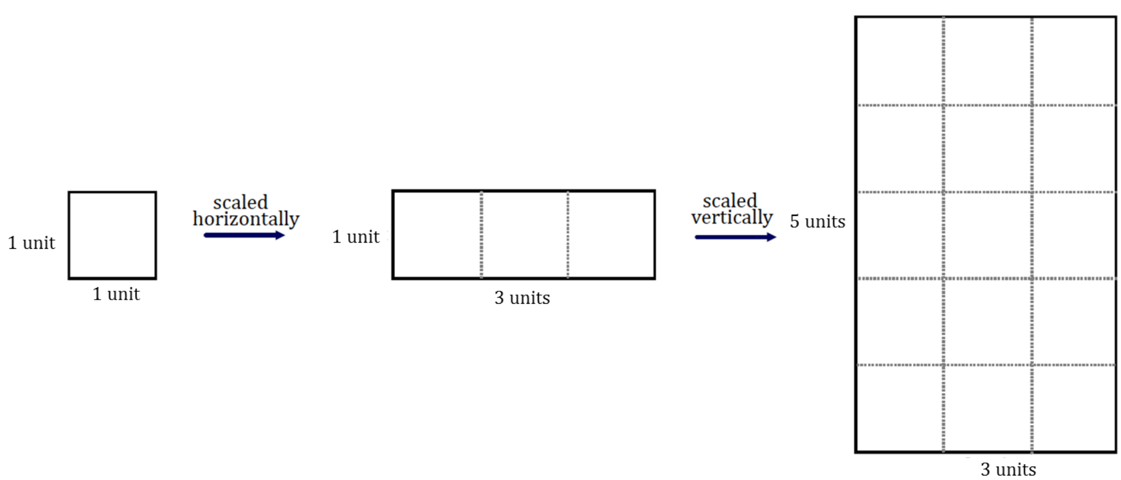
Allow students to share ideas out loud before confirming with the formal principle below.

**THE SCALING PRINCIPLE FOR AREA:**

*If similar figures $A$ and $B$ are related by a scale factor of $r$, then their respective areas are related by a factor of $r^2$.*

- The following example shows another circumstance of scaling and its effect on area:

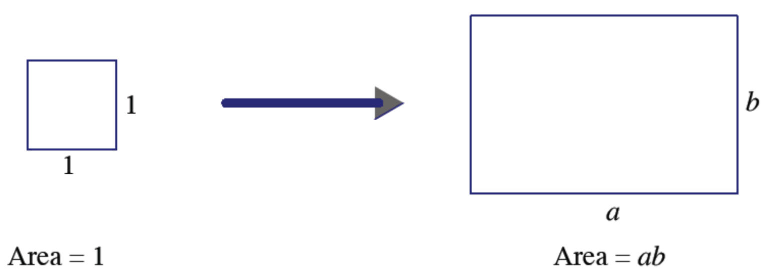
Give students 90 seconds to discuss the following sequence of images with a partner. Then ask for an explanation of what they observe.



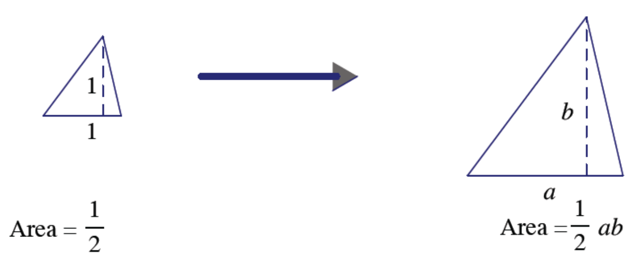

$A = 1\ \text{unit}^2$ $A = 3\ \text{units}^2$ $A = 15\ \text{units}^2$

Ask follow-up questions such as the following to encourage students to articulate what they notice:

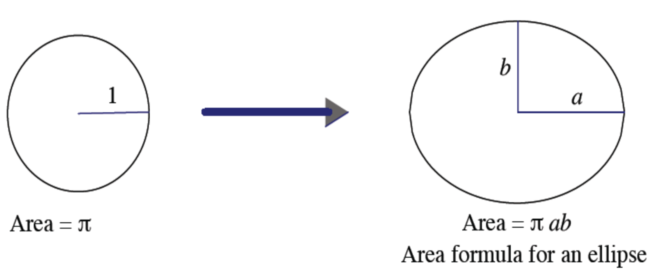
- Is the $1 \times 1$ unit square scaled in both dimensions?
  - *No, only the length was scaled and, therefore, affects the area by only the scale factor.*
- By what scale factor was the unit square scaled horizontally? How does the area of the resulting rectangle compare to the area of the unit square?
  - *The unit square was scaled horizontally by a factor of $3$, and the area is three times as much as the area of the unit square.*
- What is happening between the second image and the third image?
  - *The horizontally scaled figure is now scaled vertically by a factor of $4$. The area of the new figure is $4$ times as much as the area of the second image.*
- Notice that the directions of scaling applied to the original figure, the horizontal and vertical scaling, are perpendicular to each other. Furthermore, with respect to the first image of the unit square, the third image has $12$ times the area of the unit square. How is this related to the horizontal and vertical scale factors?
  - *The area has changed by the same factor as the product of the horizontal and vertical scale factors.*
- We generalize this circumstance: When a figure is scaled by factors $a$ and $b$ in two perpendicular directions, then its area is multiplied by a factor of $ab$:




- We see this same effect when we consider a triangle with base $1$ and height $1$, as shown below.




- We can observe this same effect with non-polygonal regions. Consider a unit circle, as shown below.



- Keep in mind that scale factors may have values between $0$ and $1$; had that been the case in the above examples, we could have seen reduced figures as opposed to enlarged ones.
- Our work in upcoming lessons will be devoted to examining the effect that dilation has on three-dimensional figures.

## Closing (2 minutes)

Ask students to summarize the key points of the lesson. Additionally, consider asking students the following questions independently in writing, to a partner, or to the whole class.

- If the scale factor between two similar figures is $1.2$, what is the scale factor of their respective areas?
  - *The scale factor of the respective areas is* $1.44$.
- If the scale factor between two similar figures is $\frac{1}{2}$, what is the scale factor of their respective areas?
  - *The scale factor of the respective areas is* $\frac{1}{4}$.
- Explain why the scaling principle for triangles is necessary to generalize to the scaling principle for polygonal regions.
  - *Each polygonal region is comprised of a finite number of non-overlapping triangles. If we know the scaling principle for triangles, and polygonal regions are comprised of triangles, then we know that what we observed for scaled triangles applies to polygonal regions in general.*

**Lesson Summary**

**THE SCALING PRINCIPLE FOR TRIANGLES: If similar triangles $S$ and $T$ are related by a scale factor of $r$, then the respective areas are related by a factor of $r^2$.**

**THE SCALING PRINCIPLE FOR POLYGONS: If similar polygons $P$ and $Q$ are related by a scale factor of $r$, then their respective areas are related by a factor of $r^2$.**

**THE SCALING PRINCIPLE FOR AREA: If similar figures $A$ and $B$ are related by a scale factor of $r$, then their respective areas are related by a factor of $r^2$.**

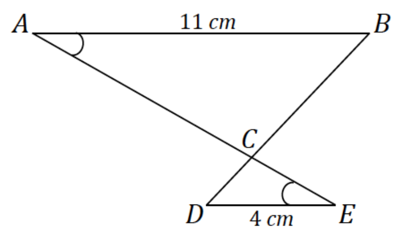
## Exit Ticket (5 minutes)

Name ___________________________________________________ Date ____________________

# Lesson 3: The Scaling Principle for Area

## Exit Ticket

In the following figure, $\overline{AE}$ and $\overline{BD}$ are segments.




a. $\triangle ABC$ and $\triangle CDE$ are similar. How do we know this?

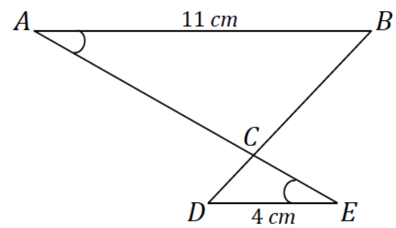
b. What is the scale factor of the similarity transformation that takes $\triangle ABC$ to $\triangle CDE$?

c. What is the value of the ratio of the area of $\triangle ABC$ to the area of $\triangle CDE$? Explain how you know.

d. If the area of $\triangle ABC$ is $30 \text{ cm}^2$, what is the area of $\triangle CDE$?

## Exit Ticket Sample Solutions

In the following figure, $\overline{AE}$ and $\overline{BD}$ are segments.




a. $\triangle ABC$ and $\triangle CDE$ are similar. How do we know this?

*The triangles are similar by the AA criterion.*

b. What is the scale factor of the similarity transformation that takes $\triangle ABC$ to $\triangle CDE$?

$r = \frac{4}{11}$

c. What is the value of the ratio of the area of $\triangle ABC$ to the area of $\triangle CDE$? Explain how you know.

$r^2 = \left(\frac{4}{11}\right)^2$*, or* $\frac{16}{121}$ *by the scaling principle for triangles.*

d. If the area of $\triangle ABC$ is $30\text{ cm}^2$, what is the approximate area of $\triangle CDE$?

$\text{Area}(\triangle CDE) = \frac{16}{121} \times 30\text{ cm}^2 \approx 4\text{ cm}^2$

## Problem Set Sample Solutions

1. A rectangle has an area of $18$. Fill in the table below by answering the questions that follow. Part of the first row has been completed for you.

| 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| Original Dimensions | Original Area | Scaled Dimensions | Scaled Area | $\frac{\text{Scaled Area}}{\text{Original Area}}$ | Area ratio in terms of the scale factor |
| $18 \times 1$ | $18$ | $9 \times \frac{1}{2}$ | $\frac{9}{2}$ | $\frac{1}{4}$ | $\frac{1}{4} = \left(\frac{1}{2}\right)^2$ |
| $9 \times 2$ | $18$ | $\frac{9}{2} \times 1$ | $\frac{9}{2}$ | $\frac{1}{4}$ | $\frac{1}{4} = \left(\frac{1}{2}\right)^2$ |
| $6 \times 3$ | $18$ | $3 \times \frac{3}{2}$ | $\frac{9}{2}$ | $\frac{1}{4}$ | $\frac{1}{4} = \left(\frac{1}{2}\right)^2$ |
| $\frac{1}{2} \times 36$ | $18$ | $\frac{1}{4} \times 18$ | $\frac{9}{2}$ | $\frac{1}{4}$ | $\frac{1}{4} = \left(\frac{1}{2}\right)^2$ |
| $\frac{1}{3} \times 54$ | $18$ | $\frac{1}{6} \times 27$ | $\frac{9}{2}$ | $\frac{1}{4}$ | $\frac{1}{4} = \left(\frac{1}{2}\right)^2$ |

a. List five unique sets of dimensions of your choice that satisfy the criterion set by the column 1 heading and enter them in column 1.

b. If the given rectangle is dilated from a vertex with a scale factor of $\frac{1}{2}$, what are the dimensions of the images of each of your rectangles? Enter the scaled dimensions in column 3.

c. **What are the areas of the images of your rectangles? Enter the areas in column 4.**

d. **How do the areas of the images of your rectangles compare to the area of the original rectangle? Write the value of each ratio in simplest form in column 5.**

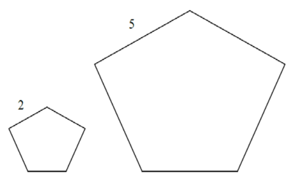
e. **Write the values of the ratios of area entered in column 5 in terms of the scale factor $\frac{1}{2}$. Enter these values in column 6.**

f. **If the areas of two unique rectangles are the same, $x$, and both figures are dilated by the same scale factor $r$, what can we conclude about the areas of the dilated images?**

*The areas of the dilated images would both be $r^2x$ and thus equal.*

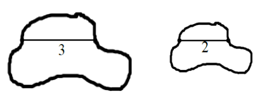
2. **Find the ratio of the areas of each pair of similar figures. The lengths of corresponding line segments are shown.**

a.



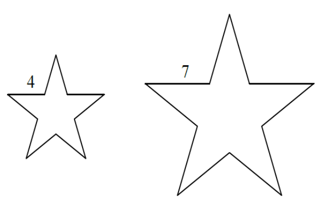

*The scale factor from the smaller pentagon to the larger pentagon is $\frac{5}{2}$. The area of the larger pentagon is equal to the area of the smaller pentagon times $\left(\frac{5}{2}\right)^2 = \frac{25}{4}$. Therefore, the ratio of the area of the smaller pentagon to the larger pentagon is $4:25$.*

b.




*The scale factor from the smaller region to the larger region is $\frac{2}{3}$. The area of the smaller region is equal to the area of the larger region times $\left(\frac{2}{3}\right)^2 = \frac{4}{9}$. Therefore, the ratio of the area of the larger region to the smaller region is $9:4$.*

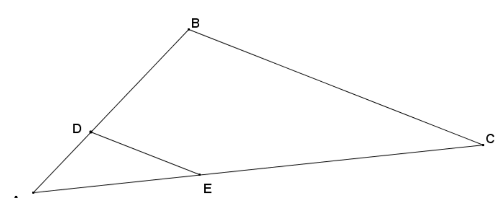
c.




*The scale factor from the small star to the large star is $\frac{7}{4}$. The area of the large star is equal to the area of the small star times $\left(\frac{7}{4}\right)^2 = \frac{49}{16}$. Therefore, the ratio of the area of the small star to the area of the large star is $16:49$.*

3. **In $\triangle ABC$, line segment $DE$ connects two sides of the triangle and is parallel to line segment $BC$. If the area of $\triangle ABC$ is $54$ and $BC = 3DE$, find the area of $\triangle ADE$.**

*The smaller triangle is similar to the larger triangle with a scale factor of $\frac{1}{3}$. So, the area of the small triangle is $\left(\frac{1}{3}\right)^2 = \frac{1}{9}$ the area of the large triangle.*



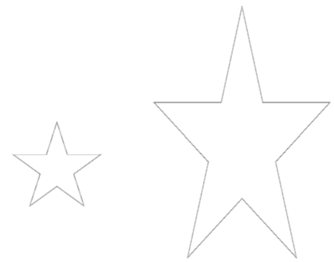

$\text{Area}(ADE) = \frac{1}{9}(54)$

$\text{Area}(ADE) = 6$

*The area of $\triangle ADE$ is $6$ square units.*

4. **The small star has an area of $5$. The large star is obtained from the small star by stretching by a factor of $2$ in the horizontal direction and by a factor of $3$ in the vertical direction. Find the area of the large star.**

*The area of a figure that is scaled in perpendicular directions is equal to the area of the original figure times the product of the scale factors for each direction. The large star therefore has an area equal to the original star times the product $3 \cdot 2$.*



$\text{Area} = 5 \cdot 3 \cdot 2$

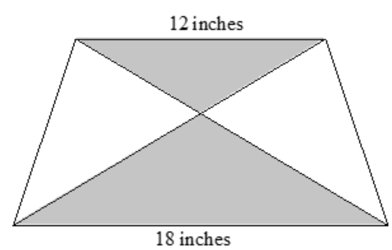
$\text{Area} = 30$

*The area of the large star is $30$ square units.*

5. **A piece of carpet has an area of $50\ \text{yd}^2$. How many square inches will this be on a scale drawing that has $1\ \text{in}.$ represent $1\ \text{yd}.$?**

*One square yard will be represented by one square inch. So, $50$ square yards will be represented by $50$ square inches.*

6. **An isosceles trapezoid has base lengths of $12\ \text{in}.$ and $18\ \text{in}.$ If the area of the larger shaded triangle is $72\ \text{in}^2$, find the area of the smaller shaded triangle.**




*The triangles must be similar by AA criterion, so the smaller triangle is the result of a similarity transformation of the larger triangle including a dilation with a scale factor of $\frac{12}{18} = \frac{2}{3}$. By the scaling principle for area, the area of the smaller triangle must be equal to the area of the larger triangle times the square of the scale factor used:*

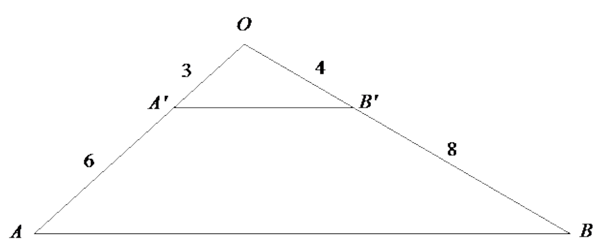
$\text{Area(small triangle)} = \left(\frac{2}{3}\right)^2 \cdot \text{Area(large triangle)}$

$\text{Area(small triangle)} = \frac{4}{9}(72)$

$\text{Area(small triangle)} = 32$

*The area of the smaller triangle with base $12\ \text{in}.$ is $32\ \text{in}^2$.*

7. **Triangle $ABO$ has a line segment $\overline{A'B'}$ connecting two of its sides so that $\overline{A'B'} \parallel \overline{AB}$. The lengths of certain segments are given. Find the ratio of the area of triangle $OA'B'$ to the area of the quadrilateral $ABB'A'$.**




*$\triangle OA'B' \sim \triangle OAB$. The area of $\triangle OA'B'$ is $\frac{1}{9}$ of the area of the area of $\triangle OAB$ because $\left(\frac{3}{3+6}\right)^2 = \left(\frac{1}{3}\right)^2 = \frac{1}{9}$. So, the area of the quadrilateral is $\frac{8}{9}$ of the area of $\triangle OAB$. The ratio of the area of triangle $OA'B'$ to the area of the quadrilateral $ABB'A'$ is $\frac{1}{9}:\frac{8}{9}$, or $1:8$.*

8. A square region $S$ is scaled parallel to one side by a scale factor $r$, $r \neq 0$, and is scaled in a perpendicular direction by a scale factor one-third of $r$ to yield its image $S'$. What is the ratio of the area of $S$ to the area of $S'$?

   *Let the sides of square $S$ be $s$. Therefore, the resulting scaled image would have lengths $rs$ and $\frac{1}{3}rs$. Then the area of square $S$ would be $s^2$, and the area of $S'$ would be $\frac{1}{3}rs(rs) = \frac{1}{3}(rs)^2 = \frac{1}{3}r^2s^2$.*

   *The ratio of areas of $S$ to $S'$ is then $s^2 : \frac{1}{3}r^2s^2$; or $1 : \frac{1}{3}r^2$, or $3 : r^2$.*

9. Figure $T'$ is the image of figure $T$ that has been scaled horizontally by a scale factor of $4$, and vertically by a scale factor of $\frac{1}{3}$. If the area of $T'$ is $24$ square units, what is the area of figure $T$?

   $\text{Area}(T') = \frac{1}{3} \cdot 4 \cdot \text{Area}(T)$

   $24 = \frac{4}{3}\text{Area}(T)$

   $\frac{3}{4} \cdot 24 = \text{Area}(T)$

   $18 = \text{Area}(T)$

   *The area of $T$ is $18$ square units.*

10. What is the effect on the area of a rectangle if …
    a. Its height is doubled and base left unchanged?

       *The area would double.*

    b. If its base and height are both doubled?

       *The area would quadruple.*

    c. If its base were doubled and height cut in half?

       *The area would remain unchanged.*

# Lesson 4: Proving the Area of a Disk

## Student Outcomes

- Students use inscribed and circumscribed polygons for a circle (or disk) of radius $r$ and circumference $C$ to show that the area of a circle is $\frac{1}{2}Cr$ or, as it is usually written, $\pi r^2$.

## Lesson Notes

In Grade 7, students studied an informal proof for the area of a circle. In this lesson, students use informal limit arguments to find the area of a circle using (1) a regular polygon inscribed within the circle and (2) a polygon similar to the inscribed polygon that circumscribes the circle (**G-GMD.A.1**). The goal is to show that the areas of the inscribed polygon and outer polygon act as upper and lower approximations for the area of the circle. As the number of sides of the regular polygon increases, each of these approximations approaches the area of the circle.

Question 6 of the Problem Set steps students through the informal proof of the circumference formula of a circle– another important aspect of **G-GMD.A.1**.

To plan this lesson over the course of two days, consider covering the Opening Exercise and the Example in the first day's lesson and completing the Discussion and Discussion Extension, or alternatively Problem Set 6 (derivation of circumference formula), during the second day's lesson.

## Classwork

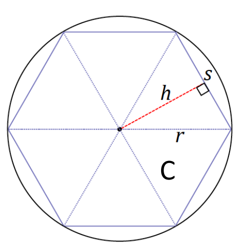
### Opening Exercise (7 minutes)

MP.2 & MP.7

Students derive the area formula for a regular hexagon inscribed within a circle in terms of the side length and height provided in the image. Then, lead them through the steps that describe the area of any regular polygon inscribed within a circle in terms of the polygon's perimeter. This will be used in the proof for the area formula of a circle.

**Opening Exercise**

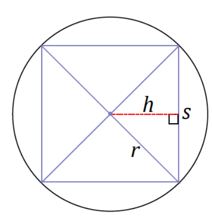
**The following image is of a regular hexagon inscribed in circle $C$ with radius $r$. Find a formula for the area of the hexagon in terms of the length of a side, $s$, and the distance from the center to a side.**




***The area formula for each of the congruent triangles is $\frac{1}{2}sh$. The area of the entire regular hexagon, which consists of 6 such triangles, is represented by the formula $3sh$.***

*Scaffolding:*

- Consider providing numeric dimensions for the hexagon (e.g., $s = 4$; therefore, $h = 2\sqrt{3}$) to first find a numeric area ($\text{Area} = 24\sqrt{3}$ provided the values above) and to generalize to the formula using variables.
- Have students (1) sketch an image and (2) write an area expression for $P_n$ when $n = 4$ and $n = 5$.
- Example: $\text{Area}(P_4) = 2sh$



Since students have found an area formula for the hexagon, lead them through the steps to write an area formula of the hexagon using its perimeter.

- The inscribed hexagon can be divided into six congruent triangles as shown in the image above. Let us call the area of one of these triangles $T$.
- Then the area of the regular hexagon, $H$, is

$$6 \times \text{Area}(T) = 6 \times \left(\frac{1}{2}s \times h\right)$$

- With some regrouping, we have

$$\begin{aligned} 6 \times \text{Area}(T) &= (6 \times s) \times \frac{h}{2} \\ &= \text{Perimeter}(H) \times \frac{h}{2} \end{aligned}$$

MP.2 & MP.7

- We can generalize this area formula in terms of perimeter for any regular inscribed polygon $P_n$. Regular polygon $P_n$ has $n$ sides, each of equal length, and the polygon can be divided into $n$ congruent triangles as in the Opening Exercise, each with area $T$.
- Then the area of $P_n$ is

$$\begin{aligned} \text{Area}(P_n) &= n \times \text{Area}(T_n) \\ &= n \times \left(\frac{1}{2}s_n \times h_n\right) \\ &= (n \times s_n) \times \frac{h_n}{2} \\ &= \text{Perimeter}(P_n) \times \frac{h_n}{2} \end{aligned}$$

*Scaffolding:*
- Consider asking students that may be above grade level to write a formula for the area outside $P_n$ but inside the circle.

*Scaffolding:*
- Consider keeping a list of notation on the board to help students make quick references as the lesson progresses.

## Example (17 minutes)

The Example shows how to approximate the area of a circle using inscribed and circumscribed polygons. Pose the following questions and ask students to consult with a partner and then share out responses.

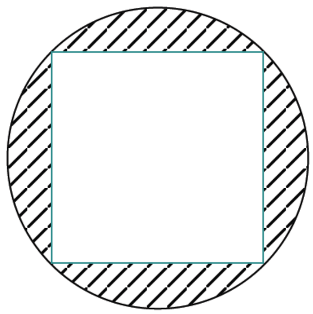
- How can we use the ideas discussed so far to determine a formula for the area of a circle? How is the area of a regular polygon inscribed within a circle related to the area of that circle?

The intention of the questions is to serve as a starting point to the material that follows. Consider prompting students further by asking them what they think the regular inscribed polygon looks like as the number of sides increases (e.g., What would $P_{100}$ look like?).

**Example**

a. **Begin to approximate the area of a circle using inscribed polygons.**

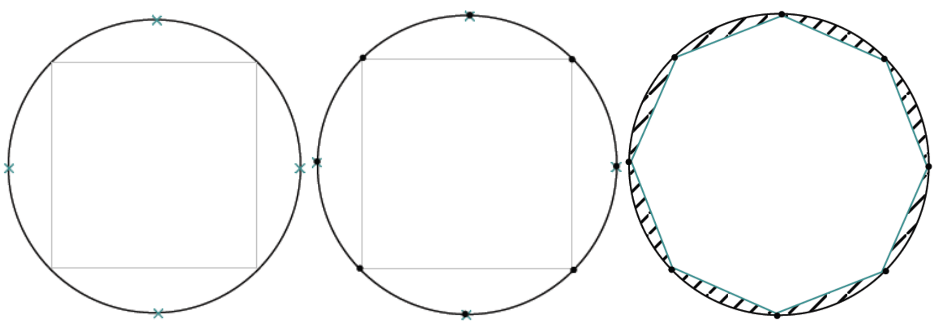
**How well does a square approximate the area of a disk? Create a sketch of $P_4$ (a regular polygon with $4$ sides, a square) in the following circle. Shade in the area of the disk that is not included in $P_4$.**



$P_4$

b. **Next, create a sketch of $P_8$ in the following circle.**

Guide students in how to locate all the vertices of $P_8$ by resketching a square, and then marking a point equally spaced between each pair of vertices; join the vertices to create a sketch of regular octagon $P_8$.



$P_8$

Have students indicate which area is greater in part (c).

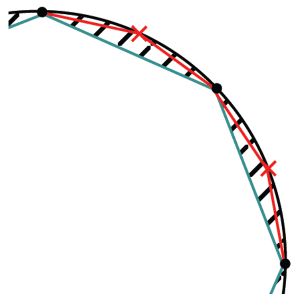
c. **Indicate which polygon has a greater area.**

$\text{Area}(P_4)$ __$<$__ $\text{Area}(P_8)$

d. **Will the area of inscribed regular polygon $P_{16}$ be greater or less than the area of $P_8$? Which is a better approximation of the area of the disk?**

*$\text{Area}(P_8) < \text{Area}(P_{16})$; the area of $P_{16}$ is a better approximation of the area of the disk.*

Share a sketch of the following portion of the transition between $P_8$ and $P_{16}$ in a disk with students. Ask students why the area of $P_{16}$ is a better approximation of the area of the circle.



A portion of polygon $P_{16}$

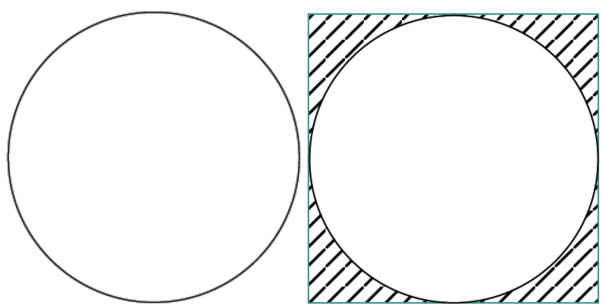
MP.3

e. **We noticed that the area of $P_4$ was less than the area of $P_8$ and that the area of $P_8$ was less than the area of $P_{16}$. In other words, $\text{Area}(P_n)$ $\underline{\ <\ }$ $\text{Area}(P_{2n})$. Why is this true?**

*We are using the $n$-polygon to create the $2n$-polygon. When we draw the segments that join each new vertex in between a pair of existing vertices, the resulting $2n$-polygon has a greater area than that of the $n$-polygon.*

f. **Now we will approximate the area of a disk using circumscribed (outer) polygons.**

**Now circumscribe, or draw a square on the outside of, the following circle such that each side of the square intersects the circle at one point. We will denote each of our outer polygons with prime notation; we are sketching $P'_4$ here.**



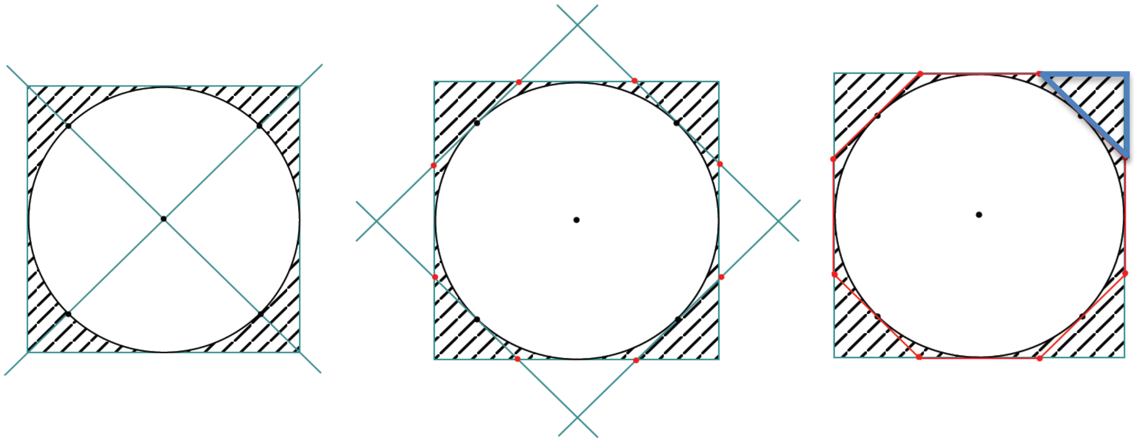
$P'_4$

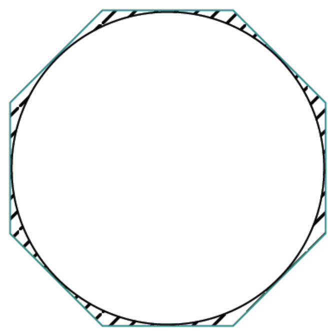
g. **Create a sketch of $P'_8$.**

- How can we create a regular octagon using the square?

Allow students a moment to share out answers and drawings of how to create the regular octagon. Provide them with the following steps if the idea is not shared out.

- *Draw rays from the center of the square to its vertices. Mark the points where the rays intersect the circle. Then draw the line that intersects the circle once through that point and only that point.*



$P'_8$

h. **Indicate which polygon has a greater area.**

$$\text{Area}(P'_4) \underline{\quad > \quad} \text{Area}(P'_8).$$

i. **Which is a better approximation of the area of the circle, $P'_4$ or $P'_8$? Explain why.**

***$P'_8$ is a better approximation of the area of the circle relative to $P'_4$ because it is closer in shape to the circle than $P'_4$.***

j. **How will $\text{Area}(P'_n)$ compare to $\text{Area}(P'_{2n})$? Explain.**

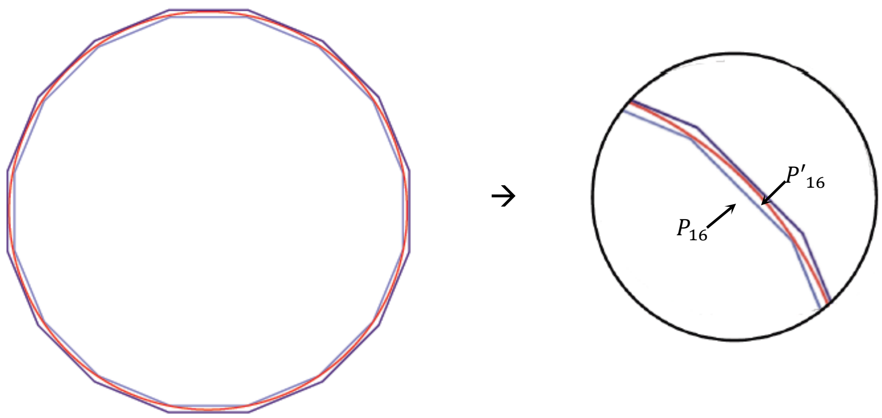
$\text{Area}(P'_n) > \text{Area}(P'_{2n})$. ***$P'_{2n}$ can be created by chipping off its vertices; therefore, the area of $P'_{2n}$ will always be less than the area of $P'_n$.***

## Discussion (10 minutes)

- How will the area of $P'_{2n}$ compare to the area of the circle? Remember that the area of the polygon includes the area of the inscribed circle.
  - $\text{Area(circle)} < \text{Area}(P'_{2n})$
- In general, for any positive integer $n \geq 3$,

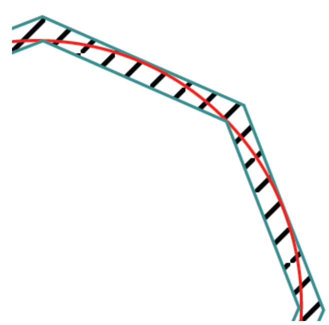
$$\text{Area}(P_n) < \text{Area(circle)} < \text{Area}(P'_n).$$

- For example, examine $P_{16}$ and $P'_{16}$, which sandwich the circle between them.




Furthermore, as $n$ gets larger and larger, or as it grows to infinity (written as $n \to \infty$ and typically read, "as $n$ approaches infinity") the difference of the area of the outer polygon and the area of the inner polygon goes to zero. An explanation of this is provided at the end of the lesson and can be used as an extension to the lesson.

- Therefore, we have *trapped* the area of the circle between the areas of the outer and inner polygons for all $n$. Since this inequality holds for every $n$, and the difference in areas between the outer and inner polygons goes to zero as $n \to \infty$, we can define the area of the circle to be the number (called the *limit*) that the areas of the inner polygons converge to as $n \to \infty$.



**LIMIT (DESCRIPTION): Given an infinite sequence of numbers, $a_1, a_2, a_3, \ldots$ , to say that *the limit of the sequence is* $A$ means, roughly speaking, that when the index $n$ is very large, then $a_n$ is very close to $A$. This is often denoted as, "As $n \to \infty$, $a_n \to A$."**

**AREA OF A CIRCLE (DESCRIPTION): The *area of a circle* is the limit of the areas of the inscribed regular polygons as the number of sides of the polygons approaches infinity.**

For example, a selection of the sequence of areas of regular $n$-gons' regions (starting with an equilateral triangle) that are inscribed in a circle of radius $1$ are as follows:

$$a_3 \approx 1.299$$
$$a_4 = 2$$
$$a_5 \approx 2.377$$
$$a_6 \approx 2.598$$
$$a_7 \approx 2.736$$
$$a_{100} \approx 3.139$$
$$a_{1000} \approx 3.141.$$

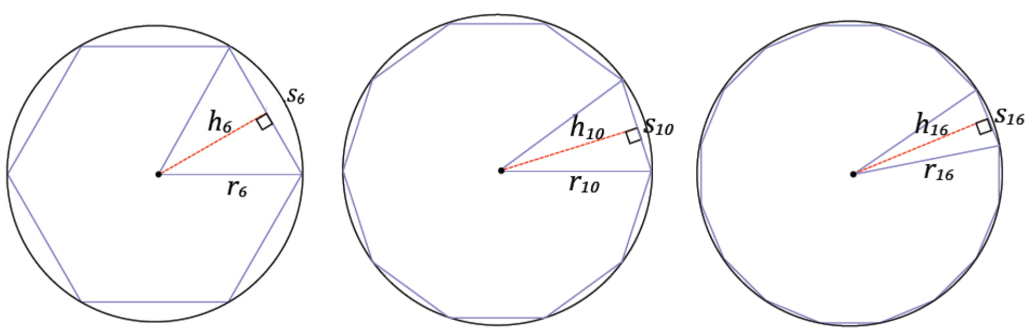
The limit of the areas is $\pi$. In fact, an inscribed regular $1000$-gon has an area very close to the area we expect to see for the area of a unit disk.

- We will use this definition to find a formula for the area of a circle.
- Recall the area formula for a regular $n$-gon:

$$\text{Area}(P_n) = [\text{Perimeter}(P_n)]\left(\frac{h_n}{2}\right).$$

- Think of the regular polygon when it is inscribed in a circle. What happens to $h_n$ and $\text{Perimeter}(P_n)$ as $n$ approaches infinity ($n \to \infty$) in terms of the radius and circumference of the circle?

Students can also refer to their sketches in part (b) of the Example for a visual of what happens as the number of sides of the polygon increases. Alternatively, consider sharing the following figures for students struggling to visualize what happens as the number of sides increases.




  - *As $n$ increases and approaches infinity, the height $h_n$ becomes closer and closer to the length of the radius (as $n \to \infty$, $h_n \to r$).*
  - *As $n$ increases and approaches infinity,* $\text{Perimeter}(P_n)$ *becomes closer and closer to the circumference of the circle (as $n \to \infty$,* $\text{Perimeter}(P_n) \to C$*).*
- Since we are defining the area of a circle as the limit of the areas of the inscribed regular polygon, substitute $r$ for $h_n$ and $C$ for $\text{Perimeter}(P_n)$ in the formulation for the area of a circle:

$$\text{Area(circle)} = \frac{1}{2}rC.$$

- Since $C = 2\pi r$, the formula becomes

$$\text{Area(circle)} = \frac{1}{2}r(2\pi r)$$
$$\text{Area(circle)} = \pi r^2.$$

- Thus, the area formula of a circle with radius $r$ is $\pi r^2$.

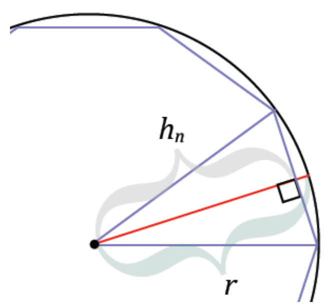
## Discussion (Extension)

Here we revisit the idea of *trapping* the area of the circle between the limits of the areas of the inscribed and outer polygons.

- As we increase the number of sides of both the inscribed and outer regular polygon, both polygons become better approximations of the circle, or in other words, each looks more and more like the circle. Then the difference of the limits of their areas should be 0:

$$\text{As } n \to \infty, [\text{Area}(P'_n) - \text{Area}(P_n)] \to 0.$$

- Let us discover why.
  - *Upon closer examination, we see that $P'_n$ can be obtained by a dilation of $P_n$.*




- What is the scale factor that takes $P_n$ to $P'_n$?
  - $\frac{r}{h_n}$
- Since the area of the dilated figure is the area of the original figure times the square of the scale factor, then

$$\text{Area}(P'_n) = \left(\frac{r}{h_n}\right)^2 \text{Area}(P_n).$$

- Now let us take the difference of the areas:

$$\text{Area}(P'_n) - \text{Area}(P_n) = \left(\frac{r}{h_n}\right)^2 \text{Area}(P_n) - \text{Area}(P_n)$$
$$\text{Area}(P'_n) - \text{Area}(P_n) = \text{Area}(P_n)\left[\left(\frac{r}{h_n}\right)^2 - 1\right].$$

- Let's consider what happens to each of the two factors on the right-hand side of the equation as $n$ gets larger and larger and approaches infinity.

- The factor $\text{Area}(P_n)$: As $n$ gets larger and larger, this value is increasing, but we know it must be less than some value. Since for every $n$, $P_n$ is contained in the square $P'_4$, its area must be less than that of $P'_4$. We know it is certainly not greater than the area of the circle (we also do not want to cite the area of the circle as this value we are approaching, since determining the area of the circle is the whole point of our discussion to begin with). So, we know the value of $\text{Area}(P_n)$ bounded by some quantity; let us call this quantity $B$:

$$\text{As } n \to \infty, \text{Area}(P_n) \to B.$$

MP.1

- What happens to the factor $\left[\left(\frac{r}{h_n}\right)^2 - 1\right]$ as $n$ approaches infinity?

Allow students time to wrestle with this question before continuing.

- The factor $\left[\left(\frac{r}{h_n}\right)^2 - 1\right]$: As $n$ gets larger and larger, the value of $\frac{r}{h_n}$ gets closer and closer to $1$. Recall that the radius is a bit more than the height, so the value of $\frac{r}{h_n}$ is greater than $1$ but shrinking in value as $n$ increases. Therefore, as $n$ approaches infinity, the value of $\left[\left(\frac{r}{h_n}\right)^2 - 1\right]$ is approaching $[(1)^2 - 1]$, or in other words, the value of $\left[\left(\frac{r}{h_n}\right)^2 - 1\right]$ is approaching $0$:

$$\text{As } n \to \infty, \left[\left(\frac{r}{h_n}\right)^2 - 1\right] \to 0.$$

- Then, as $n$ approaches infinity, one factor is never larger than $B$, while the other factor is approaching $0$. The product of these factors as $n$ approaches infinity is then approaching $0$:

$$\text{As } n \to \infty, \text{Area}(P_n)[\left(\frac{r}{h_n}\right)^2 - 1] \to 0$$

or

$$\text{As } n \to \infty, [\text{Area}(P'_n) - \text{Area}(P_n)] \to 0.$$

- Since the difference approaches $0$, each term must in fact be approaching the same thing, i.e., the area of the circle.

### Closing (3 minutes)

Ask students to summarize the key points of the lesson. Additionally, consider asking students the following questions independently in writing, to a partner, or to the whole class.

- The area of a circle can be determined by taking the limit of the area of either inscribed regular polygons or circumscribed polygons, as the number of sides $n$ approaches infinity.
- The area formula for an inscribed regular polygon is $\text{Perimeter}(P_n) \times \frac{h_n}{2}$. As the number of sides of the polygon approaches infinity, the area of the polygon begins to approximate the area of the circle of which it is inscribed. As $n$ approaches infinity, $h_n$ approaches $r$, and $\text{Perimeter}(P_n)$ approaches $C$.
- Since we are defining the area of a circle as the limit of the area of the inscribed regular polygon, we substitute $r$ for $h_n$ and $C$ for $\text{Perimeter}(P_n)$ in the formulation for the area of a circle:

$$\text{Area(circle)} = \frac{1}{2}rC.$$

- Since $C = 2\pi r$, the formula becomes

$$\text{Area(circle)} = \frac{1}{2}r(2\pi r)$$
$$\text{Area(circle)} = \pi r^2.$$

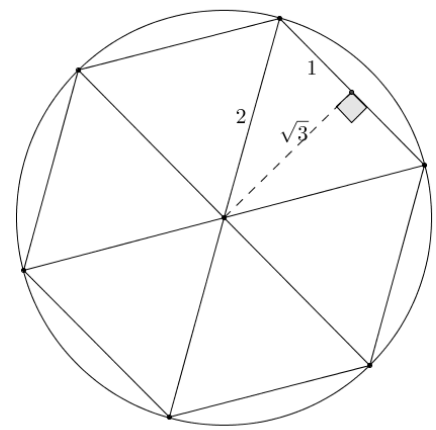
**Exit Ticket (8 minutes)**

Name ______________________________ Date ______________

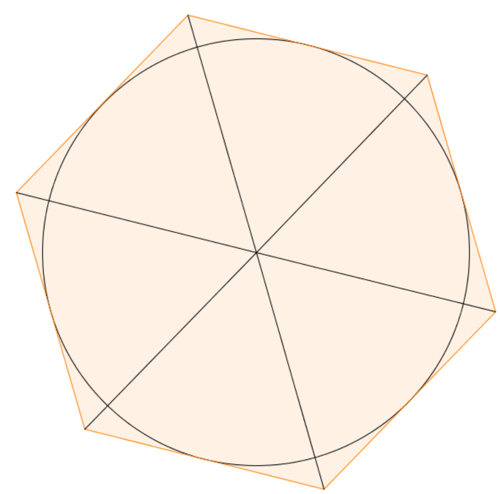
# Lesson 4: Proving the Area of a Disk

## Exit Ticket

1. Approximate the area of a disk of radius 2 using an inscribed regular hexagon.



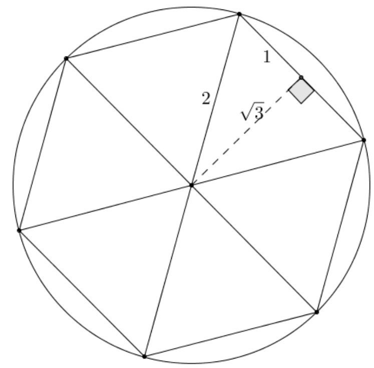

2. Approximate the area of a disk of radius 2 using a circumscribed regular hexagon.



3. Based on the areas of the inscribed and circumscribed hexagons, what is an approximate area of the given disk? What is the area of the disk by the area formula, and how does your approximation compare?

## Exit Ticket Sample Solutions

1. **Approximate the area of a disk of radius $2$ using an inscribed regular hexagon.**




*The interior of a regular hexagon can be divided into $6$ equilateral triangles, each of which can be split into two $30$-$60$-$90$ triangles by drawing an altitude. Using the relationships of the sides in a $30$-$60$-$90$ triangle, the height of each triangle is $\sqrt{3}$.*

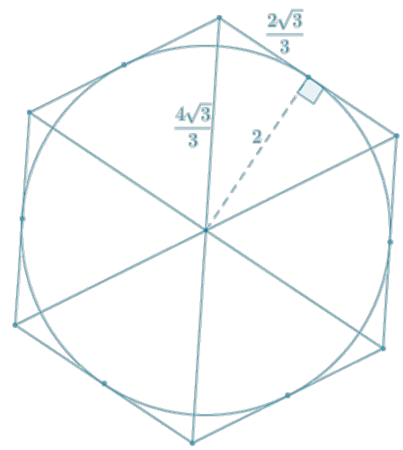
$$\text{Area} = \frac{1}{2}ph$$

$$\text{Area} = \frac{1}{2}(2 \cdot 6) \cdot \sqrt{3}$$

$$\text{Area} = 6\sqrt{3}$$

*The area of the inscribed regular hexagon is $6\sqrt{3}$.*

2. **Approximate the area of a disk of radius $2$ using a circumscribed regular hexagon.**




*Using the same reasoning for the interior of the hexagon, the height of the equilateral triangles contained in the hexagon is $2$, while the lengths of their sides are $\frac{4\sqrt{3}}{3}$.*

$$\text{Area} = \frac{1}{2}ph$$

$$\text{Area} = \frac{1}{2}\left(\frac{4\sqrt{3}}{3} \cdot 6\right) \cdot 2$$

$$\text{Area} = \left(4\sqrt{3} \cdot 2\right)$$

$$\text{Area} = 8\sqrt{3}$$

*The area of the circumscribed regular hexagon is $8\sqrt{3}$.*

3. **Based on the areas of the inscribed and circumscribed hexagons, what is an approximate area of the given disk? What is the area of the disk by the area formula, and how does your approximation compare?**

| *Average approximate area:* | *Area formula:* |
|---|---|
| $A = \frac{1}{2}(6\sqrt{3} + 8\sqrt{3})$ | $A = \pi(2)^2$ |
| $A = \frac{1}{2}(14\sqrt{3})$ | $A = \pi(4)$ |
| $A = 7\sqrt{3} \approx 12.12$ | $A = 4\pi \approx 12.57$ |

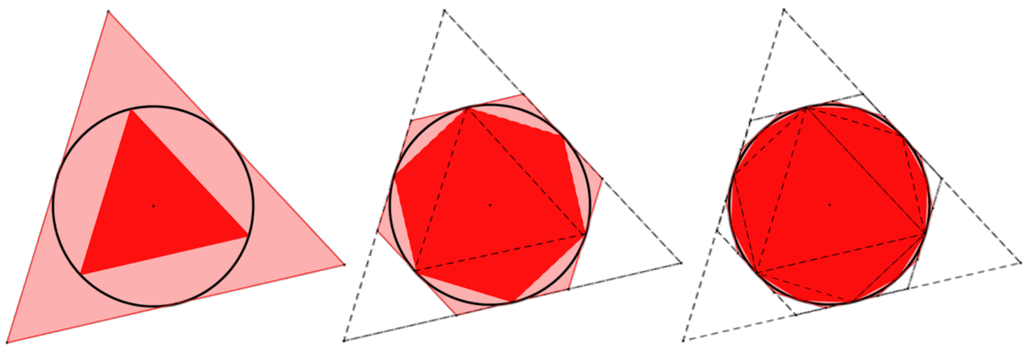
*The approximated area is close to the actual area but slightly less.*

## Problem Set Sample Solutions

1. **Describe a method for obtaining closer approximations of the area of a circle. Draw a diagram to aid in your explanation.**

   *The area of a disk can be approximated to a greater degree of accuracy by squeezing the circle between regular polygons whose areas closely resemble that of the circle. The diagrams below start on the left with inscribed and circumscribed equilateral triangles. The areas of the triangles appear to be quite different, but the disk appears to be somewhere between the area of the larger triangle and the smaller triangle.*

   *Next, double the number of sides of the inscribed and circumscribed polygons, forming inscribed and circumscribed regular hexagons, whose areas are closer to that of the disk (see the center diagram). Continue the process to create inscribed and circumscribed regular dodecagons, whose areas appear even closer yet to that of the disk (see the right diagram). The process can be performed on the dodecagons to produce regular $24$-gons, then regular $48$-gons, etc. As the number of sides of the regular inscribed and circumscribed polygons increases, the polygonal regions more closely approach the area of the disk, squeezing the area of the disk between them.*



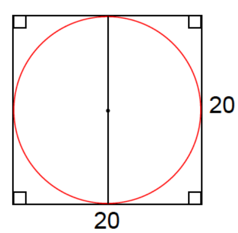
2. **What is the radius of a circle whose circumference is $\pi$?**

   *The radius is $\frac{1}{2}$.*

3. **The side of a square is $20$ cm long. What is the circumference of the circle when …**

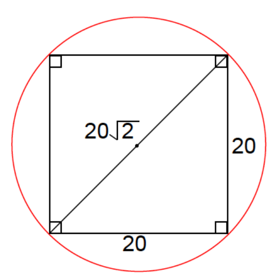
   a. **The circle is inscribed within the square?**

   *The diameter of the circle must be $20$, so the circumference is $20\pi$.*




   b. **The square is inscribed within the circle?**

   *The diameter of the circle must be $20\sqrt{2}$, so the circumference is $20\pi\sqrt{2}$.*



4. The circumference of circle $C_1 = 9$ cm, and the circumference of $C_2 = 2\pi$ cm. What is the value of the ratio of the areas of $C_1$ to $C_2$?

$C_1 = 2\pi r_1 = 9$

$r_1 = \frac{9}{2\pi}$

$\text{Area}(C_1) = \pi\left(\frac{9}{2\pi}\right)^2$

$\text{Area}(C_1) = \frac{81}{4\pi}$

$C_2 = 2\pi r_2 = 2\pi$

$r_2 = 1$

$\text{Area}(C_2) = \pi(1)^2$

$\text{Area}(C_2) = \pi$

$$\frac{\text{Area}(C_1)}{\text{Area}(C_2)} = \frac{\frac{81}{4\pi}}{\pi} = \frac{81}{4\pi^2}$$

5. The circumference of a circle and the perimeter of a square are each $50$ cm. Which figure has the greater area?

*$P_{\text{square}} = 50$; then a side has length $12.5$.*

$\text{Area(square)} = (12.5)^2 = 156.25$

*The area of the square is $156.25\text{ cm}^2$.*

*$C_{\text{circle}} = 50$; then radius is $\frac{25}{\pi}$.*

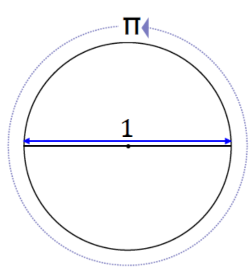
$\text{Area(circle)} = \pi\left(\frac{25}{\pi}\right)^2$

$\text{Area(circle)} = \frac{625}{\pi} \approx 198$

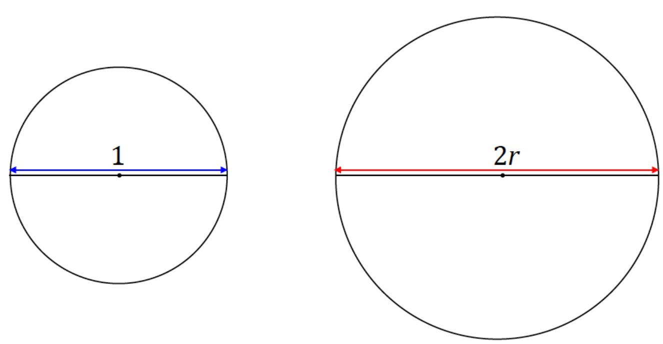
*The area of the circle is $198\text{ cm}^2$.*

*The circle has a greater area.*

6. Let us define $\pi$ to be the circumference of a circle whose diameter is $1$.




We are going to show why the circumference of a circle has the formula $2\pi r$. Circle $C_1$ below has a diameter of $d = 1$, and circle $C_2$ has a diameter of $d = 2r$.



a. All circles are similar (proved in Module 2). What scale factor of the similarity transformation takes $C_1$ to $C_2$?

*A scale factor of* $2r$*.*

b. Since the circumference of a circle is a one-dimensional measurement, the value of the ratio of two circumferences is equal to the value of the ratio of their respective diameters. Rewrite the following equation by filling in the appropriate values for the diameters of $C_1$ and $C_2$:

$$\frac{\text{Circumference}(C_2)}{\text{Circumference}(C_1)} = \frac{\text{diameter}(C_2)}{\text{diameter}(C_1)}.$$

$$\frac{\text{Circumference}(C_2)}{\text{Circumference}(C_1)} = \frac{2r}{1}$$

c. Since we have defined $\pi$ to be the circumference of a circle whose diameter is $1$, rewrite the above equation using this definition for $C_1$.

$$\frac{\text{Circumference}(C_2)}{\pi} = \frac{2r}{1}$$

d. Rewrite the equation to show a formula for the circumference of $C_2$.

$$\text{Circumference}(C_2) = 2\pi r$$

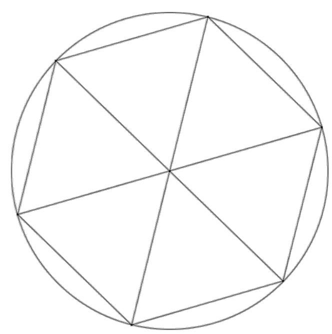
e. What can we conclude?

*Since* $C_2$ *is an arbitrary circle, we have shown that the circumference of any circle is* $2\pi r$*.*

7.

a. Approximate the area of a disk of radius $1$ using an inscribed regular hexagon. What is the percent error of the approximation?

(Remember that percent error is the absolute error as a percent of the exact measurement.)



*The inscribed regular hexagon is divided into six equilateral triangles with side lengths equal to the radius of the circle,* $1$*. By drawing the altitude of an equilateral triangle, it is divided into two* $30$*-*$60$*-*$90$ *right triangles. By the Pythagorean theorem, the altitude,* $h$*, has length* $\frac{\sqrt{3}}{2}$*.*

*The area of the regular hexagon:*

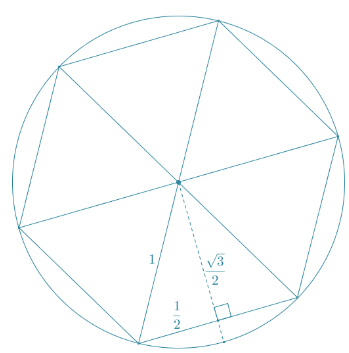
$$\text{Area} = \frac{1}{2}ph$$

$$\text{Area} = \frac{1}{2}(6 \cdot 1) \cdot \frac{\sqrt{3}}{2}$$

$$\text{Area} = \frac{3}{2}\sqrt{3} \approx 2.60$$

$$\text{Percent Error} = \frac{|x - a|}{a}$$

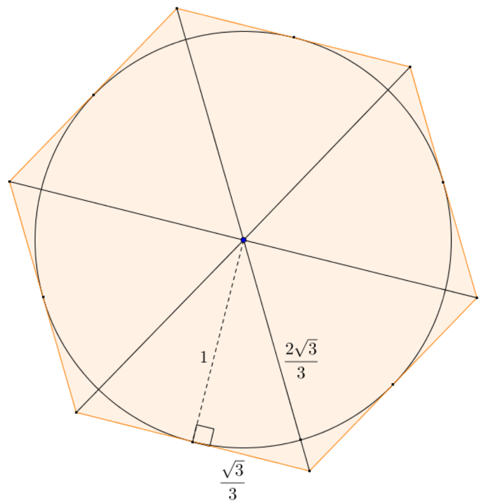
$$\text{Percent Error} = \frac{\pi - \frac{3}{2}\sqrt{3}}{\pi} \approx 17.3\%$$




*The estimated area of the disk using the inscribed regular hexagon is approximately* $2.60$ *square units with a percent error of approximately* $17.3\%$*.*

b. **Approximate the area of a circle of radius $1$ using a circumscribed regular hexagon. What is the percent error of the approximation?**

*The circumscribed regular hexagon can be divided into six equilateral triangles, each having an altitude equal in length to the radius of the circle. By the Pythagorean theorem, the sides of the equilateral triangles are $\frac{2\sqrt{3}}{3}$.*




$$\text{Area} = \frac{1}{2}ph$$

$$\text{Area} = \frac{1}{2}\left(6 \cdot \frac{2\sqrt{3}}{3}\right) \cdot 1$$

$$\text{Area} = 2\sqrt{3} \approx 3.46$$

$$\text{Percent Error} = \frac{|x-a|}{a}$$

$$\text{Percent Error} = \frac{2\sqrt{3}-\pi}{\pi} \approx 10.3\%$$

*The estimated area of the disk using the circumscribed regular hexagon is approximately $3.46$ square units with a percent error of approximately $10.3\%$.*

c. **Find the average of the approximations for the area of a circle of radius $1$ using inscribed and circumscribed regular hexagons. What is the percent error of the average approximation?**

*Let $A_v$ represent the average approximation for the area of the disk, and let $A_X$ be the exact area of the disk using the area formula.*

$$A_v \approx \frac{1}{2}\left(\frac{3}{2}\sqrt{3} + 2\sqrt{3}\right) \qquad A_X = \pi(1)^2$$

$$A_v \approx \frac{7}{4}\sqrt{3} \approx 3.03 \qquad A_X = \pi \approx 3.14$$

$$\text{Percent Error} = \frac{\text{Absolute Error}}{\text{Exact Area}}$$

$$\text{Percent Error} \approx \frac{|3.14-3.03|}{3.14} \times 100\% \qquad \textit{(using } \pi \approx 3.14\textit{)}$$

$$\text{Percent Error} \approx \frac{0.11}{3.14} \times 100\%$$

$$\text{Percent Error} \approx 3.5\%$$

8. **A regular polygon with $n$ sides each of length $s$ is inscribed in a circle of radius $r$. The distance $h$ from the center of the circle to one of the sides of the polygon is over $98\%$ of the radius. If the area of the polygonal region is $10$, what can you say about the area of the circumscribed regular polygon with $n$ sides?**

*The circumscribed polygon has area $\left(\frac{r}{h}\right)^{2}10$.*

*Since $0.98r < h < r$, by inversion $\frac{1}{0.98r} > \frac{1}{h} > \frac{1}{r}$.*

*By multiplying by $r$, $\frac{r}{0.98r} = \frac{1}{0.98} > \frac{r}{h} > 1$.*

*The area of the circumscribed polygon is $\left(\frac{r}{h}\right)^2 10 < \left(\frac{1}{0.98}\right)^2 10 < 10.42$.*

*The area of the circumscribed polygon is less than $10.42$ square units.*

# Topic B:

# Volume

## G-GMD.A.1, G-GMD.A.3, G-GMD.B.4, G-MG.A.1, G-MG.A.2, G-MG.A.3

| | | |
|---|---|---|
| **Focus Standards:** | G-GMD.A.1 | Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. *Use dissection arguments, Cavalieri's principle, and informal limit arguments.* |
| | G-GMD.A.3 | Use volume formulas for cylinders, pyramids, cones and spheres to solve problems.★ |
| | G-GMD.B.4 | Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects. |
| | G-MG.A.1 | Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder).★ |
| | G-MG.A.2 | Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot).★ |
| | G-MG.A.3 | Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios).★ |
| **Instructional Days:** | 9 | |
| **Lesson 5:** | Three-Dimensional Space (E)[1] | |
| **Lesson 6:** | General Prisms and Cylinders and Their Cross-Sections (E) | |
| **Lesson 7:** | General Pyramids and Cones and Their Cross-Sections (S) | |
| **Lesson 8:** | Definition and Properties of Volume (S) | |
| **Lesson 9:** | Scaling Principle for Volumes (P) | |
| **Lesson 10:** | The Volume of Prisms and Cylinders and Cavalieri's Principle (S) | |
| **Lesson 11:** | The Volume Formula of a Pyramid and Cone (E) | |
| **Lesson 12:** | The Volume Formula of a Sphere (S) | |
| **Lesson 13:** | How Do 3D Printers Work? (S) | |

[1] Lesson Structure Key: **P**-Problem Set Lesson, **M**-Modeling Cycle Lesson, **E**-Exploration Lesson, **S**-Socratic Lesson

With a reference to area established in Topic A, students study volume in Topic B. In Grade 8, volume is treated independent of the subtle problems that arise when we attempt to measure the volume of figures other than rectangular solids. From an advanced mathematical perspective, area and volume are conceptually very close in that Jordan measure provides a good foundation, but there are profound differences between area and volume that show up mathematically only when we consider the problem of cutting bodies along planes and reassembling them. Two bodies of the same volume might not be "equi-decomposable" in this sense. This, of course, is much more advanced an idea than anything in the curriculum, but it is one of the mathematical reasons Cavalieri's principle is indispensable. In contrasting Grade 8 with Module 3, the role of this principle is a prominent difference. More generally, understanding and predicting the shapes of cross-sections of three-dimensional figures—though it was done in Grade 7—is a complex skill that needs a lot of work to fully develop. We return to that with a level of sophistication that was absent in Grade 7.

In Lesson 5, students study the basic properties of two-dimensional and three-dimensional space, noting how ideas shift between the dimensions. For example, in two-dimensional space, two lines perpendicular to the same line are parallel, but in three-dimensional space we consider how two *planes* perpendicular to the same line are parallel. In Lesson 6, students learn that general cylinders are the parent category for prisms, circular cylinders, right cylinders, and oblique cylinders (MP.6). Students also study why the cross-section of a cylinder is congruent to its base (**G-GMD.B.4**). In Lesson 7, students study the explicit definition of a cone and learn what distinguishes pyramids from general cones. Students also see how dilations explain why a cross-section taken parallel to the base of a cone is similar to the base (**G-GMD.B.4**, MP.7). Lesson 8 demonstrates the properties of volume, which are analogous to the properties of area (seen in Lesson 2). Students reason why the volume of any right triangular prism has the same volume formula as that of a right triangular prism with a right triangle as a base. This leads to the generalization of the volume formula for any right cylinder (**G-GMD.A.1**, **G-GMD.A.3**). In Lesson 9, students examine the scaling principle for volume (they have seen the parallel situation regarding area in Lesson 3) and see that a solid scaled by factors $a$, $b$, and $c$ in three perpendicular directions will result in a volume multiplied by a factor of $abc$. In Lesson 10, students learn Cavalieri's principle, which describes the relationship between cross-sections of two solids and their respective volumes. If two solids are included between two parallel planes, and cross-sections taken parallel to the bases are of equal area at every level, then the volumes of the solids must be equal. Cavalieri's principle is used to reason why the volume formula of any cylinder is area of $\text{base} \times \text{height}$ (**G-GMD.A.1**). Lesson 11 focuses on the derivation of the volume formulas for cones, and Lesson 12 focuses on the derivation of the volume formula for spheres, which depends partly on the volume formula of a cone (**G-GMD.A.1**). Lesson 13 is a look at 3D printers and ultimately how the technology is linked to Cavalieri's principle.

Module 3 is a natural place to see geometric concepts in modeling situations. Modeling-based problems are found throughout Topic B and include the modeling of real-world objects, the application of density, the occurrence of physical constraints, and issues regarding cost and profit (**G-MG.A.1**, **G-MG.A.2**, **G-MG.A.3**).

# Lesson 5: Three-Dimensional Space

## Student Outcomes

- Students describe properties of *points, lines,* and *planes* in *three-dimensional space.*

## Lesson Notes

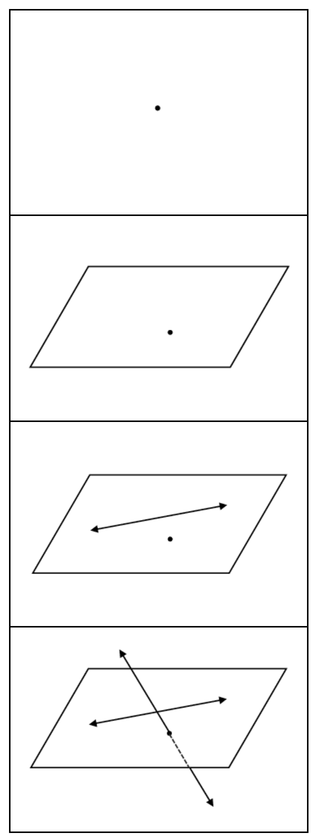
A strong intuitive grasp of three-dimensional space, and the ability to visualize and draw, is crucial for upcoming work with volume as described in **G-GMD.A.1**, **G-GMD.A.2**, **G-GMD.A.3**, and **G-GMD.B.4**. By the end of the lesson, we want students to be familiar with some of the basic properties of points, lines, and planes in three-dimensional space. The means of accomplishing this objective: Draw, draw, and draw! The best evidence for success with this lesson is to see students persevere through the drawing process of the properties. No proof is provided for the properties; therefore, it is imperative that students have the opportunity to verify the properties experimentally with the aid of manipulatives such as cardboard boxes and uncooked spaghetti.

In the case that the lesson requires two days, it is suggested that everything that precedes the Exploratory Challenge is covered on the first day and the Exploratory Challenge itself is covered on the second day.

## Classwork

### Opening Exercise (5 minutes)

The terms *point, line,* and *plane* are first introduced in Grade 4. It is worth emphasizing to students that they are undefined terms, meaning they are part of our assumptions as a basis of the subject of geometry, and we can build the subject once we use these terms as a starting place. Therefore, we give these terms intuitive descriptions and should be clear that the concrete representation is just that—a representation.



Have students sketch, individually or with partners, as you call out the following figures. Consider allowing a moment for students to attempt the sketch and following suit and sketching the figure once they are done.

- Sketch a point.
- Sketch a plane containing the point.

If students struggle to draw a plane, observe that just like lines, when we draw a plane on a piece of paper, we can really only represent part of it, since a plane is a flat surface that extends forever in all directions. For that reason, we will show planes with edges to make the illustrations easier to follow.

- Draw a line that lies on the plane.
- Draw a line that intersects the plane in the original point you drew.

### Exercise (5 minutes)

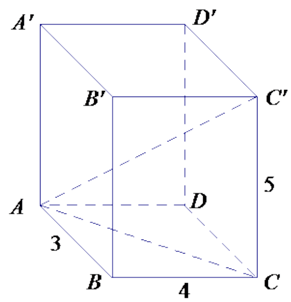
Allow students to wrestle with the Exercise independently or with partners.

> *Scaffolding:*
> - Consider providing small groups or partner pairs with manipulatives for Exercise 1 and the following Discussion: boxes (for example, a tissue box or a small cardboard box) or nets to build right rectangular prisms (G6-M5-L15) and uncooked spaghetti or pipe cleaners to model lines. Students should use manipulatives to help visualize and respond to Discussion questions.
> - Consider listing Discussion questions on the board and allowing for a few minutes of partner conversation to help start the group discussion.

**Exercise**

**The following three-dimensional right rectangular prism has dimensions $3 \times 4 \times 5$. Determine the length of $\overline{AC'}$. Show a full solution.**

***By the Pythagorean theorem, the length of $\overline{AC}$ is $5$ units long. So, triangle $ACC'$ is a right triangle whose legs both have length $5$. In other words, it is $45$-$45$-$90$ right triangle. So, $\overline{AC'}$ has length $5\sqrt{2}$.***




### Discussion (10 minutes)

Use the figure in Exercise 1 to ask follow-up questions. Allow students to share ideas before confirming answers. The questions are a preview to the properties of points, lines, and planes in three-dimensional space.

Ask the following questions regarding lines in relation to each other as a whole class. Lead students to consider whether each pair of lines is in the same or different planes.

- Would you say that $\overleftrightarrow{AB}$ is parallel to $\overleftrightarrow{A'B'}$? Why?
  - *Yes, they are parallel; since the figure is a right rectangular prism, $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$ are each perpendicular to $\overleftrightarrow{A'A'}$ and $\overleftrightarrow{B'B'}$, so $\overleftrightarrow{AB}$ must be parallel to $\overleftrightarrow{A'B'}$.*
- Do $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$ lie in the same plane?
  - *Yes, $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$ lie in the plane $ABB'A'$.*
- Consider line $\overleftrightarrow{AC}$. Will it ever run into the top face?
  - *No, because $\overleftrightarrow{AC}$ lies in the plane $ABCD$, which is a plane parallel to the top plane.*
- Can we say that the faces of the figure are parallel to each other?
  - *Not all the faces are parallel to each other; only the faces directly opposite each other are parallel.*
- Would you say that $\overleftrightarrow{BB'}$ is perpendicular to the base?
  - *We know that $\overleftrightarrow{BB'}$ is perpendicular to $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$, so we are guessing that it is also perpendicular to the base.*
- Is there any way we could draw another line through $B$ that would also be perpendicular to the base?
  - *No, because if we tried modeling $\overleftrightarrow{BB'}$ with a pencil over a piece of paper (the plane), there is only one way to make the pencil perpendicular to the plane.*

- What is the distance between the top face and the bottom face of the figure? How would you measure it?
  - *The distance can be measured by the length of $\overline{AA'}$, $\overline{BB'}$, $\overline{CC'}$, or $\overline{DD'}$.*
- Do lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{B'C'}$ meet? Would you say that they are parallel?
  - *The lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{B'C'}$ do not meet, but they do not appear to be parallel either.*
- Can lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{B'C'}$ lie in the same plane?
  - *Both lines cannot lie in the same plane.*
- Name a line that intersects plane $ABB'A'$.
  - *Answers may vary. One possible line is $\overleftrightarrow{BC}$.*
- Name a line that will not intersect plane $ABB'A'$.
  - *Answers may vary. One possible line is $\overleftrightarrow{CD}$.*

### Exploratory Challenge (18 minutes)

For each of the properties in the table below, have students draw a diagram to illustrate the property. Students will use their desktops, sheets of papers for planes, and pencils for lines to help model each property as they illustrate it. Consider dividing the class into small groups to sketch diagrams to illustrate a few of the properties, and then have students share their sketches and explanations of the properties with the class. Alternatively, if it is best to facilitate each property, lead students in the review of each property, and have them draw as you do so. By the end of the lesson, students should have completed the table at the end of the lesson with illustrations of each property.

- We are now ready to give a summary of properties of points, lines, and planes in three-dimensional space.

See the Discussion following the table for points to address as students complete the table. This could begin with questions as simple as, “What do you notice?” or “What do you predict will be true about this diagram?”

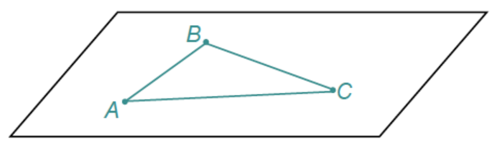
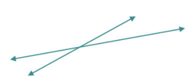
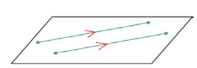
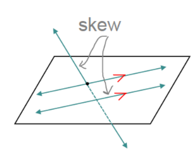
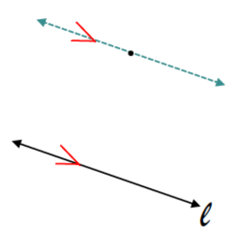
An alternative table (Table 2) is provided following the lesson. In Table 2, several images are filled in, and the description of the associated property is left blank. If the teacher elects to use Table 2, the goal will be to elicit descriptions of the property illustrated in the table.

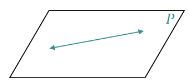
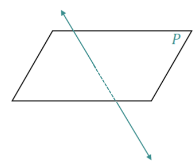
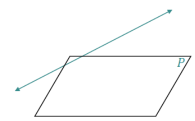
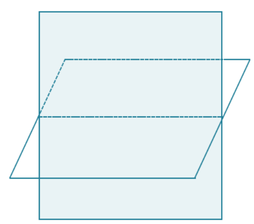
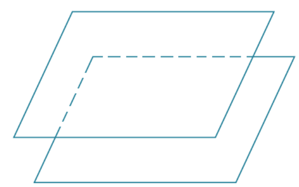
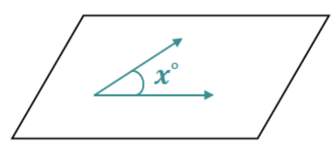
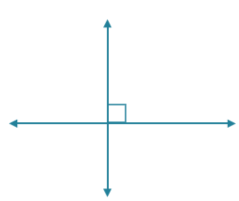
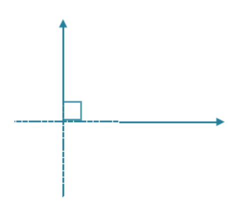
*Scaffolding:*

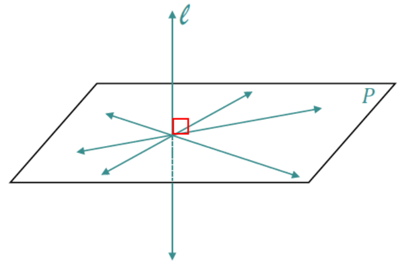
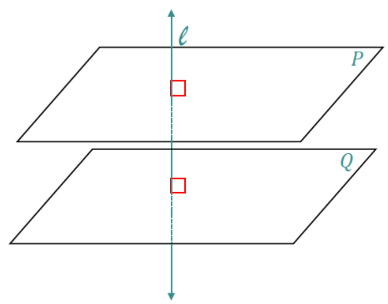
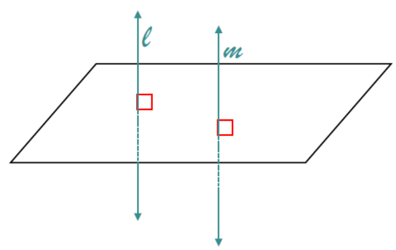
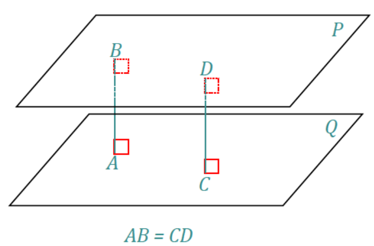
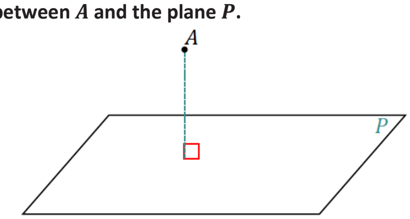
- For a class struggling with spatial reasoning, consider assigning one to two properties to a small group and having groups present their models and sketches.
- Once each group has presented, review the provided highlights of each property (see Discussion).
- For ELL students, teachers should read the descriptions of each property out loud and encourage students to rehearse the important words in each example chorally (such as point, collinear, plane, etc.).

**Exploratory Challenge**

Table 1: Properties of Points, Lines, and Planes in Three-Dimensional Space

| | Property | Diagram |
|---|---|---|
| 1 | Two points $P$ and $Q$ determine a distance $PQ$, a line segment $PQ$, a ray $PQ$, a vector $PQ$, and a line $PQ$. |   |
| 2 | Three non-collinear points $A$, $B$, and $C$ determine a plane $ABC$ and, in that plane, determine a triangle $ABC$. | Given a picture of the plane below, sketch a triangle in that plane.   |
| 3 | Two lines either meet in a single point or they do not meet. Lines that do not meet and lie in a plane are called *parallel*. Skew lines are lines that do not meet and are not parallel. | (a) Sketch two lines that meet in a single point. <br>(b) Sketch lines that do not meet and lie in the same plane; i.e., sketch parallel lines. <br>(c) Sketch a pair of skew lines.   |
| 4 | Given a line $\ell$ and a point not on $\ell$, there is a unique line through the point that is parallel to $\ell$. |   |

| | | | | |
|---|---|---|---|---|
| 5 | Given a line $\ell$ and a plane $P$, then $\ell$ lies in $P$, $\ell$ meets $P$ in a single point, or $\ell$ does not meet $P$, in which case we say $\ell$ is *parallel* to $P$. (Note: This implies that if two points lie in a plane, then the line determined by the two points is also in the plane.) | (a) Sketch a line $\ell$ that lies in plane $P$.   | (b) Sketch a line $\ell$ that meets $P$ in a single point.   | (c) Sketch a line $\ell$ that does not meet $P$; i.e., sketch a line $\ell$ parallel to $P$.   |
| 6 | Two planes either meet in a line or they do not meet, in which case we say the planes are *parallel.* | (a) Sketch two planes that meet in a line.  | (b) Sketch two planes that are parallel.  | |
| 7 | Two rays with the same vertex form an *angle.* The angle lies in a plane and can be measured by degrees. | Sketch the example in the following plane:   | | |
| 8 | Two lines are *perpendicular* if they meet, and any of the angles formed between the lines is a *right angle.* Two segments or rays are perpendicular if the lines containing them are perpendicular lines. |  |  | |

| | | |
|---|---|---|
| 9 | **A line $\ell$ is perpendicular to a plane $P$ if they meet in a single point, and the plane contains two lines that are perpendicular to $\ell$, in which case every line in $P$ that meets $\ell$ is perpendicular to $\ell$. A segment or ray is perpendicular to a plane if the line determined by the ray or segment is perpendicular to the plane.** | **Draw an example of a line that is perpendicular to a plane. Draw several lines that lie in the plane that pass through the point where the perpendicular line intersects the plane.**   |
| 10 | **Two planes perpendicular to the same line are parallel.** |   |
| 11 | **Two lines perpendicular to the same plane are parallel.** | **Sketch an example that illustrates this statement using the following plane:**   |
| 12 | **Any two line segments connecting parallel planes have the same length if they are each perpendicular to one (and hence both) of the planes.** | **Sketch an example that illustrates this statement using parallel planes $P$ and $Q$.**   |
| 13 | **The *distance between a point and a plane* is the length of the perpendicular segment from the point to the plane. The distance is defined to be zero if the point is on the plane. The *distance between two planes* is the distance from a point in one plane to the other.** | **Sketch the segment from $A$ that can be used to measure the distance between $A$ and the plane $P$.**   |

**Discussion (simultaneous with Exploratory Challenge):**

Pause periodically to discuss the properties as students complete diagrams in the table.

**Regarding Properties 1 and 2:** Make sure students understand what it means for *three points to determine a plane*. Consider having two students each hold a cotton ball (or an eraser, candy, etc.) to represent two points, and demonstrate how a sheet of paper can pass through the two points in an infinite number of ways. Note that in two-dimensional space, the fact that three points determine a plane is not a very interesting property to discuss because the plane is all that exists anyway, but in three-dimensional space this fact is quite important!

**Regarding Property 3:** Again, contrast two-dimensional versus three-dimensional context: In two dimensions, do two lines have to intersect? What do we call lines that do not intersect? How is this different in *three-dimensional space*? If two lines do *not* intersect, does the same set of possible situations exist in three dimensions as in two dimensions?

**Regarding Property 4:** Consider a model of this property in two dimensions (all modeling objects lying flat on the desk) as opposed to a model in three dimensions (say a piece of spaghetti modeling the line lays on the desk and a marble is held above it). When the parallel line through the point is found, a sheet of paper (a plane) can be passed through the two parallel lines—in two dimensions or three dimensions.

**Regarding Property 5:** In the universe of two dimensions, a line always lies in the plane. In three dimensions, we have these other situations, where the line can intersect the plane in one point or not at all, in which case it is parallel to the plane.

**Regarding Property 6:** Students may want to cut a notch in a sheet of paper so that another sheet can be placed into the notch, and the intersection can be easily observed.

**Regarding Property 9:** Consider modeling this by holding a pencil (representing a line) perpendicular to a piece of paper (representing the plane) and showing that any other standard piece of paper (such as an index card or anything with a right angle corner) fits into the angle formed by the pencil and the paper. Compare this property to the Discussion question regarding $\overleftrightarrow{BB'}$ and whether it was perpendicular to the base of the right rectangular prism.

**Regarding Properties 10 and 11:** In two dimensions, we saw a similar situation when we considered two lines perpendicular to the same line. Now we see in three dimensions where two parallel planes are perpendicular to the same line, and two parallel lines are perpendicular to the same plane.

**Regarding Property 12:** Consider a similar context in two dimensions. Imagine two parallel lines. If each line were perpendicular to one (and hence both) of the lines, would any two line segments have the same length?

**Regarding Property 13:** Again, imagine a similar situation in two dimensions: Instead of the distance between a point and a plane, imagine the distance between a line and a point not on the line.

**Closing (2 minutes)**

Ask students to summarize the key points of the lesson. Additionally, consider asking students the following questions independently in writing, to a partner, or to the whole class.

Consider sketching a property on the board and asking students to identify the property that describes the illustration.

- How did your knowledge of the properties of points and lines in the plane help you understand points and lines in three-dimensional space?

- Observe that 2 points still determine a line, 3 points determine a triangle, and that for many of the other properties, the relationship hinges on whether or not two objects (such as lines) are contained in a common plane. For example, two lines are parallel if they do not intersect and lie in a common plane.

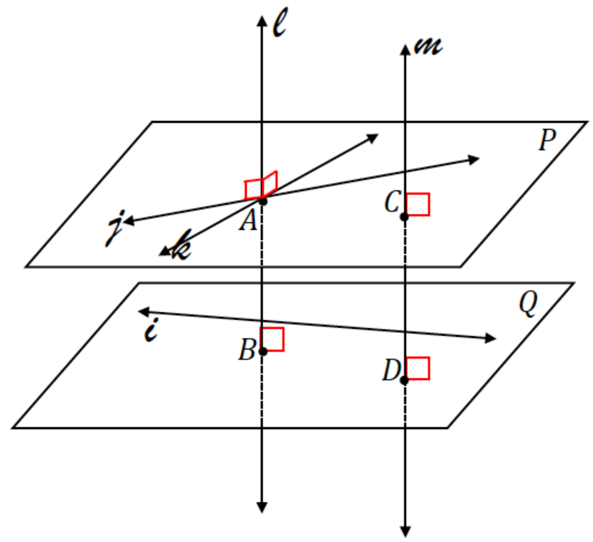
> Lesson Summary
>
> **SEGMENT: The *segment between points $A$ and $B$* is the set consisting of $A$, $B$, and all points on the line $\overleftrightarrow{AB}$ between $A$ and $B$. The segment is denoted by $\overline{AB}$, and the points $A$ and $B$ are called the *endpoints.***
>
> **LINE PERPENDICULAR TO A PLANE: A line $L$ intersecting a plane $E$ at a point $P$ is said to be *perpendicular to the plane $E$* if $L$ is perpendicular to every line that (1) lies in $E$ and (2) passes through the point $P$. A segment is said to be perpendicular to a plane if the line that contains the segment is perpendicular to the plane.**

## Exit Ticket (5 minutes)

Name ___________________________________________ Date ____________________

# Lesson 5: Three-Dimensional Space

## Exit Ticket




1. What can be concluded about the relationship between line $\ell$ and plane $P$? Why?

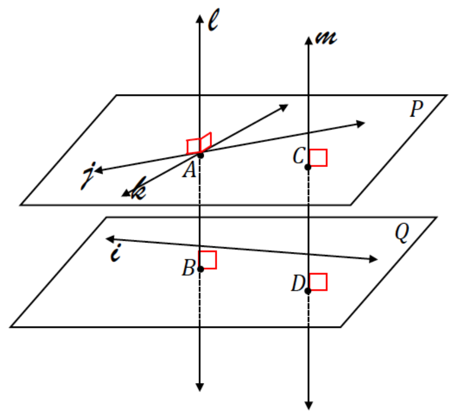
2. What can be concluded about the relationship between planes $P$ and $Q$? Why?

3. What can be concluded about the relationship between lines $\ell$ and $m$? Why?

4. What can be concluded about segments $\overline{AB}$ and $\overline{CD}$?

5. Line $j$ lies in plane $P$, and line $i$ lies in plane $Q$. What can be concluded about the relationship between lines $i$ and $j$?

## Exit Ticket Sample Solutions




1. What can be concluded about the relationship between line $\ell$ and plane $P$? Why?

   *Since line $\ell$ is perpendicular to two lines that lie in plane $P$, line $\ell$ must be perpendicular to plane $P$.*

2. What can be concluded about the relationship between planes $P$ and $Q$? Why?

   *Since line $m$ is perpendicular to both planes $P$ and $Q$, planes $P$ and $Q$ must be parallel to each other.*

3. What can be concluded about the relationship between lines $\ell$ and $m$? Why?

   *Since lines $\ell$ and $m$ are both perpendicular to both planes $P$ and $Q$, lines $\ell$ and $m$ must be parallel to each other.*

4. What can be concluded about segments $\overline{AB}$ and $\overline{CD}$?

   $AB = CD$

5. Line $j$ lies in plane $P$, and line $i$ lies in plane $Q$. What can be concluded about the relationship between lines $i$ and $j$?

   *Lines $i$ and $j$ are skew lines.*

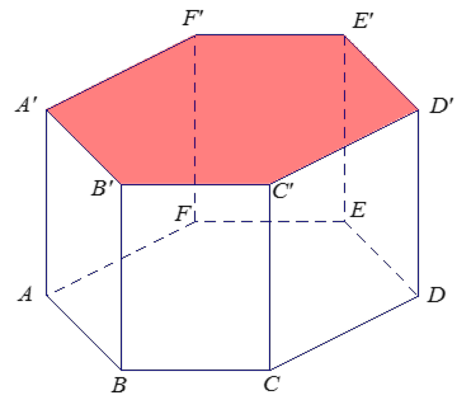
## Problem Set Sample Solutions

1. Indicate whether each statement is always true (A), sometimes true (S), or never true (N).
   a. If two lines are perpendicular to the same plane, the lines are parallel. *A*
   b. Two planes can intersect in a point. *N*
   c. Two lines parallel to the same plane are perpendicular to each other. *S*
   d. If a line meets a plane in one point, then it must pass through the plane. *A*
   e. Skew lines can lie in the same plane. *N*
   f. If two lines are parallel to the same plane, the lines are parallel. *S*
   g. If two planes are parallel to the same line, they are parallel to each other. *A*
   h. If two lines do not intersect, they are parallel. *S*

2. Consider the right hexagonal prism whose bases are regular hexagonal regions. The top and the bottom hexagonal regions are called the *base faces*, and the side rectangular regions are called the *lateral faces*.

a. List a plane that is parallel to plane $C'D'E'$.

*Plane $ABC$*




b. List all planes shown that are not parallel to plane $CDD'$.

*Students may define planes using different points. A correct sample response is shown:*

*Planes $ABB'$, $BBC'$, $DEE'$, $EFF'$, $FAA'$, $A'B'C'$, and $ABC$.*

c. Name a line perpendicular to plane $ABC$.

*$\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, $\overleftrightarrow{CC'}$, $\overleftrightarrow{DD'}$, $\overleftrightarrow{EE'}$, or $\overleftrightarrow{FF'}$.*

d. Explain why $AA' = CC'$.

*The bases of the right prism are parallel planes, which means that the lateral faces are perpendicular to the bases; hence, the lines contained in the lateral faces are perpendicular to the base planes. Any two line segments connecting parallel planes have the same length if they are each perpendicular to one (and hence both) of the planes.*

e. Is $\overleftrightarrow{AB}$ parallel to $\overleftrightarrow{DE}$? Explain.

*Yes. $\overleftrightarrow{AB}$ and $\overleftrightarrow{DE}$ lie in the same base plane and are opposite sides of a regular hexagon.*

f. Is $\overleftrightarrow{AB}$ parallel to $\overleftrightarrow{C'D'}$? Explain.

*No. The lines do not intersect; however, they are not in the same plane and are, therefore, skew.*

g. Is $\overleftrightarrow{AB}$ parallel to $\overleftrightarrow{D'E'}$? Explain.

*$\overleftrightarrow{D'E'}$ is parallel to $\overleftrightarrow{DE}$ since the lines contain opposite sides of a parallelogram (or rectangle). Together with the result of part (e), if two lines are parallel to the same line, then those two lines are also parallel. Even though the lines do not appear to be on the same plane in the given figure, there is a plane that is determined by points $A$, $B$, $E'$, and $D'$.*

h. If line segments $\overline{BC'}$ and $\overline{C'F'}$ are perpendicular, then is $\overleftrightarrow{BC}$ perpendicular to plane $C'A'F'$? Explain.

*No. For a line to be perpendicular to a plane, it must be perpendicular to two (and thus all) lines in the plane. The given information only provides one pair of perpendicular lines.*

i. One of the following statements is false. Identify which statement is false and explain why.

(i) $\overleftrightarrow{BB'}$ is perpendicular to $\overleftrightarrow{B'C'}$.

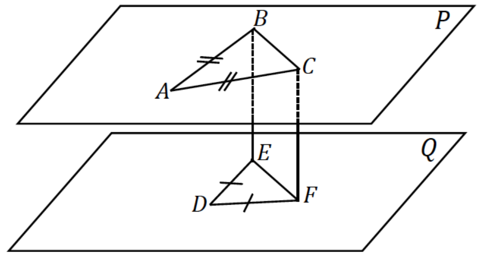
(ii) $\overleftrightarrow{EE'}$ is perpendicular to $\overleftrightarrow{EF}$.

(iii) $\overleftrightarrow{CC'}$ is perpendicular to $\overleftrightarrow{E'F'}$.

(iv) $\overleftrightarrow{BC}$ is parallel to $\overleftrightarrow{F'E'}$.

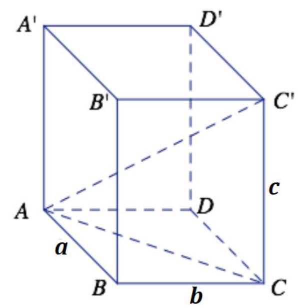
*Statement (iii) is incorrect because even though $\overleftrightarrow{CC'}$ and $\overleftrightarrow{E'F'}$ lie in perpendicular planes, the lines do not intersect, so the lines are skew.*

3. In the following figure, $\triangle ABC$ is in plane $P$, $\triangle DEF$ is in plane $Q$, and $BCFE$ is a rectangle. Which of the following statements are true?




   a. $\overline{BE}$ is perpendicular to plane $Q$. *True*
   b. $BF = CE$. *True*
   c. Plane $P$ is parallel to plane $Q$. *True*
   d. $\triangle ABC \cong \triangle DEF$. *False*
   e. $AE = AF$. *True*

4. Challenge: The following three-dimensional right rectangular prism has dimensions $a \times b \times c$. Determine the length of $\overline{AC'}$.




   *By the Pythagorean theorem, the length of $\overline{AC}$ is $\sqrt{a^2+b^2}$ units long.*

   *$\triangle ACC'$ is a right triangle with $AC = \sqrt{a^2+b^2}$ and $CC' = 5$.*

   *Then,*

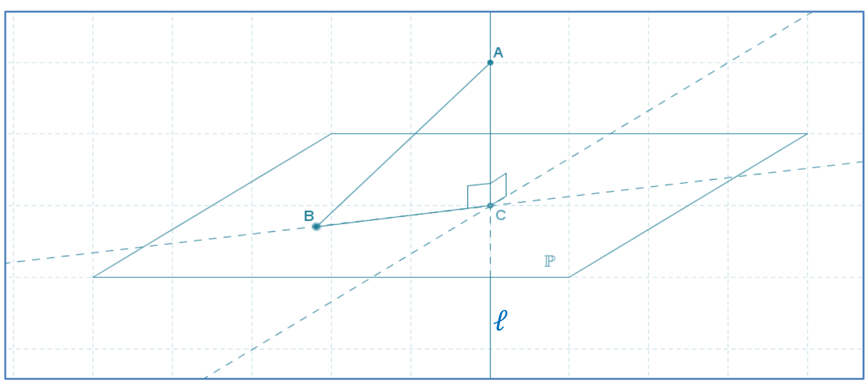
   $$\overline{AC'} = \sqrt{\left(\sqrt{a^2+b^2}\right)^2 + c^2}$$

   *or*

   $$\overline{AC'} = \sqrt{a^2+b^2+c^2}.$$

5. A line $\ell$ is perpendicular to plane $P$. The line and plane meet at point $C$. If $A$ is a point on $\ell$ different from $C$, and $B$ is a point on $P$ different from $C$, show that $AC < AB$.

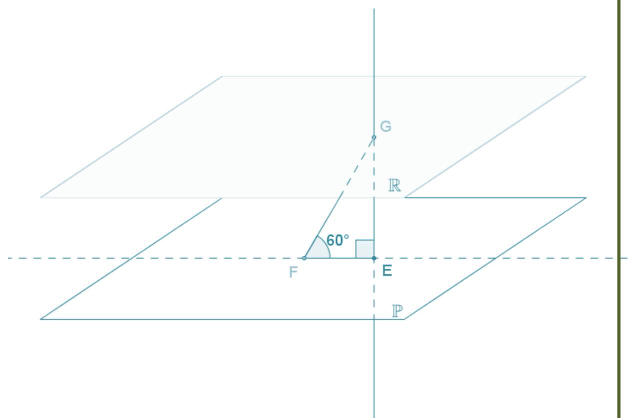
   *Consider $\triangle ABC$. Since $\ell$ is perpendicular to $P$, $\angle ACB$ is a right angle no matter where $B$ and $C$ lie on the plane. So, $\overline{AB}$ is the hypotenuse of a right triangle, and $\overline{AC}$ is a leg of the right triangle. By the Pythagorean theorem, the length of either leg of a right triangle is less than the length of the hypotenuse. Thus, $AC < AB$.*



6. Given two distinct parallel planes $P$ and $R$, $\overleftrightarrow{EF}$ in $P$ with $EF = 5$, point $G$ in $R$, $m\angle GEF = 9°$, and $m\angle EFG = 60°$, find the minimum and maximum distances between planes $P$ and $R$, and explain why the actual distance is unknown.

*Triangle $EFG$ is a $30$-$60$-$90$ triangle with its short leg of length $5$ units. The length of the longer leg must be $\sqrt{3}$, or approximately $8.7$. The maximum distance between the planes is approximately $8.7$.*

*If plane $EFG$ is perpendicular to plane $P$, then the distance between the planes is equal to the length of the longer leg of the right triangle. If plane $EFG$ is not perpendicular to plane $P$ (and plane $R$), then the distance between the planes must be less than $5\sqrt{3}$. Furthermore, the distance between the distinct parallel planes must be greater than zero, or the planes would coincide.*




*The actual distance is unknown because the plane determined by points $E$, $F$, and $G$ may or may not be perpendicular to planes $P$ and $R$.*

7. The diagram below shows a right rectangular prism determined by vertices $A$, $B$, $C$, $D$, $E$, $F$, $G$, and $H$. Square $ABCD$ has sides with length $5$, and $AE = 9$. Find $DF$.

*The adjacent faces on a right rectangular prism are perpendicular, so angle $AEF$ must then be a right angle.*

*Using the Pythagorean theorem:*

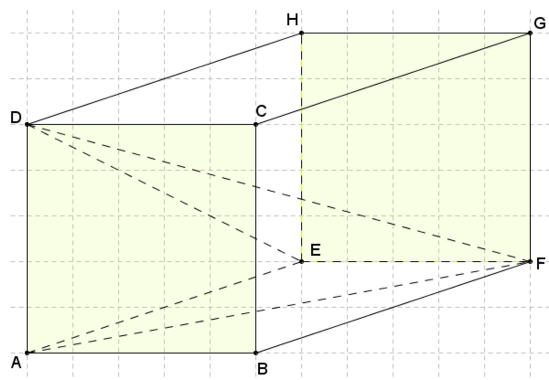
$AF^2 = AE^2 + EF^2$

$AF^2 = 9^2 + 5^2$

$AF^2 = 81 + 25$

$AF^2 = 106$

$AF = \sqrt{106}$




*$\overline{DA}$ is perpendicular to plane $AEFB$ since it is perpendicular to both $\overline{AB}$ and $\overline{AE}$; therefore, it is perpendicular to all lines in plane $AEFB$. Then triangle $AFD$ is a right triangle with legs of length $AD = 5$ and $AF = \sqrt{106}$.*

*Using the Pythagorean theorem:*

$DF^2 = DA^2 + AF^2$

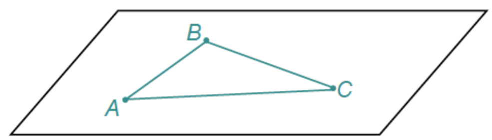
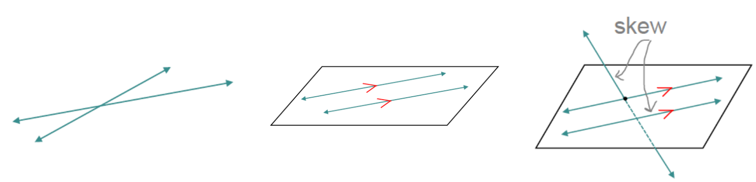
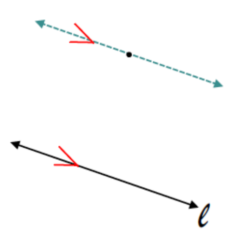
$DF^2 = 5^2 + \left(\sqrt{106}\right)^2$

$DF^2 = 25 + 106$

$DF^2 = 131$

$DF = \sqrt{131} \approx 11.4$ *The distance $DF$ is approximately $11.4$.*

**Table 2: Properties of Points, Lines, and Planes in Three-Dimensional Space**

| | Property | Diagram |
|---|---|---|
| **1** | |  |
| **2** | |  |
| **3** | | **(a)** **(b)** **(c)**  |
| **4** | |  |

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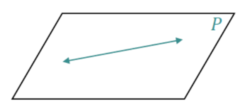
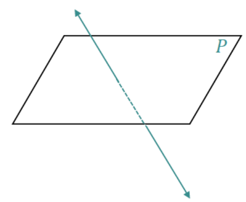
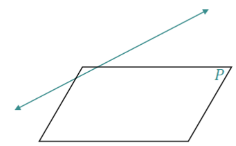
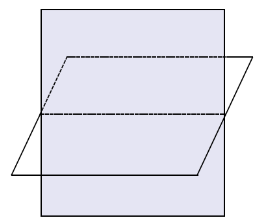
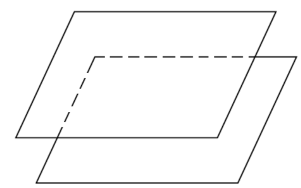
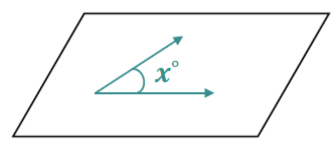
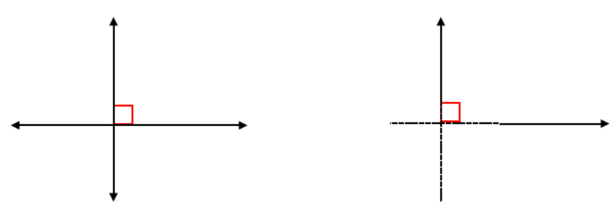
<tr><td>5</td><td></td><td>(a)<br></td><td>(b)<br></td><td>(c)<br></td></tr>
<tr><td>6</td><td></td><td colspan="3">(a)<br><br>(b)<br></td></tr>
<tr><td>7</td><td></td><td colspan="3"></td></tr>
<tr><td>8</td><td></td><td colspan="3"></td></tr>
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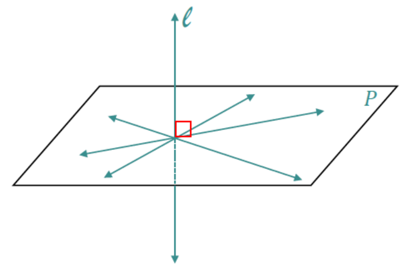
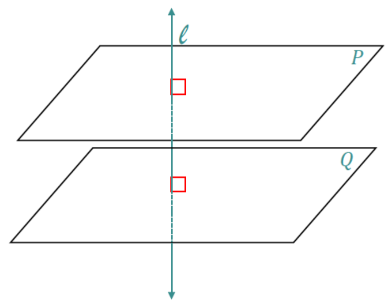
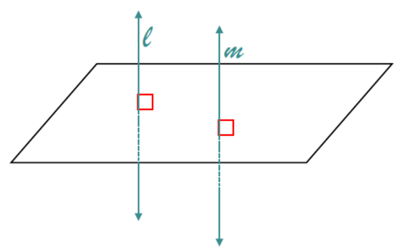
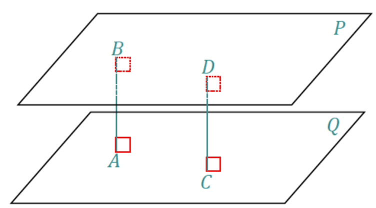
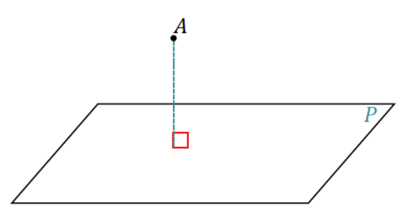
| | | |
|---|---|---|
| 9 | |   |
| 10 | |   |
| 11 | |   |
| 12 | |   $AB = CD$ |
| 13 | |   |

# Lesson 6: General Prisms and Cylinders and Their Cross-Sections

## Student Outcomes

- Students understand the definitions of a *general prism* and a *cylinder* and the distinction between a *cross-section* and a *slice.*

## Lesson Notes

In Lesson 6, students are reintroduced to several solids as a lead into establishing the volume formulas for the figures (**G-GMD.A.1**). They begin with familiar territory, reexamine a *right rectangular prism,* and generalize into the idea of *general cylinders.* Students should feel comfortable with the hierarchy of figures by the close of the lesson, aided by the provided graphic organizer or chart. Students are asked to unpack formal definitions with sketches. Toward the close of the lesson, students learn the difference between a slice and a cross-section and identify *two-dimensional cross-sections of three-dimensional objects,* as well as identify the three-dimensional object generated by the *rotation of a rectangular region about an axis* (**G-GMD.B.4**). Teachers may choose to plan the lesson to accommodate the included Extension, where students use cross-sections to establish why the bases of general cylinders are *congruent* to each other. This is important to the upcoming work on Cavalieri's principle in Lesson 10.

## Classwork

### Opening Exercise (3 minutes)

> **Opening Exercise**
>
> **Sketch a right rectangular prism.**
>
> ***Sketches may vary. Note whether students use dotted lines to show hidden edges, and ask students with sketches showing no hidden edges to compare images with students who do have hidden edges shown.***

- Is a right rectangular prism hollow? That is, does it include the points inside?

Allow students to share thoughts, and confirm the correct answer in the following Discussion.

> *Scaffolding:*
>
> - For struggling learners unfamiliar with the term *right rectangular prism*, rephrase the prompt to say, "Sketch a box."
> - As an additional step for advanced learners, ask them to sketch a cylinder and to observe similarities and differences in the structures of the two figures.

### Discussion (12 minutes)

- In your study of right rectangular prisms in Grade 6, Module 5 (see the Module Overview), you have examined their properties, interpreted their volume, and studied slices. Let us take a moment to review how we precisely define a *right rectangular prism*.

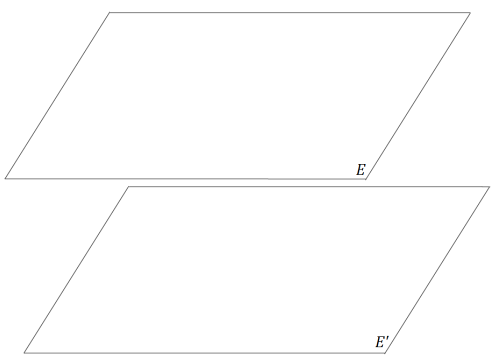
RIGHT RECTANGULAR PRISM: Let $E$ and $E'$ be two parallel planes. Let $B$ be a rectangular region[1] in the plane $E$. At each point $P$ of $B$, consider the segment $\overline{PP'}$ perpendicular to $E$, joining $P$ to a point $P'$ of the plane $E'$. The union of all these segments is called a *right rectangular prism*.

Allow students time to work in partners to unpack the definition by attempting to sketch what is described by the definition. Consider projecting or rewriting the definition in four numbered steps to structure students' sketches:

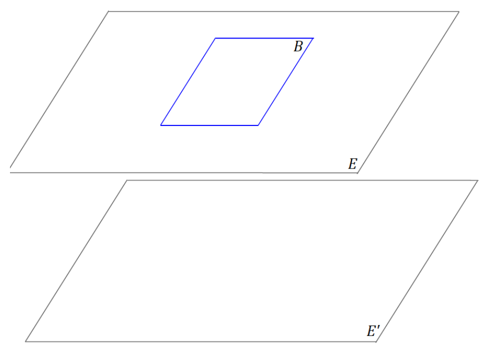
**RIGHT RECTANGULAR PRISM: 1. Let $E$ and $E'$ be two parallel planes. 2. Let $B$ be a rectangular region in the plane $E$. 3. At each point $P$ of $B$, consider the segment $\overline{PP'}$ perpendicular to $E$, joining $P$ to a point $P'$ of the plane $E'$. 4. The union of all these segments is called a *right rectangular prism*.**

At Step 3, tell students that the regions $B$ and $B'$ are called the *base faces* (or just *bases*) of the prism. Then walk around the room and ask pairs to show one example of $\overline{PP'}$. Make sure the whole class agrees what this means and looks like before students show a few more examples of segments to model Step 4.

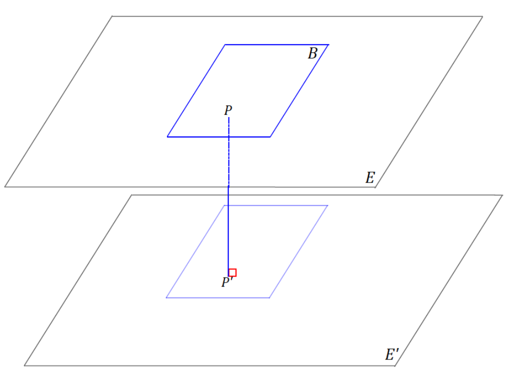
**Step 1**



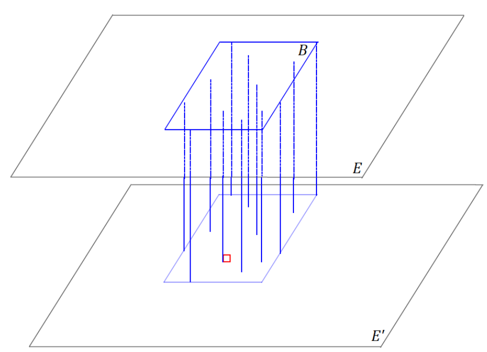

**Step 2**




**Step 3**




**Step 4**

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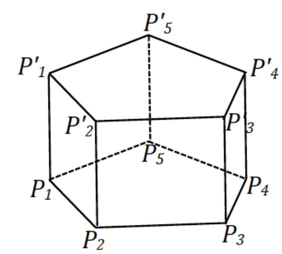
[1] (Fill in the blank.) A rectangular region is the union of a rectangle and <u>*its interior*</u>.

Alternatively, students can build a 3D model based on the definition. Consider providing partner pairs or small groups with a box of angel hair pasta to model the step-by-step process, using the uncooked pasta itself in addition to the box (the box represents the overall frame of the prism, each piece of pasta represents the segment joining the two base regions). Ask students to describe what each part of the model represents: Each piece of paper represents $E$ and $E'$, the intersection of the box and the papers represents $B$ and $B'$, and each piece of pasta represents the segment $\overline{PP'}$. It may be worth gluing pasta along the outside of the box for visual emphasis.

Use the figure to the right to review the terms *edge* and *lateral face of a prism* with students.




- Look at segments $\overline{P_1P_2}$ and $\overline{P_1{}'P_2{}'}$. If we take these 2 segments together with all of the vertical segments joining them, we get a lateral face. Similarly, the segment joining $P_1$ to $P_2$ is called a *lateral edge*.

After discussing edge and lateral face, the discussion shifts to general cylinders. Prior to this Geometry course, general cylinders are first addressed in Grade 8, Module 5, Lesson 10.

- We will define a more general term under which a right rectangular prism is categorized.

**GENERAL CYLINDER: (See Figure 1.) Let $E$ and $E'$ be two parallel planes, let $B$ be a region[2] in the plane $E$, and let $L$ be a line which intersects $E$ and $E'$ but not $B$. At each point $P$ of $B$, consider the segment $\overline{PP'}$ parallel to $L$, joining $P$ to a point $P'$ of the plane $E'$. The union of all these segments is called a *general cylinder with base $B$*.**

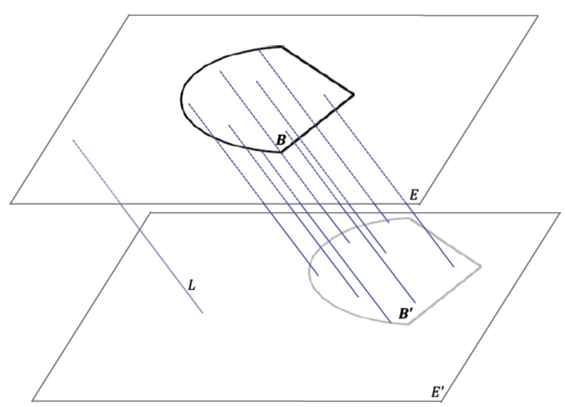



**Figure 1**

Have students discuss the following question in partner pairs:

- Compare the definitions of *right rectangular prism* and *general cylinder*. Are they very different? What is the difference?
  - *The definitions are not very different. In the definition of a right rectangular prism, the region $B$ is a rectangular region. In the definition of a general cylinder, the shape of $B$ is not specified.*
- As the region $B$ is not specified in the definition of general cylinder, we should understand that to mean that the base can take on a polygonal shape, a curved shape, an irregular shape, etc.

**[2] In Grade 8, a *region* refers to a *polygonal region* (triangle, quadrilateral, pentagon, and hexagon) or a *circular region,* or regions that can be decomposed into such regions.**

In most calculus courses, we usually drop the word *general* from the name and just talk about cylinders to refer to all types of bases, circular or not.

- Notice that in a general cylinder, at each point $P$ of $B$, the segment $\overline{PP'}$ is not required to be perpendicular to the base planes. When the segments $\overline{PP'}$ are not perpendicular to the base, the resulting solid is called *oblique* (slanted). Solids where the segments $\overline{PP'}$ are perpendicular to the base planes are categorized as *right* (i.e., as in how it is used for *right* rectangular prism).
- Another way of saying the same thing is to say that if the lateral edges of a general cylinder are perpendicular to the base, the figure is a *right* figure; if the lateral edges are not perpendicular to the base, the figure is *oblique.*
- A general cylinder is qualified and named by its base. If the base is a polygonal region, then the general cylinder is usually called a *prism*.
- A general cylinder with a disk (circle) for a base is called a *circular cylinder*. We will continue to use the term *cylinder* to refer to circular cylinder as was done at the elementary level and use *general cylinder* to specify when the base region is a general region.

## Exploratory Challenge (15 minutes)

Teachers may complete this exploration in one of three ways. (1) Use the following series of questions to help guide students into filling out a blank graphic organizer (found at the close of the lesson) on general cylinders. (2) Have students draw a sketch based on the description of each figure in the chart found at the close of the lesson. (3) Have students fill in the description of each figure in the chart found at the close of the lesson.

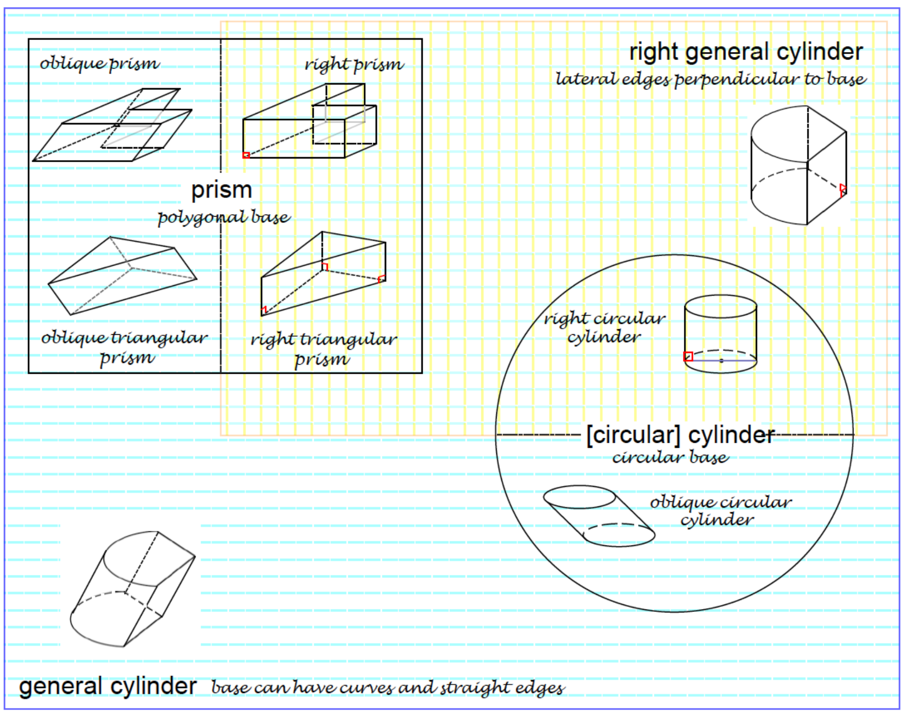
**Option 1.** Students will fill in the graphic organizer with any relevant examples per category; the following completed model is a mere model and is not the solution. Ask the following questions as they complete the graphic organizer to help them distinguish how the different types of general cylinders are related to each other.

- Draw an example for each category in the graphic organizer. Write down the qualifiers of each subcategory as shown in the example graphic organizer.
- What is the term that has the broadest meaning in this graphic organizer? What does it imply about the other terms listed on the sheet?
    - *The broadest term is general cylinder, and since the other terms are smaller sections of the sheet, they are subcategories of general cylinder.*
- What are the other subcategories of the general cylinder listed on the sheet?
    - *The subcategories are right general cylinder, circular cylinder (right and oblique), and prism (right and oblique).*
- What are major distinguishing properties between a general cylinder and its subcategories?
    - *A general cylinder with a polygonal base is called a prism.*
    - *A general cylinder with a circular base is called a cylinder.*
    - *A general cylinder with lateral edges perpendicular to the base is a right general cylinder.*
    - *A general cylinder with lateral edges not perpendicular to the base is an oblique general cylinder.*
- What do you know about the shape of the base of a general cylinder?
    - *It can be curved or have straight edges or both, or it can be irregular.*

Have students draw an example of a general cylinder; share the model's example if needed. Have students write down a brief descriptor for a general cylinder; for example, "A base can have curves and straight edges." The example should be oblique since there is a separate space to draw right general cylinders. Ask students to check their neighbor's drawing and walk around the room to ensure that students are on track.

Next, have students draw an example of a right general cylinder. Consider asking them to use the same base as used for their general cylinder but to now make it a *right* general cylinder. Ensure that they write a descriptor to qualify the significance of the subcategory.

Then move onto the prism and circular cylinder subcategories. Note that the model shows two sets of examples for the prism subcategory. This is to illustrate that a polygonal base can mean something with a basic shape for a base, such as a triangle, or it can be a composite shape, such as the top two images in the prism subcategory.




### Discussion (3 minutes)

Slices, when a plane intersects with a solid, are first discussed in Grade 7, Module 6, Topic C.

- What is a *cross-section* of a solid?
    - *Students may describe a cross-section as a slice. Accept reasonable responses, and confirm the following answer.*
- We describe a *cross-section* of a general cylinder as the intersection of the general cylinder with a plane parallel to the plane of the base.

**Discussion**

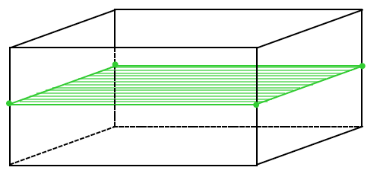


**Figure 2**

Example of a cross-section of a prism, where the intersection of a plane with the solid is parallel to the base.

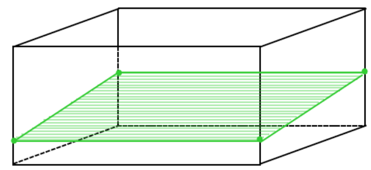


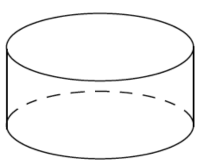
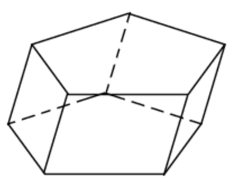
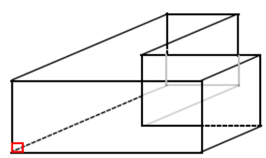
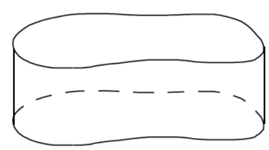
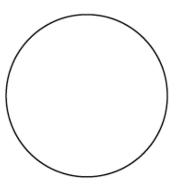
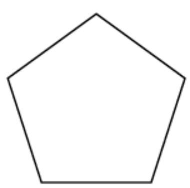
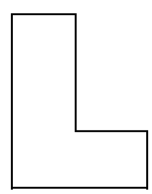
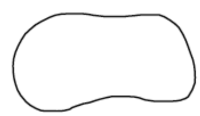
**Figure 3**

A general intersection of a plane with a prism, which is sometimes referred to as a slice.

## Exercise (5 minutes)

**Exercise**

**Sketch the cross-section for the following figures:**

| **a.** | **b.** | **c.** | **d.** |
|---|---|---|---|
|  |  |  |  |
|  |  |  |  |

Ask students to draw the cross-section of each figure in their graphic organizer or chart as part of their homework.

Provided any remaining time, continue with a brief discussion on how a cylinder can be generated from rotating a rectangle.

- We close with the idea of, not a cross-section, but in a way, a slice of a figure. What would happen if a rectangle were rotated about one of its sides? What figure would be outlined by this rotation?

Model what this looks like by taping an edge of a rectangular piece of paper (or even an index card) to a pencil and spinning the pencil between the palms of your hands. Students should see that the rotation sweeps out a cylinder. This will prepare students for Problem Set questions 6(a) and 6(b).

## Extension

The following Extension prepares students for the informal argument regarding Cavalieri's Principle in Lesson 10.

- Consider the following general cylinder in Figure 4 and the marked cross-section. Does the cross-section have any relationship with the base of the prism?

**Extension**

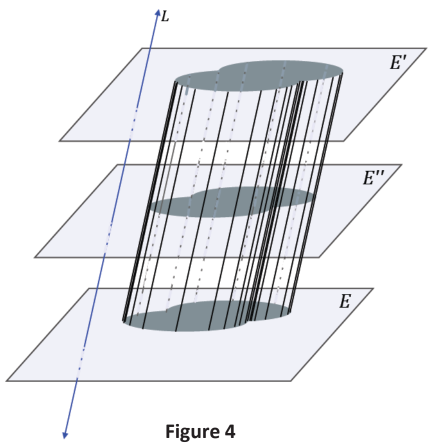



**Figure 4**

- They look like they are congruent.
- Let us make the claim that all cross-sections of a general cylinder are congruent to the base. How can we show this to be true?

Allow students time to discuss with a partner how they could demonstrate this (i.e., a rough argument). How can we use what we know about the base regions being congruent to show that a cross-section is congruent to its respective base? Review the following argument after students have attempted the informal proof and shared their ideas:

- Take a plane $E''$ between $E$ and $E'$ so that it is parallel to both.
- The top portion of the cylinder is another cylinder and, hence, has congruent bases.
- Thus, the cross-section lying in $E''$ is congruent to both of the bases.

Consider modeling this idea using a deck of playing cards or a stack of coins. Will a cross-section of either group, whether stacked perpendicularly or skewed, be congruent to a base?

For triangular prisms, we can make the argument more precise.

- How have we determined whether two triangles are congruent or not in the past? What do we know about the parts of each of the triangles in the image, and what more do we need to know?

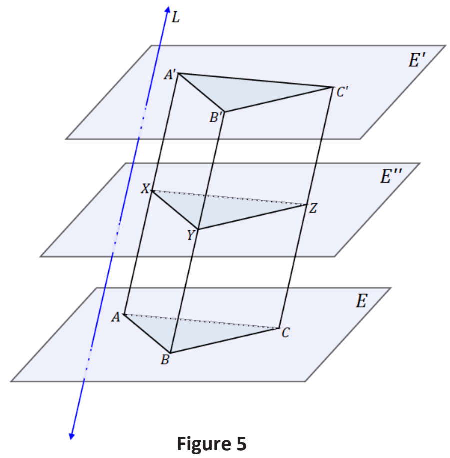



Figure 5

Review the following argument after students have attempted the informal proof and shared their ideas:

- As we know from earlier in the lesson, a prism is the totality of all segments $\overline{PP'}$ parallel to line $L$ from each point $P$ from the base region joining $P$ to a point $P'$ of the plane $E'$.
- Points $X$, $Y$, and $Z$ are the points where $E''$ intersects $\overline{AA'}$, $\overline{BB'}$, and $\overline{CC'}$.

  Then $\overline{AX} \parallel \overline{BY}$ because both segments are parallel to line $L$.

  Also, $\overline{AB} \parallel \overline{XY}$ since lateral face $ABB'A$ intersects parallel planes (i.e., the lateral face intersects parallel planes $E$ and $E''$); the intersection of a plane with two parallel planes is two parallel lines.
- We can then conclude that $ABYX$ is a parallelogram.

  Therefore, $AB = XY$.
- We can make similar arguments to show $BC = YZ$, and $AC = XZ$.

  By SSS, $\triangle ABC \cong \triangle XYZ$.
- How does this argument allow us to prove that any prism, no matter what polygon the base is, has cross-sections congruent to the base?
  - *We can decompose the base into triangles and use those triangles to decompose the prism into triangular prisms.*

### Closing (2 minutes)

Ask students to summarize the key points of the lesson. Additionally, consider asking students the following questions independently in writing, to a partner, or to the whole class.

- Describe how oblique and right prisms and oblique and right cylinders are related to general cylinders. What distinguishes prisms and cylinders from general cylinders?
- What is a cross-section (as opposed to a slice)?

**Lesson Summary**

**RIGHT RECTANGULAR PRISM: Let $E$ and $E'$ be two parallel planes. Let $B$ be a rectangular region in the plane $E$. At each point $P$ of $B$, consider the segment $\overline{PP'}$ perpendicular to $E$, joining $P$ to a point $P'$ of the plane $E'$. The union of all these segments is called a *right rectangular prism*.**

**LATERAL EDGE AND FACE OF A PRISM: Suppose the base $B$ of a prism is a polygonal region and $P_i$ is a vertex of $B$. Let $P'_i$ be the corresponding point in $B'$ such that $\overline{P_iP'_i}$ is parallel to the line $L$ defining the prism. The segment $\overline{P_iP'_i}$ is called a *lateral edge of the prism*. If $\overline{P_iP_{i+1}}$ is a base edge of the base $B$ (a side of $B$), and $F$ is the union of all segments $\overline{PP'}$ parallel to $L$ for which $P$ is in $\overline{P_iP_{i+1}}$ and $P'$ is in $B'$, then $F$ is a *lateral face of the prism*. It can be shown that a lateral face of a prism is always a region enclosed by a parallelogram.**

**GENERAL CYLINDER: Let $E$ and $E'$ be two parallel planes, let $B$ be a region in the plane $E$, and let $L$ be a line which intersects $E$ and $E'$ but not $B$. At each point $P$ of $B$, consider the segment $\overline{PP'}$ parallel to $L$, joining $P$ to a point $P'$ of the plane $E'$. The union of all these segments is called a *general cylinder with base $B$*.**

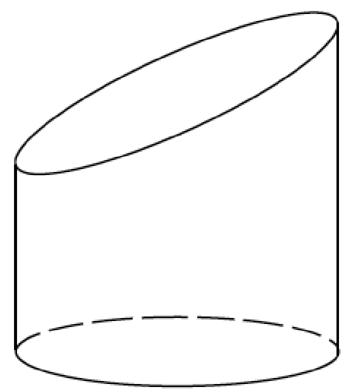
## Exit Ticket (5 minutes)

Name ___________________________________________ Date ____________________

# Lesson 6: General Prisms and Cylinders and Their Cross-Sections

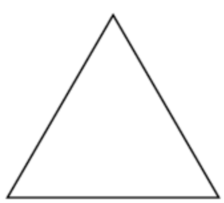
## Exit Ticket

1. Is this a cylinder? Explain why or why not.

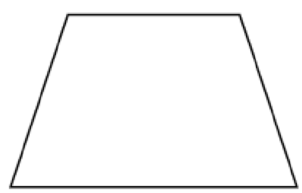


2. For each of the following cross-sections, sketch the figure from which the cross-section was taken.

a.



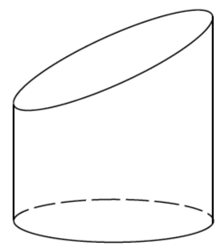
b.



## Exit Ticket Sample Solutions

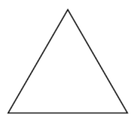
1. Is this a cylinder? Explain why or why not.

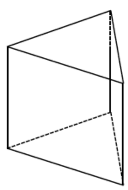
   *The figure is not a cylinder because the bases are not parallel to each other.*



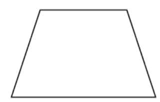
2. For each of the following cross-sections, sketch the figure from which the cross-section was taken.

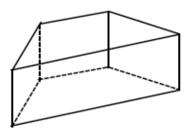
   a.





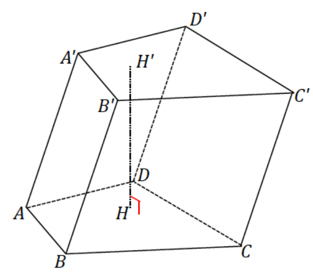
   b.





## Problem Set Sample Solutions

1. Complete each statement below by filling in the missing term(s).




   a. The following prism is called a(n) ______ prism.

   *Oblique*

   b. If $\overline{AA'}$ were perpendicular to the plane of the base, then the prism would be called a(n) ______ prism.

   *Right*

   c. The regions $ABCD$ and $A'B'C'D'$ are called the ______ of the prism.

   *Bases*

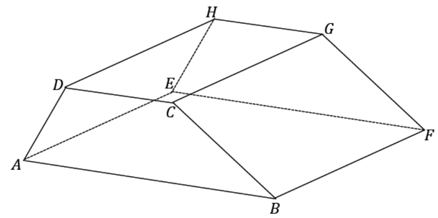
   d. $\overline{AA'}$ is called a(n) ______.

   *Edge*

   e. Parallelogram region $BB'C'C$ is one of four______ ______.

   *Lateral faces*

2. The following right prism has trapezoidal base regions; it is a right trapezoidal prism. The lengths of the parallel edges of the base are $5$ and $8$, and the nonparallel edges are $4$ and $6$; the height of the trapezoid is $3.7$. The lateral edge length $DH$ is $10$. Find the surface area of the prism.




$\text{Area(bases)} = 2 \times \left(\frac{5+8}{2}\right)(3.7) = 48.1$

$\text{Area}(DEGH) = 5(10) = 50$

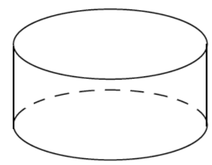
$\text{Area}(BCGF) = 6(10) = 60$

$\text{Area}(ABFE) = 8(10) = 80$

$\text{Area}(ADHE) = 4(10) = 40$

$\text{Total Surface Area} = 48.1 + 50 + 60 + 80 + 40$
$= 278.1$

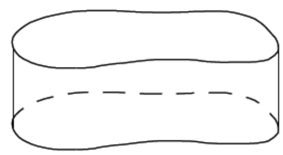
3. The base of the following right cylinder has a circumference of $5\pi$ and a lateral edge of $8$. What is the radius of the base? What is the lateral area of the right cylinder?



*The radius of the base is* $2.5$.

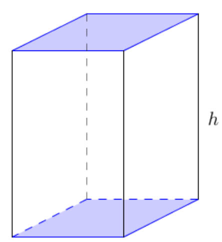
*The lateral area is* $5\pi(8) = 40\pi$.

4. The following right general cylinder has a lateral edge of length $8$, and the perimeter of its base is $27$. What is the lateral area of the right general cylinder?



*The lateral area is* $27(8) = 216$.

5. A right prism has base area $5$ and volume $30$. Find the prism's height, $h$.

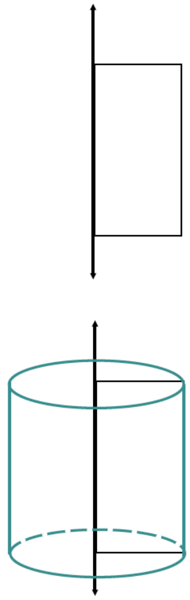



$\text{Volume} = (\text{area of base}) \times (\text{height})$
$30 = (5)h$
$6 = h$

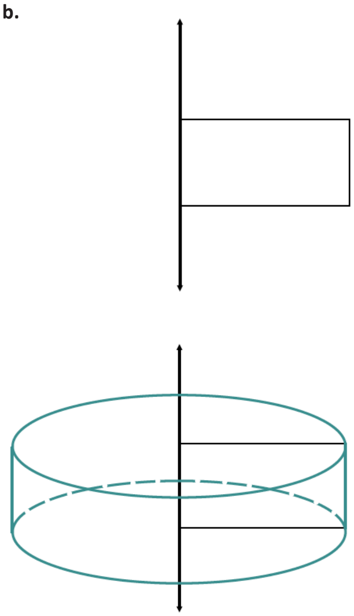
*The height of the prism is* $6$.

6. Sketch the figures formed if the rectangular regions are rotated around the provided axis.

a.



b.



7. A cross-section is taken parallel to the bases of a general cylinder and has an area of $18$. If the height of the cylinder is $h$, what is the volume of the cylinder? Explain your reasoning.

*If the cross-section is parallel to the bases of the cylinder, then it is congruent to the bases; thus, the area of the base of the cylinder is $18$. The volume of a general cylinder is the product of the area of the cylinder's base times the height of the cylinder, so the volume of the general cylinder is $18h$.*

8. A general cylinder has a volume of $144$. What is one possible set of dimensions of the base and height of the cylinder if all cross-sections parallel to its bases are …

a. Rectangles?

*Answers will vary.*

$\textbf{Volume} = (\textbf{Area of base}) \times (\textbf{height})$
$\textbf{Volume} = \mathbf{144}$
$\textbf{Volume} = (\mathbf{12})(\mathbf{12})$
$\textbf{Volume} = (\mathbf{4 \cdot 3})(\mathbf{12})$

*The base of the cylinder (rectangular prism) could be $4 \times 3$, and the cylinder could have a height of $12$.*

b. Right triangles?

*Answers will vary.*

$\textbf{Volume} = (\textbf{area of base}) \times (\textbf{height})$
$\textbf{Volume} = \mathbf{144}$
$\textbf{Volume} = (\mathbf{12})(\mathbf{12})$
$\textbf{Volume} = \frac{\mathbf{1}}{\mathbf{2}}(\mathbf{6 \cdot 4})(\mathbf{12})$

*The base of the cylinder (triangular prism) could be a right triangle with legs of length $6$ and $4$, and the cylinder could have a height of $12$.*

c. Circles?

*Answers will vary.*

$\text{Volume} = (\text{area of base}) \times (\text{height})$
$\text{Volume} = 144$
$\text{Volume} = (12)(12)$

$$\text{Volume} = \left(\pi\left(\sqrt{\frac{12}{\pi}}\right)^2\right) \times (12)$$

*The base of the cylinder (circular cylinder) could have a radius of* $\sqrt{\frac{12}{\pi}}$*, and the cylinder could have a height of* $12$*.*

9. A general hexagonal prism is given. If $P$ is a plane that is parallel to the planes containing the base faces of the prism, how does $P$ meet the prism?

*If $P$ is between the planes containing the base faces, then $P$ meets the prism in a hexagonal region that is congruent to the bases of the prism; otherwise, $P$ does not meet the prism.*

10. Two right prisms have similar bases. The first prism has height $5$ and volume $100$. A side on the base of the first prism has length $2$, and the corresponding side on the base of the second prism has length $3$. If the height of the second prism is $6$, what is its volume?

*The scale factor of the base of the second prism is* $\frac{3}{2}$*, so its area is* $\left(\frac{3}{2}\right)^2$ *, the area of the base of the first prism.*

$\text{Volume} = (\text{Area of base}) \times (\text{height})$
$100 = (\text{Area of base}) \times (5)$
$\text{Area of base} = 20$

*The area of the base of the first prism is* $20$*.*

*The area of the base of the second prism is then* $\left(\frac{3}{2}\right)^2 (20) = 45$*.*

$\text{Volume} = (\text{Area of base}) \times (\text{height})$
$\text{Volume} = (45) \times (6)$
$\text{Volume} = 270$

*The volume of the second prism is* $270$*.*

11. **A tank is the shape of a right rectangular prism with base $2$ ft. $\times$ $2$ ft. and height $8$ ft. The tank is filled with water to a depth of $6$ ft. A person of height $6$ ft. jumps in and stands on the bottom. About how many inches will the water be over the person's head? Make reasonable assumptions.**

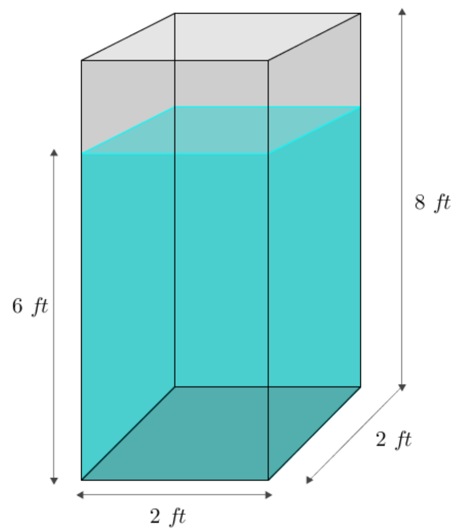
*Model the human as a right cylinder with height $6$ ft. and base area $\frac{1}{2}$ $\text{ft}^2$. The volume of the human is then $3$ $\text{ft}^3$.*

*The depth of the water will be increased as the human displaces a volume of $3$ $\text{ft}^3$ of the water in the tank.*

*Let $x$ represent the increase in depth of the water in feet.*

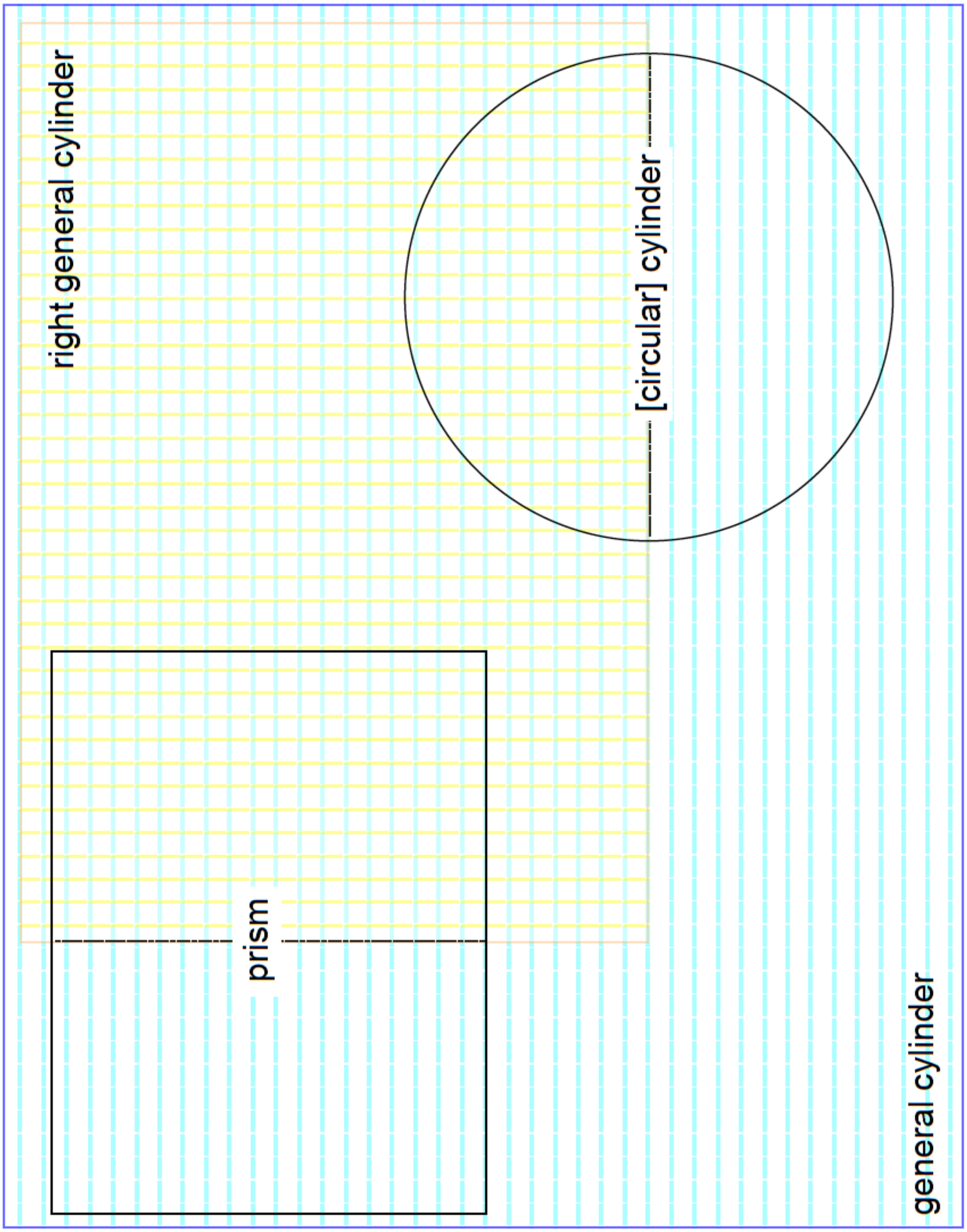
$$\text{Volume} = (\text{area of base}) \times (\text{height})$$
$$3\ \text{ft}^3 = (4\ \text{ft}^2)(x)$$
$$\frac{3}{4}\ \text{ft.} = x$$

*The water will rise by $\frac{3}{4}$ ft. $= 9$ in., so the water will be approximately $9$ in. over the human's head.*



## Exploratory Challenge

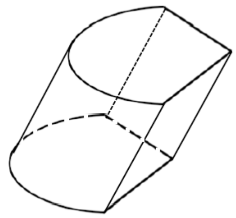
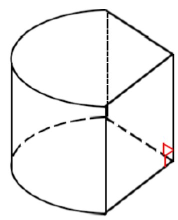
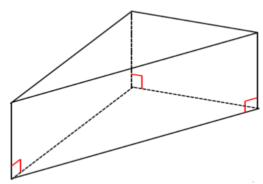
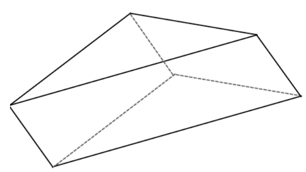
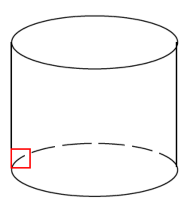
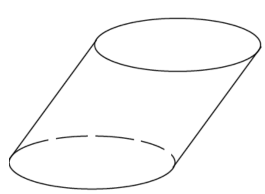
**Option 1**



**Option 2**

| | Figure and Description | Sketch of Figure | Sketch of Cross-Section |
|---|---|---|---|
| 1. | **General Cylinder**<br>Let $E$ and $E'$ be two parallel planes, let $B$ be a region in the plane $E$, and let $L$ be a line which intersects $E$ and $E'$ but not $B$. At each point $P$ of $B$, consider the segment $\overline{PP'}$ parallel to $L$, joining $P$ to a point $P'$ of the plane $E'$. The union of all these segments is called a *general cylinder with base* $B$. | | |
| 2. | **Right General Cylinder**<br>A general cylinder whose lateral edges are perpendicular to the bases. | | |
| 3. | **Right Prism**<br>A general cylinder whose lateral edges are perpendicular to a polygonal base. | | |
| 4. | **Oblique Prism**<br>A general cylinder whose lateral edges are not perpendicular to a polygonal base. | | |
| 5. | **Right Cylinder**<br>A general cylinder whose lateral edges are perpendicular to a circular base. | | |
| 6. | **Oblique Cylinder**<br>A general cylinder whose lateral edges are not perpendicular to a circular base. | | |

**Option 3**

| | Figure and Description | Sketch of Figure | Sketch of Cross-Section |
|---|---|---|---|
| 1. | **General Cylinder** |  | |
| 2. | **Right General Cylinder** |  | |
| 3. | **Right Prism** |  | |
| 4. | **Oblique Prism** |  | |
| 5. | **Right Cylinder** |  | |
| 6. | **Oblique Cylinder** |  | |

# Lesson 7: General Pyramids and Cones and Their Cross-Sections

## Student Outcomes

- Students understand the definition of a *general pyramid* and *cone,* and that their respective *cross-sections* are similar to the base.
- Students show that if two cones have the same base area and the same height, then cross-sections for the cones the same distance from the vertex have the same area.

## Lesson Notes

In Lesson 7, students examine the relationship between a cross-section and the base of a general cone. Students understand that pyramids and circular cones are subsets of general cones just as prisms and cylinders are subsets of general cylinders. In order to understand why a cross-section is similar to the base, there is discussion regarding a *dilation in three-dimensions,* explaining why a dilation maps a plane to a parallel plane. Then, a more precise argument is made using triangular pyramids; this parallels what is seen in Lesson 6, where students are presented with the intuitive idea of why bases of general cylinders are congruent to cross-sections using the notion of a translation in three dimensions, followed by a more precise argument using a triangular prism and *SSS triangle congruence.* Finally, students prove the *general cone cross-section theorem,* which we will later use to understand *Cavalieri's principle*.

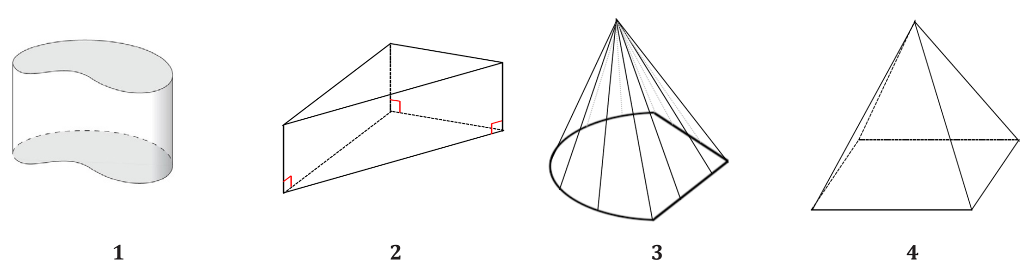
## Classwork

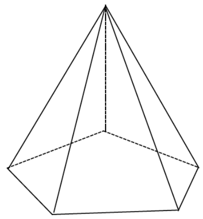
### Opening Exercise (3 minutes)

The goals of the Opening Exercise are to remind students of the parent category of general cylinders, how prisms are a subcategory under general cylinders, and how to draw a parallel to figures that "come to a point" (or, they might initially describe these figures); some of the figures that come to a point have polygonal bases, while some have curved bases. This is meant to be a quick exercise; allow 90 seconds for students to jot down ideas and another 90 seconds to share out responses.

**Opening Exercise**

**Group the following images by shared properties. What defines each of the groups you have made?**





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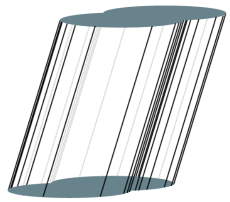

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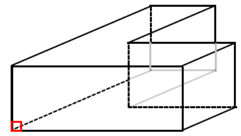

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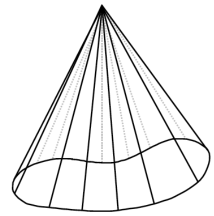


8

*Group A: General cylinders, images $1$ and $6$.*

*Group B: Prisms, images $2$ and $7$.*

*Group C: Figures that come to a point with a polygonal base, images $4$ and $5$.*

*Group D: Figures that come to a point with a curved or semi-curved region as a base, images $3$ and $8$.*

## Discussion (10 minutes)

Spend 6–7 minutes on the definitions of cone and rectangular pyramid in relation to the Opening Exercise. Provide time for students to generate and write down their own definitions in partner pairs (for both rectangular pyramid and general cone) before reviewing the precise definitions; encourage students to refer to the language used in Lesson 6 for general cylinder.

Leave the balance of time for the discussion on why cross-sections of pyramids are similar to the base and how to find the scale factor of the dilation that maps the base to the cross-section.

*Scaffolding:*

- Consider using the Frayer model as part of students' notes for the definition.
- Consider choral repetition of the vocabulary.

- Image 4 of the Opening Exercise is an example of a rectangular pyramid. What would be your definition of a rectangular pyramid?
- Images 2, 4, 5, and 8 are examples of general cones. What would be your definition of a general cone?

Have the definition of *general cylinder* prominently visible in the classroom as a point of comparison to the definition of a *general cone*.

**RECTANGULAR PYRAMID: Given a rectangular region $B$ in a plane $E$ and a point $V$ not in $E$, the *rectangular pyramid with base $B$ and vertex $V$* is the collection of all segments $\overline{VP}$ for any point $P$ in $B$.**

**GENERAL CONE: Let $B$ be a region in a plane $E$ and $V$ be a point not in $E$. The *cone with base $B$ and vertex $V$* is the union of all segments $\overline{VP}$ for all points $P$ in $B$ (See Figures 1 and 2).**

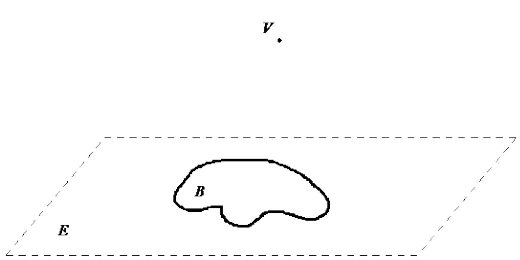



**Figure 1**

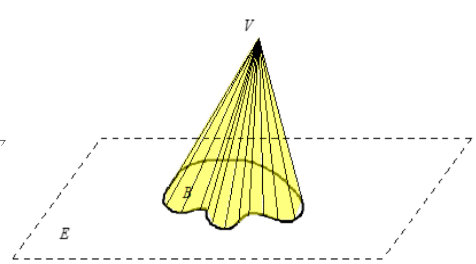



**Figure 2**

- You have seen rectangular pyramids before. Look at the definition again, and compare and contrast it with the definition of general cone.
- The definitions are essentially the same. The only difference is that a rectangular pyramid has a rectangular base. A general cone can have any region for a base.
- Much like a general cylinder, a general cone is named by its base.
- A general cone with a disk as a base is called a *circular cone*.
- A general cone with a polygonal base is called a *pyramid.* Examples of this include a rectangular pyramid or a triangular pyramid.
- A general cone whose vertex lies on the perpendicular line to the base and that passes through the center of the base is a *right cone* (or a *right pyramid* if the base is polygonal). Figure 4 shows a right rectangular pyramid, while Figure 3 shows a rectangular pyramid that is not right.

> *Scaffolding:*
> - Consider highlighting the terms *lateral faces* and *edges* with the use of nets for rectangular pyramids that are available in the supplemental materials of Grade 7, Module 6.

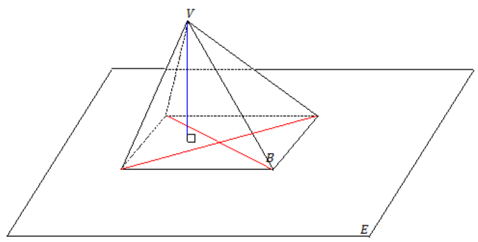


**Figure 3**

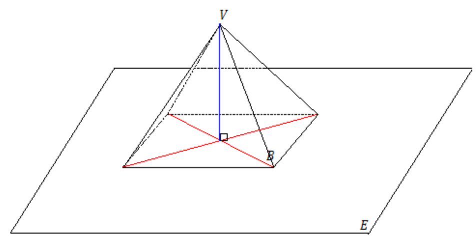


**Figure 4**

- A *right circular cone* has been commonly referred to as a *cone* since the elementary years; we will continue to use *cone* to refer to a *right circular cone.*

For pyramids, in addition to the base, we have lateral faces and edges.

- Name a lateral face and edge in Figure 5 and explain how you know it is a lateral face.

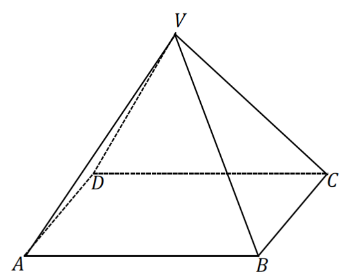


**Figure 5**

  - *The triangular region $AVB$ is defined by a side of the base, $AB$, and vertex $V$ and is an example of a lateral face.*
  - *The segments $\overline{AV}$, $\overline{BV}$, $\overline{CV}$, and $\overline{DV}$ are all lateral edges.*

Once the definitions of general cone and rectangular pyramid have been discussed, begin the discussion on how the cross-section is similar to the base.

- Observe the general cone in Figure 6. The plane $E'$ is parallel to $E$ and is between the point $V$ and the plane $E$. The intersection of the general cone with $E'$ gives a cross-section $B'$.

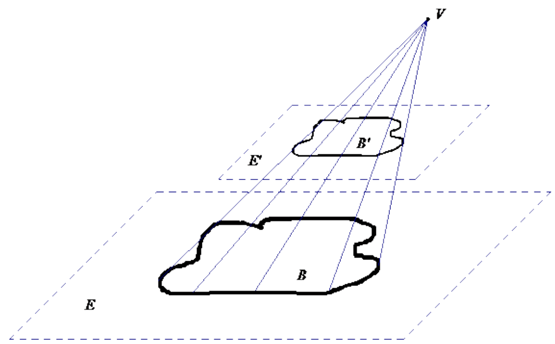


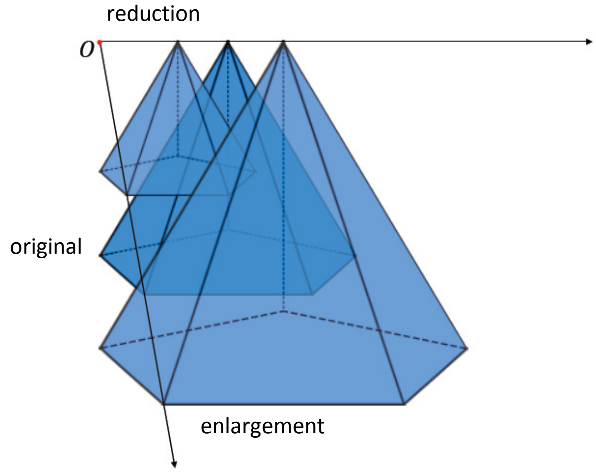

**Figure 6**

*Scaffolding:*
- Remind students of their study of cross-sections of general cylinders in Lesson 6; a cross-section is a slice taken parallel to the base of the solid.
- Ask students, "Can a cross-section of a pyramid be made at an angle, or does it have to be parallel to the base?" (It must be parallel to the base.)

- We can show the existence of a dilation that maps $B$ to $B'$; this similarity transformation would imply that the cross-section is similar to the base.
- We have only studied dilations in two dimensions, or in the plane, but it turns out that dilations behave similarly in three-dimensional space.
- A *dilation* of three-dimensional space with center $O$ and scale factor $r$ is defined the same way it is in the plane. The dilation maps $O$ to itself and maps any other point $X$ to the point $X'$ on ray $OX$ so that $OX' = r \cdot OX$.

Emphasize that we already knew that a dilation is thought of as two points at a time: the center and a point being dilated. This still holds true in three dimensions.

A visual may help establish an intuitive sense of what a dilation of a three-dimensional figure looks like. This can be easily done using interactive whiteboard software that commonly includes images of prisms. By enlarging and reducing the image of a prism, students can get a feel of what is happening during the dilation of a 3D figure. Snapshots are provided below.



## Example 1 (8 minutes)

- We have made the claim that a cross-section of a general cone is similar to its base. Use the following question as a means of demonstrating why this should be true.

> **Example 1**
>
> **In the following triangular pyramid, a plane passes through the pyramid so that it is parallel to the base and results in the cross-section $\triangle A'B'C'$. If the area of $\triangle ABC$ is $25\text{ mm}^2$, what is the area of $\triangle A'B'C'$?**
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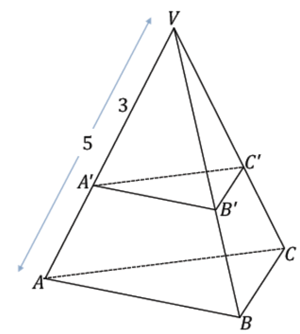
> 
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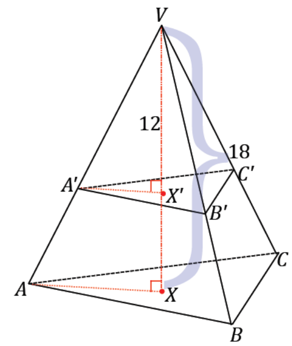
- Based on the fact that the cross-section is parallel to the base, what can you conclude about $\overline{AB}$ and $\overline{A'B'}$?
  - *They must be parallel.*
- Since $\overline{AB} \parallel \overline{A'B'}$, by the *triangle side splitter theorem,* we can conclude that $\overline{A'B'}$ splits $\triangle ABV$ proportionally. Then a dilation maps $A$ to $A'$ and $B$ to $B'$ by the same scale factor. What is the center and scale factor $k$ of this dilation?
  - *The center must be $V$, and the scale factor must be $k = \frac{3}{5}$.*
- What does the *dilation theorem* tell us about the length relationship between $\overline{AB}$ and $\overline{A'B'}$?
  - $A'B' = \frac{3}{5}AB$
- Furthermore, since the cross-section is parallel to the base, what conclusions can we draw about the relationship between $\overline{BC}$ and $\overline{B'C'}$, and $\overline{AC}$ and $\overline{A'C'}$?
  - *$\overline{BC} \parallel \overline{B'C'}$ and $\overline{AC} \parallel \overline{A'C'}$ and, just as with $\overline{AB}$ and $\overline{A'B'}$, a dilation with center $V$ and the scale factor $k = \frac{3}{5}$ maps $B$ to $B'$ and $C$ to $C'$. $B'C' = \frac{3}{5}BC$ and $A'C' = \frac{3}{5}AC$.*
- If each of the lengths of $\triangle A'B'C'$ is $\frac{3}{5}$ the corresponding lengths of $\triangle ABC$, what can be concluded about the relationship between $\triangle A'B'C'$ and $\triangle ABC$?
  - *The triangles are similar by the SSS similarity criterion.*
- What is the relationship between the areas of these similar figures?
  - $\text{Area}(\triangle A'B'C') = \left(\frac{3}{5}\right)^2 \text{Area}(\triangle ABC)$
- Find the area of $\triangle A'B'C'$.
  - $\text{Area}(\triangle \text{A'B'C'}) = \left(\frac{3}{5}\right)^2 (25)$
  - $\text{Area}(\triangle A'B'C') = \left(\frac{9}{25}\right)(25) = 9$*; the area of* $\triangle A'B'C'$ *is* $9\text{ mm}^2$.
- Based on what we knew about the cross-section of the pyramid, we were able to determine that the cross-section is in fact similar to the base and use that knowledge to determine the area of the cross-section.

### Example 2 (7 minutes)

- In Example 1, we found evidence to support our claim that the cross-section of the pyramid was similar to the base of the pyramid. We were able to find the area of the cross-section because the lengths provided along an edge of the pyramid allowed us to find the scale factor of the dilation involved.
- Can we solve a similar problem if the provided lengths are along the altitude of the pyramid?

**Example 2**

**In the following triangular pyramid, a plane passes through the pyramid so that it is parallel to the base and results in the cross-section $\triangle A'B'C'$. The altitude from $V$ is drawn; the intersection of the altitude with the base is $X$, and the intersection of the altitude with the cross-section is $X'$. If the distance from $X$ to $V$ is $18$ mm, the distance from $X'$ to $V$ is $12$ mm, and the area of $\triangle A'B'C'$ is $28$ mm$^2$, what is the area of $\triangle ABC$?**




Allow students time to wrestle with the question in partner pairs or small groups. Triangles $\triangle AVX$ and $\triangle A'V'X'$ can be shown to be similar by the AA Similarity. The argument to show that the cross-section is similar to the base is the same as that presented in Example 1. The difference here is how to determine the scale factor of the dilation. Since the corresponding sides of the right triangles are proportional in length, the scale factor is $k = \frac{VX'}{VX} = \frac{2}{3}$. The area of the base can be calculated as follows:

$$\text{Area}(\triangle A'B'C') = \left(\frac{2}{3}\right)^2 \text{Area}(\triangle ABC)$$

$$28 = \left(\frac{2}{3}\right)^2 \text{Area}(\triangle ABC)$$

$$\text{Area}(\triangle ABC) = 63$$

The area of $\triangle ABC$ is $63$ mm$^2$.

Before moving to either the Extension or Exercise 1, have a brief conversation on how a cone can be generated from rotating a right triangle about either leg of the triangle.

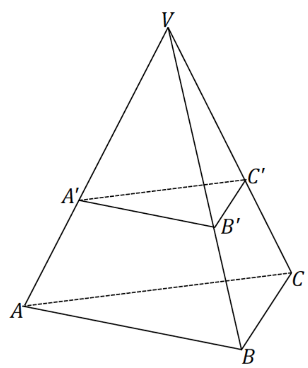
- As we saw in Lesson 6, it is possible to generate a three-dimensional figure from a two-dimensional figure. What would happen if a right triangle were rotated about one of its sides? What figure would be outlined by this rotation?

Model what this looks like by taping one leg of a piece of paper cut in the shape of a right triangle to a pencil and spinning the pencil between the palms of your hands. Students should see that the rotation sweeps out a cone. Question 4, parts (a) and (b) in the Problem Set relate to this concept.

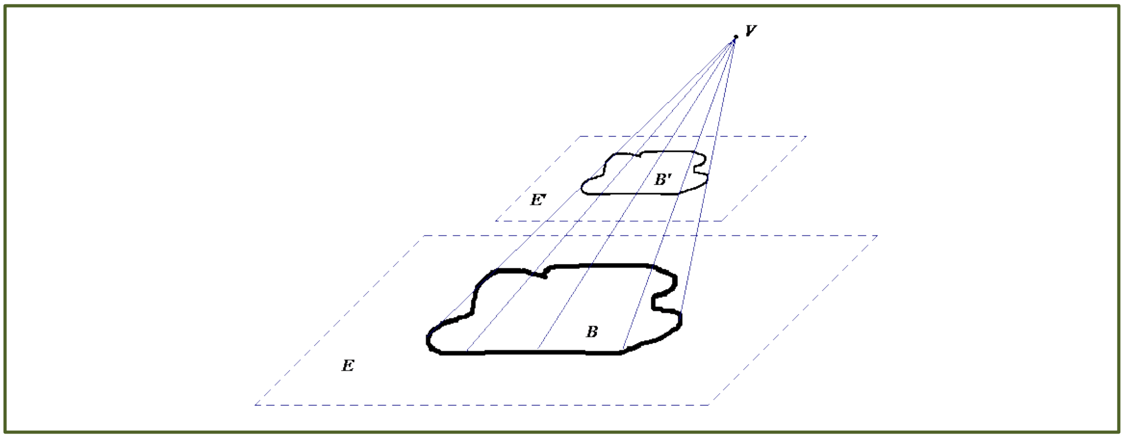
**Extension:** This Extension generalizes the argument of why cross-sections are similar to their bases. This section could be used as a substitute for Example 1. If this Extension is not being used in the lesson, proceed to Exercise 1.

- Now, let's look at a special case: a triangular pyramid. In this case, we can use what we know about triangles to prove that cross-sections are similar to the base.

**Extension**



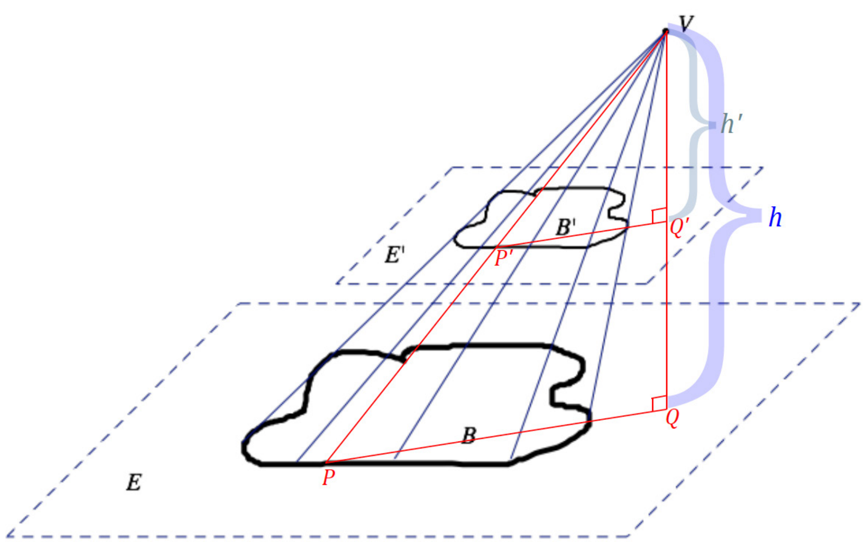

- Look at plane $ABV$. Can you describe a dilation of this plane that would take $\overline{AB}$ to $\overline{A'B'}$? Remember to specify a center and a scale factor.
  - *The dilation would have center $V$ and scale factor $k = \frac{A'V}{AV} = \frac{B'V}{BV}$.*
- Do the same for plane $BCV$.
  - *The scale factor for this dilation would also be $k = \frac{B'V}{BV} = \frac{C'V}{CV}$.*
- What about plane $CAV$?
  - *The scale factor is still $k = \frac{C'V}{CV} = \frac{A'V}{AV}$.*
- Since corresponding sides are related by the same scale factor, what can you conclude about triangles $\triangle ABC$ and $\triangle A'B'C'$?
  - *$\triangle ABC \sim \triangle A'B'C'$ by the SSS similarity criterion.*
- How can this result be used to show that any pyramid (i.e., those with polygonal bases rather than those with triangular bases) has cross-sections similar to the base?
  - *Whatever polygon represents the base of the pyramid, we can cut the pyramid up into a bunch of triangular regions. Then, the cross-section will be a bunch of triangles that are similar to the corresponding triangles in the base. So, the cross-section as a whole is similar to the base.*
- Observe that while we've only proven the result for pyramids, it does generalize for general cones, just as we suspected when we discussed dilations.




- Since $B' \sim B$, we know that $\text{Area}(B') = k^2\text{Area}(B)$, where $k$ is the scale factor of the dilation.
- How can we relate the scale factor to the height of the general cone?

Allow students a moment to consider before continuing.

- Draw an altitude for the cone, and let $Q$ and $Q'$ be the points where it intersects planes $E$ and $E'$, respectively. Call the distance between $V$ and $Q$ as $h$ and the distance between $V$ and $Q'$ as $h'$.
- Choose a point $P$ in $B$ and draw $\overline{PV}$. Let $P'$ be the intersection of this segment with $E'$.
- Consider plane $PQV$. What is the scale factor taking segment $\overline{PQ}$ to segment $\overline{P'Q'}$?
  - *A dilation with scale factor $k = \frac{h'}{h}$ and center $V$ maps $B$ to $B'$.*




- Use this scale factor to compare the $\text{Area}(B')$ to the $\text{Area}(B)$.
  - *The area of the similar region should be the area of the original figure times the square of the scale factor:* $\text{Area}\ (B') = \left(\frac{h'}{h}\right)^2 \text{Area}(B)$.

### Exercise 1 (3 minutes)

**Exercise 1**

**The area of the base of a cone is $16$, and the height is $10$. Find the area of a cross-section that is distance $5$ from the vertex.**

***$\text{Area}(\text{cross-section}) = \left(\frac{5}{10}\right)^2 \cdot 16 = 4$; the area of the cross-section that is a distance $5$ from the vertex is $4$ $\text{units}^2$.***

## Example 3 (5 minutes)

Guide students to prove the *general cone cross-section theorem,* using what they have learned regarding the area of the base and the area of a cross-section.

**GENERAL CONE CROSS-SECTION THEOREM: If two general cones have the same base area and the same height, then cross-sections for the general cones the same distance from the vertex have the same area.**

**State the theorem in your own words.**

***The theorem is saying that if two cones have the same base area and the same height, then cross-sections of both solids that are the same height from the vertex should have the same area.***

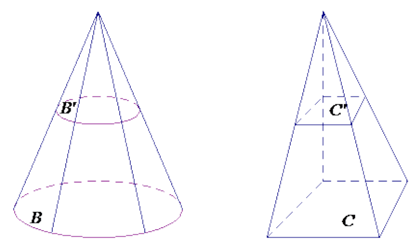



**Figure 8**

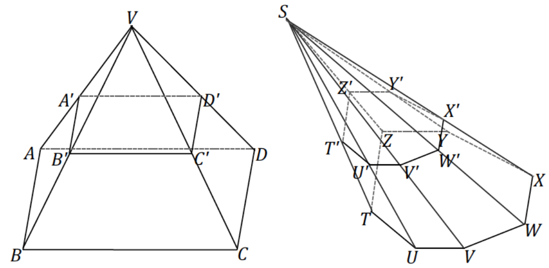
**Use the space below to prove the *general cone cross-section theorem.***

- Let the bases of the cones $B$ and $C$ in Figure 8 be such that (1) $\text{Area}(B) = \text{Area}(C)$, (2) the height of each cone is $h$, and (3) the distance from each vertex to $B'$ and to $C'$ are both $h'$.
- How can we show that $\text{Area}(B') = \text{Area}(C')$?
  - $\text{Area}(B') = \left(\frac{h'}{h}\right)^2 \text{Area}(B)$.
  - $\text{Area}(C') = \left(\frac{h'}{h}\right)^2 \text{Area}(C)$.
- Since $\text{Area}(B) = \text{Area}(C)$, then $\text{Area}(B') = \text{Area}(C')$.

### Exercise 2 (3 minutes)

**Exercise 2**

**The following pyramids have equal altitudes, and both bases are equal in area and are coplanar. Both pyramids' cross-sections are also coplanar. If $BC = 3\sqrt{2}$ and $B'C' = 2\sqrt{3}$, and the area of $TUVWXYZ$ is $30$ units$^2$, what is the area of cross-section $A'B'C'D'$?**




$\left(\frac{2\sqrt{3}}{3\sqrt{2}}\right)^2 \cdot 30 = 20$; *the area of the cross-section* $A'B'C'D'$ *is* $20$ units$^2$.

### Closing (1 minute)

Ask students to summarize the key points of the lesson. Additionally, consider asking students the following questions independently in writing, to a partner, or to the whole class.

- What distinguishes a pyramid from a general cone?
  - *Pyramids are general cones with polygonal bases.*
- What is the relationship between the base and the cross-section of a general cone? If the height of a general cone is $h$, what is the relationship between the area of a base region and a cross-section taken at a height $h'$ from the vertex of the general cone?
  - *The cross-section is similar to the base and has an area* $\left(\frac{h'}{h}\right)^2$ *times that of the area of the base.*
- Restate the *general cone cross-section theorem* in your own words.

**Lesson Summary**

**CONE: Let $B$ be a region in a plane $E$ and $V$ be a point not in $E$. The *cone with base $B$ and vertex $V$* is the union of all segments $\overline{VP}$ for all points $P$ in $B$.**

**If the base is a polygonal region, then the *cone* is usually called a *pyramid*.**

**RECTANGULAR PYRAMID: Given a rectangular region $B$ in a plane $E$ and a point $V$ not in $E$, the *rectangular pyramid with base $B$ and vertex $V$* is the union of all segments $\overline{VP}$ for points $P$ in $B$.**

**LATERAL EDGE AND FACE OF A PYRAMID: Suppose the base $B$ of a pyramid with vertex $V$ is a polygonal region and $P_i$ is a vertex of $B$. The segment $\overline{P_iV}$ is called a *lateral edge* of the pyramid. If $\overline{P_iP_{i+1}}$ is a base edge of the base $B$ (a side of $B$), and $F$ is the union of all segments $\overline{PV}$ for $P$ in $\overline{P_iP_{i+1}}$, then $F$ is called a *lateral face* of the pyramid. It can be shown that the face of a pyramid is always a triangular region.**

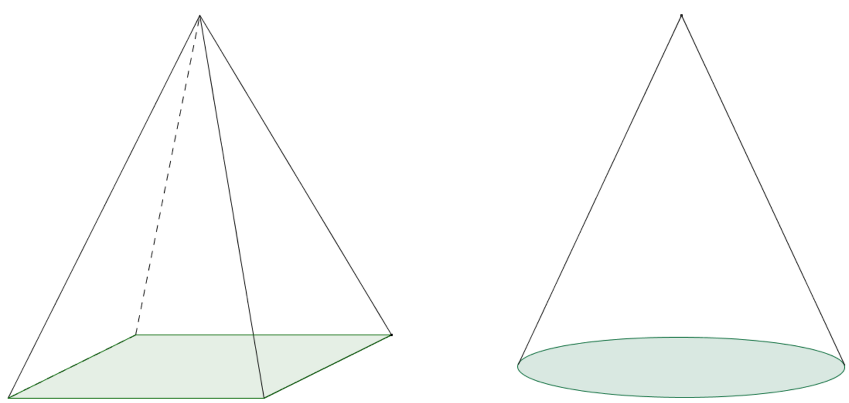
### Exit Ticket (5 minutes)

Name ___________________________________________ Date ____________________

# Lesson 7: General Pyramids and Cones and Their Cross-Sections

## Exit Ticket

The diagram below shows a circular cone and a general pyramid. The bases of the cones are equal in area, and the solids have equal heights.

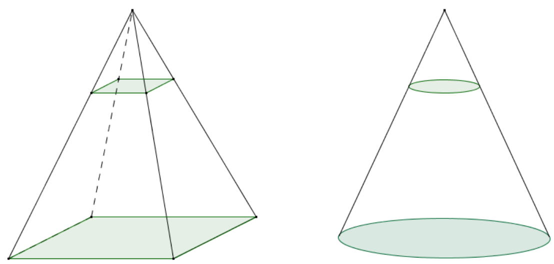


a. Sketch a slice in each cone that is parallel to the base of the cone and $\frac{2}{3}$ closer to the vertex than the base plane.

b. If the area of the base of the circular cone is $616 \text{ units}^2$, find the exact area of the slice drawn in the pyramid.

## Exit Ticket Sample Solutions

The diagram below shows a circular cone and a general pyramid. The bases of the cones are equal in area, and the solids have equal heights.



a. Sketch a slice in each cone that is parallel to the base of the cone and $\frac{2}{3}$ closer to the vertex than the base plane.

b. If the area of the base of the circular cone is $616 \text{ units}^2$, find the exact area of the slice drawn in the pyramid.

*The distance from the slice to the vertex is $\frac{1}{3}$ the height of the cone, so the scale factor from the base to the slice is $\frac{1}{3}$. The areas of the planar regions are related by the square of the scale factor, or $\frac{1}{9}$.*

$$\text{Area(slice)} = \frac{1}{9}(\text{Area(base)})$$

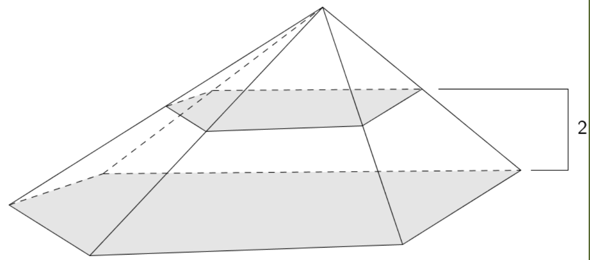
$$\text{Area(slice)} = \frac{1}{9}(616)$$

$$\text{Area(slice)} = \frac{616}{9}$$

*If two cones have the same base area and the same height, then cross-sections for the cones the same distance from the vertex have the same area, so the area of the slice from the pyramid is $\frac{616}{9} \text{ units}^2$.*

## Problem Set Sample Solutions

1. **The base of a pyramid has area $4$. A cross-section that lies in a parallel plane that is distance of $2$ from the base plane has an area of $1$. Find the height, $h$, of the pyramid.**




*The cross-section is similar to the base with scale factor $r$, and the areas of the similar slices are related by the square of the scale factor, $r^2$.*

$$r^2 \cdot 4 = 1$$
$$r^2 = \frac{1}{4}$$
$$r = \frac{1}{2}$$

*The scale factor of the cross-section is $\frac{1}{2}$.*

*Let $h'$ = the distance from the cross-section (or slice) to the vertex. Then*

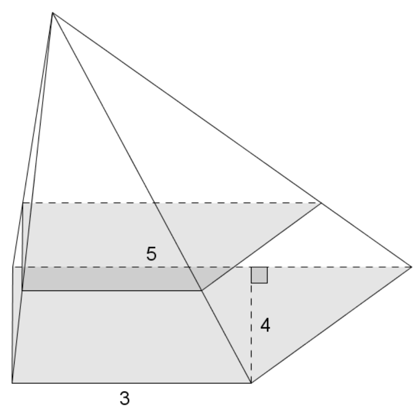
*$h' = h - 2$, and the heights of the slices are also related by the scale factor, so*

$$\frac{h-2}{h} = \frac{1}{2}$$
$$h = 2(h-2)$$
$$h = 2h - 4$$
$$h = 4$$

*The height of the pyramid is $4$ units.*

2. **The base of a pyramid is a trapezoid. The trapezoidal bases have lengths of $3$ and $5$, and the trapezoid's height is $4$. Find the area of the parallel slice that is three-fourths of the way from the vertex to the base.**

*The area of the trapezoidal base:*




$$\text{Area} = \frac{1}{2}(b_1 + b_2)h$$
$$\text{Area} = \frac{1}{2}(3+5) \cdot 4$$
$$\text{Area} = 16$$

*The area of the base of the pyramid is $16$ units$^2$.*

*Let $h$ be the height of the pyramid. The distance from the vertex to the given slice is $\frac{3}{4}h$. The slice is similar to the base with scale factor $\frac{\frac{3}{4}h}{h} = \frac{3}{4}$.*

*The areas of similar figures are related by the square of the scale factor relating their dimensions, so*

$\left(\frac{3}{4}\right)^2 \cdot 16 = 9$ *The area of the slice is $9$ units$^2$.*

3. A cone has base area $36\text{ cm}^2$. A parallel slice $5\text{ cm}$ from the vertex has area $25\text{ cm}^2$. Find the height of the cone.

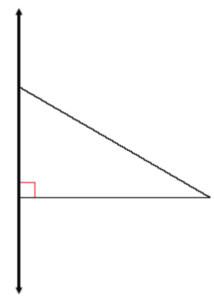
*Let $h$ be the cone's height in centimeters.*

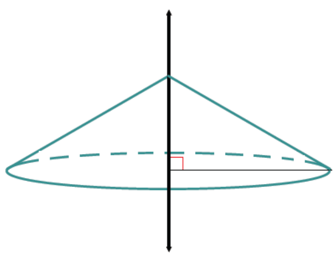
$$\left(\frac{5}{h}\right)^2 \cdot 36 = 25$$
$$\left(\frac{5}{h}\right)^2 = \frac{25}{36}$$
$$\frac{5}{h} = \frac{5}{6}$$
$$h = 6$$

*The cone has height $6\text{ cm}$.*

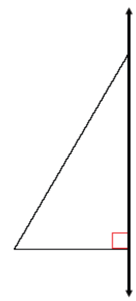
4. Sketch the figures formed if the triangular regions are rotated around the provided axis:

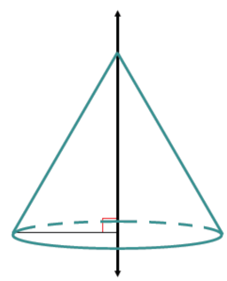
a.





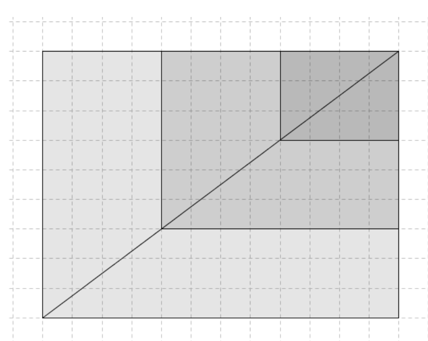
b.





5. Liza drew the top view of a rectangular pyramid with two cross-sections as shown in the diagram and said that her diagram represents one, and only one, rectangular pyramid. Do you agree or disagree with Liza? Explain.

*Liza's statement is not valid. As shown, the top cross-section has dimensions that are $\frac{1}{3}$ the corresponding lengths of the base, and the lower cross-section has dimensions that are $\frac{2}{3}$ the corresponding lengths of the base; therefore, the cross-sections are $\frac{1}{3}$ and $\frac{2}{3}$ of the distance from the vertex to the base, respectively. However, since no information was given about the distance between cross-sections or the height of the pyramid, the only conclusion we can draw is that the distance between each given consecutive cross-section is $\frac{1}{3}$ the height of the pyramid.*



6. A general hexagonal pyramid has height $10\text{ in.}$ A slice $2\text{ in}$. above the base has area $16\text{ in}^2$. Find the area of the base.

*Let $A$ be the area of the base in square inches. Two inches above the base is $\frac{4}{5}$ from the vertex.*

$$\left(\frac{4}{5}\right)^2 A = 16$$
$$A = 25$$

*The base has area $25\text{ in}^2$.*

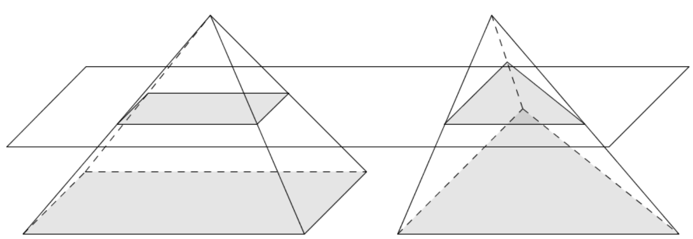
7. A general cone has base area $3\text{ units}^2$. Find the area of the slice of the cone that is parallel to the base and $\frac{2}{3}$ of the way from the vertex to the base.

*Let $h$ represent the height of the cone. Then the distance from the vertex to the slice of the cone is $\frac{2}{3}h$. The slice is similar to the base with a scale factor of $\frac{\frac{2}{3}h}{h} = \frac{2}{3}$. The area of the slice is equal to the area of the base times the square of the scale factor.*

$$\text{Area(slice)} = \left(\frac{2}{3}\right)^2 \cdot 36$$
$$\text{Area(slice)} = \frac{4}{9} \cdot 36$$
$$\text{Area(slice)} = 16$$

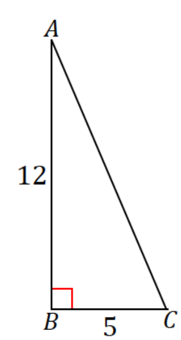
*The area of the slice of the cone is $1\text{ unit}^2$.*

8. A rectangular cone and a triangular cone have bases with the same area. Explain why the cross-sections for the cones halfway between the base and the vertex have the same area.



*Let $A$ be the area of the bases. The cross-sections are each similar to the base with scale factor $\frac{1}{2}$. So, each has area $\frac{1}{4}A$. Thus, they are equal.*

9. The following right triangle is rotated about side $AB$. What is the resulting figure, and what are its dimensions?




*The resulting figure is a circular cone with radius $5$ and height $12$.*

# Lesson 8: Definition and Properties of Volume

## Student Outcomes

- Students understand the precise language that describes the properties of volume.
- Students understand that the *volume of any right cylinder* is given by the formula $\text{area of base} \times \text{height}$.

## Lesson Notes

Students have been studying and understanding the *volume properties* since Grade 5 (**5.MD.C.3**, **5.MD.C.4**, **5.MD.C.5**). We review the idea that the volume properties are analogous to the *area properties* by doing a side-by-side comparison in the table. Since the essence of the properties is not new, the idea is for students to become comfortable with the formal language. Images help illustrate the properties. The goal is to review the properties briefly and spend the better part of the lesson demonstrating why the volume of any right general cylinder is $\text{area of base} \times \text{height}$. It will be important to draw on the parallel between the *approximation process used for area* (in Lesson 1) and in this lesson. Just as rectangles and triangles were used for the upper and lower approximations to help determine area, so we can show that *right rectangular prisms* and *triangular prisms* are used to make the same kind of approximation for the volume of a general right cylinder.

## Classwork

### Opening (6 minutes)

- Today, we examine properties of volume (much as we examined the properties of area in Lesson 2) and why the volume of a right cylinder can be found with the formula $\text{Volume} = \text{area of base} \times \text{height}$.
- Just as we approximated the area of curved or irregular regions by using rectangles and triangles to create upper and lower approximations, we can approximate the volume of general cylinders by using rectangular and triangular prisms to create upper and lower approximations.

Spend a few moments on a discussion of what students believe *volume* means (consider having a solid object handy as a means of reference). Students will most likely bring up the idea of "amount of space" or "how much water" an object holds as part of their descriptions. If students cite volume as "how much water" (or sand, air, etc.) an object holds, point out that to measure the volume of water, we would have to discuss volume yet again and be stuck in circular reasoning. Conclude with the idea that, just like area, we leave volume as an undefined term, but we can list what we believe is true about volume; the list below contains assumptions we make regarding volume.

There are two options teachers can take in the discussion on volume properties. (1) Show the area properties as a point of comparison and simply ask students to describe properties that come to mind when they think of volume (students have been studying volume and its properties since Grade 5). (2) Provide students with the handout at the close of the lesson and ask them to describe in their own words what they think the analogous properties are in three dimensions for volume. Whichever activity is selected, keep it brief.

Then share the following table that lists the volume properties with precise language. Ask students to compare the list of area properties to the list of volume properties; elicit from students that the properties for volume parallel the properties for area. The goal is to move through this table at a brisk pace, comparing the two columns to each other, and supporting the language with the images or describing a potential problem using the images. The following statements are to help facilitate the language of each property during the discussion:

Regarding Property 1: We assign a numerical value greater than or equal to zero that quantifies the size but not the shape of an object in three dimensions.

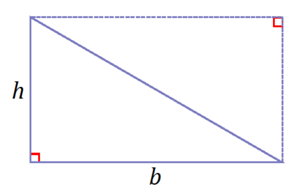
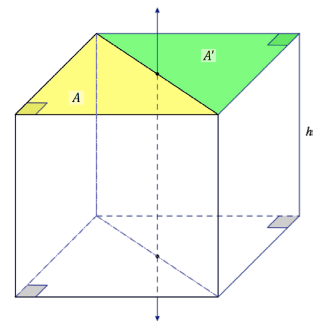
Regarding Property 2: Just as we declared that the area of a rectangle is given by the formula $\text{length} \times \text{width}$, we are making a statement saying that the volume of a box, or a right rectangular or triangular prism, is $\text{area of base} \times \text{height}$.

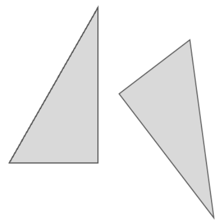
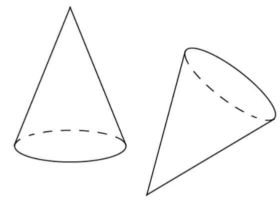
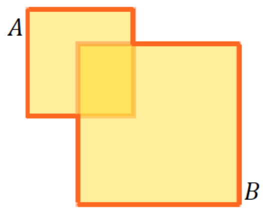
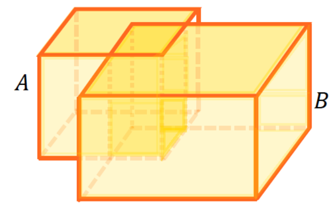
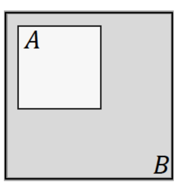
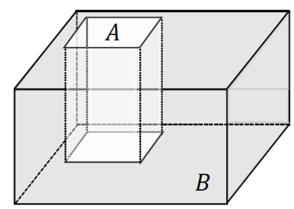
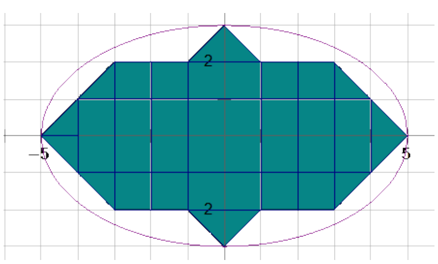
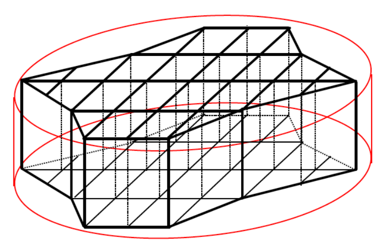
Regarding Property 3: In two dimensions, we identify two figures as congruent if a sequence of rigid motions maps one figure onto the other so that the two figures coincide. We make the same generalization for three dimensions. Two solids, such as the two cones, are congruent if there is a series of three-dimensional rigid motions that will map one onto the other.

Regarding Property 4: The volume of a composite figure is the sum of the volumes of the individual figures minus the volume of the overlap of the figures.

Regarding Property 5: When a figure is formed by carving out some portion, we can find the volume of the remaining portion by subtraction.

Regarding Property 6: Just as in Lesson 1, we used upper and lower approximations comprised of rectangles and triangles to get close to the actual area of a figure, so we will do the same for the volume of a curved or irregular general right cylinder but with the use of triangular and rectangular prisms.

| Area Properties | Volume Properties |
|---|---|
| **1. The area of a set in two dimensions is a number greater than or equal to zero that measures the size of the set and not the shape.** | **1. The volume of a set in three dimensions is a number greater than or equal to zero that measures the size of the set and not the shape.** |
| **2. The area of a rectangle is given by the formula $\text{length} \times \text{width}$. The area of a triangle is given by the formula $\frac{1}{2}\,\text{base} \times \text{height}$. A polygonal region is the union of finitely many non-overlapping triangular regions and has area the sum of the areas of the triangles.**   | **2. A right rectangular or triangular prism has volume given by the formula $\text{area of base} \times \text{height}$. A right prism is the union of finitely many non-overlapping right rectangular or triangular prisms and has volume the sum of the volumes of the prisms.**   |

| | |
|---|---|
| **3. Congruent regions have the same area.**<br> | **3. Congruent solids have the same volume.**<br> |
| **4. The area of the union of two regions is the sum of the areas minus the area of the intersection:**<br>$\mathbf{Area}(A \cup B) = \mathbf{Area}(A) + \mathbf{Area}(B) - \mathbf{Area}(A \cap B)$<br><br> | **4. The volume of the union of two solids is the sum of the volumes minus the volume of the intersection:**<br>$\mathbf{Vol}(A \cup B) = \mathbf{Vol}(A) + \mathbf{Vol}(B) - \mathbf{Vol}(A \cap B)$<br><br> |
| **5. The area of the difference of two regions where one is contained in the other is the difference of the areas:**<br>**If $A \subseteq B$, then $\mathbf{Area}(B - A) = \mathbf{Area}(B) - \mathbf{Area}(A)$.**<br><br> | **5. The volume of the difference of two solids where one is contained in the other is the difference of the volumes:**<br>**If $A \subseteq B$, then $\mathbf{Vol}(B - A) = \mathbf{Vol}(B) - \mathbf{Vol}(A)$.**<br><br> |
| **6. The area $a$ of a region $A$ can be estimated by using polygonal regions $S$ and $T$ so that $S$ is contained in $A$ and $A$ is contained in $T$. Then $\mathbf{Area}(S) \le a \le \mathbf{Area}(T)$.**<br><br> | **6. The volume $v$ of a solid $W$ can be estimated by using right prism solids $S$ and $T$ so that $S \subseteq W \subseteq T$. Then $\mathbf{Vol}(S) \le v \le \mathbf{Vol}(T)$.**<br> |

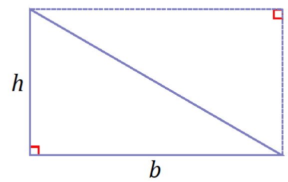
### Opening Exercise (6 minutes)

For the following questions, students are expected to use general arguments to answer each prompt. Allow students to complete the questions independently or in small groups.

MP.3 & MP.7

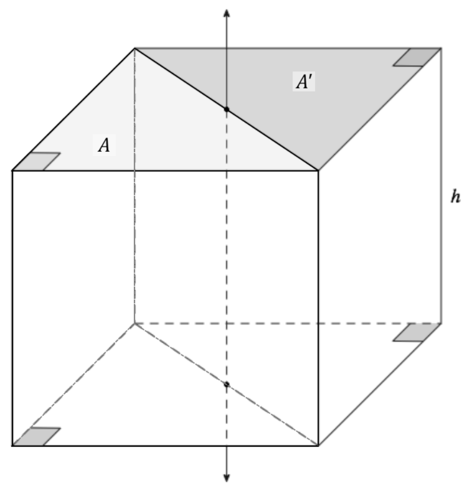
**Opening Exercise**

a. **Use the following image to reason why the area of a right triangle is $\frac{1}{2}bh$ (Area Property 2).**




*The right triangle may be rotated about the midpoint of the hypotenuse so that two triangles together form a rectangle. The area of the rectangle can be found by the formula $bh$. Since two congruent right triangles together have an area described by $bh$, one triangle can be described by half that value, or $\frac{1}{2}bh$.*

b. **Use the following image to reason why the volume of the following triangular prism with base area $A$ and height $h$ is $Ah$ (Volume Property 2).**

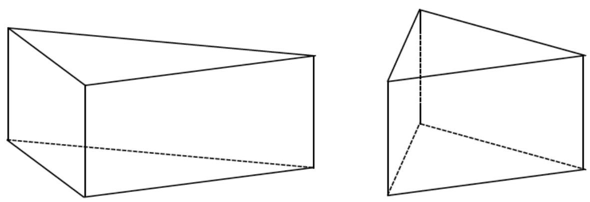



*The copy of a triangular prism with a right triangle base can be rotated about the axis shown so that the two triangular prisms together form a rectangular prism with volume $2Ah$. Since two congruent right triangular prisms together have a volume described by $2Ah$, the triangular prism has volume half that value, or $Ah$.*

Note that the response incorporates the idea of a rotation in three dimensions. It is not necessary to go into great detail about rigid motions in three dimensions here, as the idea can be applied easily without drawing much attention to it. However, should students ask, it is sufficient to say that rotations, reflections, and translations behave much as they do in three dimensions as they do in two dimensions.

### Discussion (10 minutes)

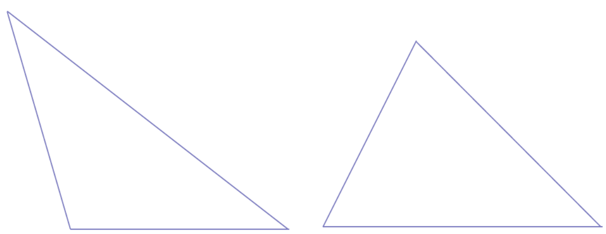
- As part of our goal to approximate the volume of a general right cylinder with rectangular and triangular prisms, we will focus our discussion on finding the volume of different types of triangular prisms.
- The image in Opening Exercise, part (b) makes use of a triangular prism with a right triangle base.
- Can we still make the argument that *any* triangular prism (i.e., a triangular prism that does not necessarily have a *right* triangle as a base) has a volume described by the formula $\text{area of base} \times \text{height}$?



*Scaffolding:*
- Consider using the triangular prism nets provided in Grade 6, Module 5, Lesson 15 during this Discussion.
- Show how triangular prisms that do not have right triangle bases can be broken into right triangular prisms.

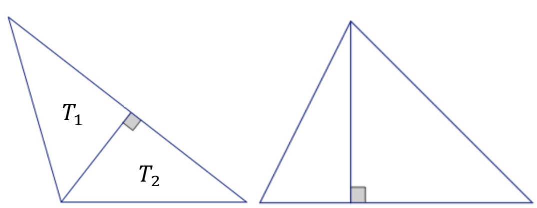
MP.3 & MP.7

- Consider the following obtuse and acute triangles. Is there a way of showing that a prism with either of the following triangles as bases will still have the volume formula $Ah$?



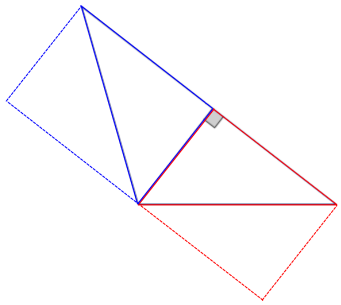
Allow students a few moments to consider the argument will hold true. Some may try to form rectangles out of the above triangles. Continue the discussion after allowing them time to understand the essential question: Can the volume of any triangular prism be found using the formula $\text{Volume} = Ah$ (where $A$ represents the area of the base, and $h$ represents the height of the triangular prism)?

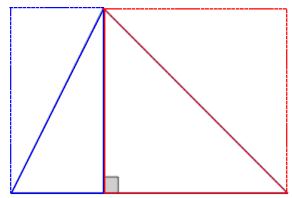
- *Any* triangle can be shown to be the union of two right triangles. Said more formally, any triangular region $T$ is the union $T_1 \cup T_2$ of two right triangular regions so that $T_1 \cap T_2$ is a side of each triangle.



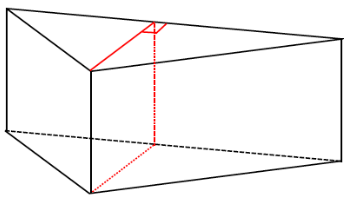
- Then we can show that the area of either of the triangles is the sum of the area of each sub-triangle:

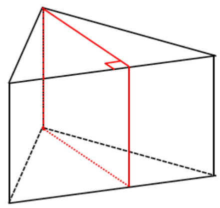
$$\text{Area} = T_1 + T_2 = \frac{1}{2}b_1h_1 + \frac{1}{2}b_2h_2$$





- Just as we can find the total area of the triangle by adding the areas of the two smaller triangles, we can find the total volume of the triangular prism by adding the volumes of each sub-triangular prism:





- What do we now know about any right prism with a triangular base?
    - *Any triangular region can be broken down into smaller right triangles. Each of the sub-prisms formed with those right triangles as bases has a volume formula of $Ah$. Since the height is the same for both sub-prisms, the volume of an entire right prism with triangular base can be found by taking the sum of the areas of the right triangular bases times the height of the prism.*

Allow students to consider and share ideas on this with a partner. The idea can be represented succinctly in the following way and is a great example of the use of the *distributive property*:

- Any right prism $P$ with a triangular base $T$ and height $h$ is the union $P_1 \cup P_2$ of right prisms with bases made up of right triangular regions $T_1$ and $T_2$, respectively, so that $P_1 \cap P_2$ is a rectangle of zero volume (Volume Property 4).

$$\text{Vol}(P) = \text{Vol}(P_1) + \text{Vol}(P_2) - \text{Vol}(\text{rectangle})$$

$$\text{Vol}(P) = (\text{Area}(T_1) \cdot h) + (\text{Area}(T_2) \cdot h)$$

$$\text{Vol}(P) = \text{Area}(T_1 + T_2) \cdot h$$

$$\text{Vol}(P) = \text{Area}(T) \cdot h$$

Note that we encountered a similar situation in Lesson 2 when we took the union of two squares and followed the respective Area Property 4 in that lesson: $\text{Area}(A \cup B) = \text{Area}(A) + \text{Area}(B) - \text{Area}(A \cap B)$. In that case, we saw that the area of the intersection of two squares was a line segment, and the area had to be $0$. The same sort of thing happens here, and the volume of a rectangle must be $0$. Remind students that this result allows us to account for all cases, but that when the area or volume of the overlap is $0$, we usually ignore it in the calculation.

- How is a general right prism related to a right triangular prism?
  - *The base of a general right prism is a polygon, which can be divided into triangles.*

Note: The *right* in right triangular prism qualifies the prism (i.e., we are referring to a prism whose lateral edges are perpendicular to the bases). To cite a triangular prism with a right triangle base, the base must be described separately from the prism.

- Given that we know that the base of a general right prism is a polygon, what will the formula for its volume be? Explain how you know.

Allow students a few moments to articulate this between partner pairs, and then have them share their ideas with the class. Students may say something to the following effect:

  - *We know that any general right prism has a polygonal base, and any polygon can be divided into triangles. The volume of a triangular prism can be calculated by taking the area of the base times the height. Then we can picture a general prism being made up of several triangular prisms, each of which has a volume of area of the base times the height. Since all the triangular prisms would have the same height, this calculation is the same as taking the sum of all the triangles' areas times the height of the general prism. The sum of all the triangles' areas is just the base of the prism, so the volume of the prism is area of the base times the height.*

Confirm with the following explanation:

- The base $B$ of a general right prism $P$ with height $h$ is always a polygonal region. We know that the polygonal region is the union of finitely many non-overlapping triangles:

$$B = T_1 \cup T_2 \cup \cdots \cup T_n$$

- Let $P_1, P_2, \cdots, P_n$ be the right triangular prisms with height $h$ with bases $T_1, T_2, \cdots, T_n$, respectively. Then $P = P_1 \cup P_2 \cup \cdots \cup P_n$ of non-overlapping triangular prisms, and the volume of right prism $P$ is

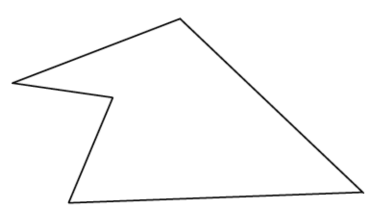
$$\mathrm{Vol}(P) = \mathrm{Vol}(P_1) + \mathrm{Vol}(P_2) + \cdots + \mathrm{Vol}(P_n)$$
$$\mathrm{Vol}(P) = \mathrm{Area}(T_1) \cdot h + \mathrm{Area}(T_2) \cdot h + \cdots + \mathrm{Area}(T_n) \cdot h$$
$$\mathrm{Vol}(P) = \mathrm{Area}(T_1 + T_2 + \cdots + T_n) \cdot h$$
$$\mathrm{Vol}(P) = \mathrm{Area}(B) \cdot h\,.$$

## Exercises 1–2 (4 minutes)

**Exercises 1–2**

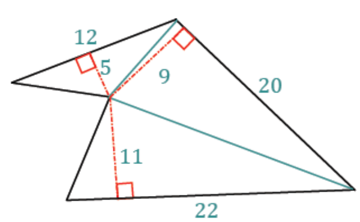
**Complete Exercises 1–2, and then have a partner check your work.**

**1. Divide the following polygonal region into triangles. Assign base and height values of your choice to each triangle, and determine the area for the entire polygon.**



***Sample response:***

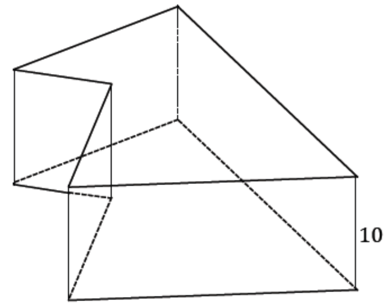
$$\frac{1}{2}(12)(5) + \frac{1}{2}(20)(9) + \frac{1}{2}(22)(11) = 241 \text{ units}^2$$



2. **The polygon from Exercise 1 is used here as the base of a general right prism. Use a height of 10 and the appropriate value(s) from Exercise 1 to determine the volume of the prism.**

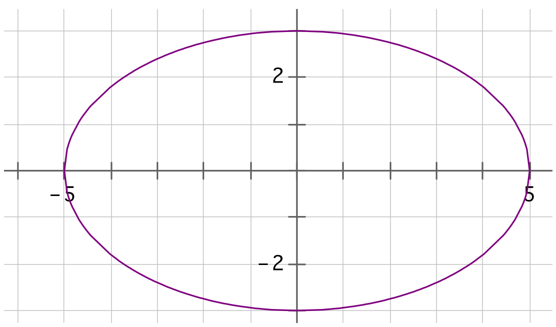
*Sample response:*

$$(241)(10) = 2,410 \text{ units}^3$$



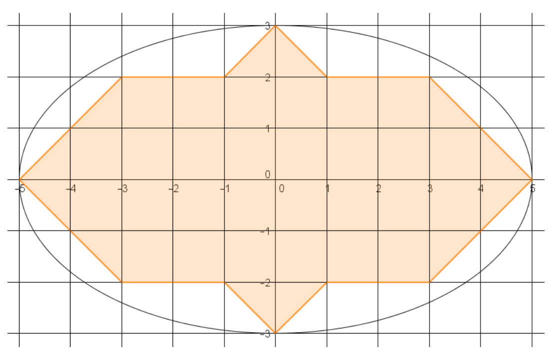

## Discussion (8 minutes)

- What have we learned so far in this lesson?
  - *We reviewed the properties of volume.*
  - *We determined that the volume formula for any right triangular prism and any general right prism is $Ah$, where $A$ is the area of the base of the prism and $h$ is the height.*
- What about the volume formula for a general right cylinder? What do you think the volume formula for a general right cylinder will be?
- Think back to Lesson 1 and how we began to approximate the area of the ellipse. We used whole squares and half squares—regions we knew how to calculate the areas of—to make upper and lower area approximations of the curved region.

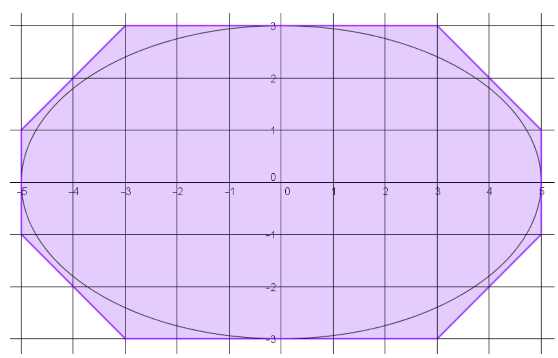



- Which image shows a lower approximation? Which image shows an upper approximation?

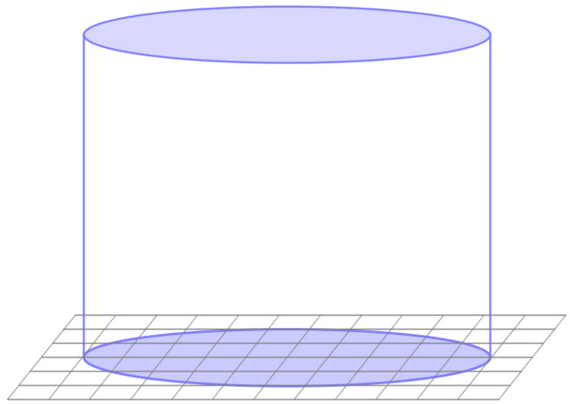
*Lower approximation*



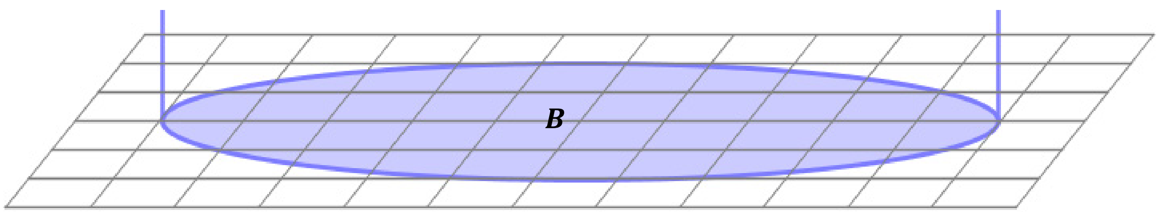
*Upper approximation*



- Now imagine a similar situation, but in three dimensions.

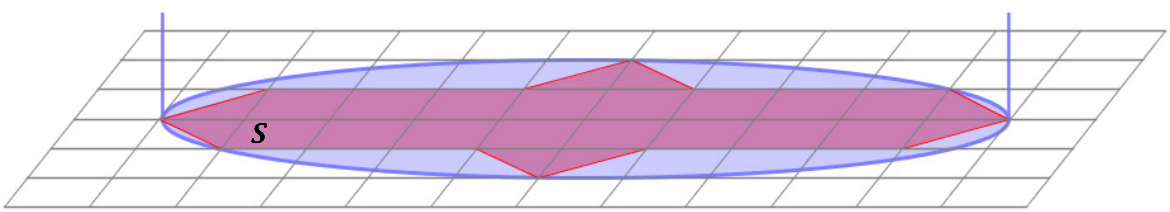


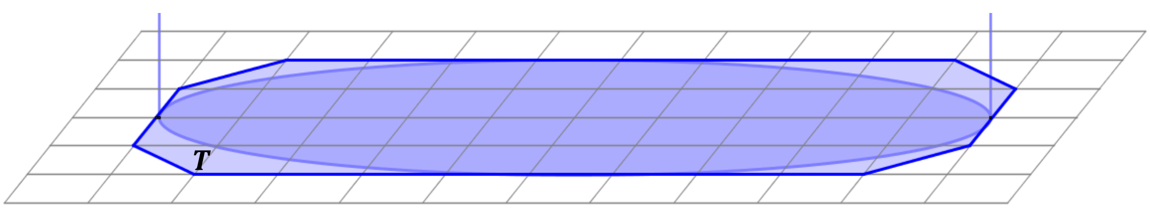
- To approximate the volume of this elliptical cylinder, we will use rectangular prisms and triangular prisms (because we know how to find their volumes) to create upper and lower volume approximations. The prisms we use are determined by first approximating the area of the base of the elliptical cylinder, and projecting prisms over these area approximations using the same height as the height of the elliptical cylinder:




- For any lower and upper approximations, $S$ and $T$, of the base, the following inequality holds:

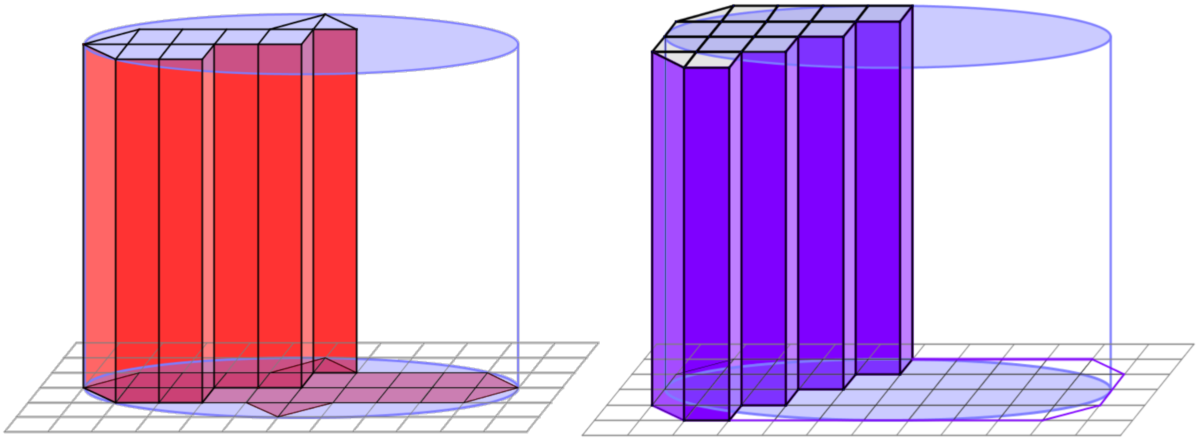
$$\mathrm{Area}(S) \leq \mathrm{Area}(B) \leq \mathrm{Area}(T)$$




- Since this is true no matter how closely we approximate the region, we see that the area of the base of the elliptical cylinder, $\mathrm{Area}(B)$, is the unique value between the area of any lower approximation, $\mathrm{Area}(S)$, and the area of any upper approximation, $\mathrm{Area}(T)$.

- What would we have to do in order to determine the volume of the prisms built over the areas of the upper and lower approximations, given a height $h$ of the elliptical cylinder?
  - *We would have to multiply the area of the base times the height.*



- Then the volume formula of the prism over the area of the lower approximation is $\text{Area}(S) \cdot h$ and the volume formula of the prism over the area of the upper approximation is $\text{Area}(T) \cdot h$.
- Since $\text{Area}(S) \le \text{Area}(B) \le \text{Area}(T)$ and $h > 0$, we can then conclude that

$$\text{Area}(S) \cdot h \le \text{Area}(B) \cdot h \le \text{Area}(T) \cdot h$$

- This inequality holds for any pair of upper and lower approximations of the base, so we conclude that the volume of the elliptical cylinder will be the unique value determined by the area of its base times its height, or $\text{Area}(B) \cdot h$.
- The same process works for any general right cylinder. Hence, the volume formula for a general right cylinder is area of the base times the height, or $Ah$.

## Exercises 3–4 (4 minutes)

With any remaining time available, have students try the following problems.

**Exercises 3–4**

**We can use the formula $\text{density} = \dfrac{\text{mass}}{\text{volume}}$ to find the density of a substance.**

**3. A square metal plate has a density of $10.2\ \text{g/cm}^3$ and weighs $2.193\ \text{kg}$.**

**a. Calculate the volume of the plate.**

$$2.193\ kg = 2193\ g$$
$$10.2 = \frac{2193}{v}$$
$$v = 215$$

*The volume of the plate is* $215\ \text{cm}^3$.

b. **If the base of this plate has an area of $25\text{ cm}^2$, determine its thickness.**

*Let $h$ represent the thickness of the plate in centimeter.*

$$\text{Volume} = A \cdot h$$
$$215 = 25h$$
$$h = 8.6$$

*The thickness of the plate is $8.6\text{ cm}$.*

4. **A metal cup full of water has a mass of $1,000\text{ g}$. The cup itself has a mass of $214.6\text{ g}$. If the cup has both a diameter and a height of $10\text{ cm}$, what is the approximate density of water?**

*The mass of the water is the difference of the mass of the full cup and the mass of the empty cup, which is $785.4\text{ g}$.*

*The volume of the water in the cup is equal to the volume of the cylinder with the same dimensions.*

$$\text{Volume} = Ah$$
$$\text{Volume} = (\pi(5)^2) \cdot (10)$$
$$\text{Volume} = 250\pi$$

*Using the density formula,*

$$\text{density} = \frac{785.4}{250\pi}$$
$$\text{density} = \frac{3.1416}{\pi} \approx 1$$

*The density of water is approximately $1\text{ g/cm}^3$.*

## Closing (2 minutes)

Ask students to summarize the key points of the lesson. Additionally, consider asking students the following questions independently in writing, to a partner, or to the whole class.

- What is the volume formula for any triangular prism? Briefly describe why.
  - *The volume formula is area of the base times height, or $Ah$. This is because we can break up any triangle into sub-triangles that are right triangles. We know that the volume formula for a triangular prism with a right triangle base is $Ah$; then the volume of a prism formed over a composite base made up of right triangles is also $Ah$.*
- Compare the role of rectangular and triangular prisms in approximating the volume of a general right cylinder to that of rectangles and triangles in approximating the area of a curved or irregular region.
  - *Since we do not know how to calculate the area of a curved or irregular region, we use figures we do know how to calculate the area for (rectangles and triangles) to approximate those regions. This is the same idea behind using rectangular and triangular prisms for approximating volumes of general right prisms with curved or irregular bases.*
- What is the volume formula for any general right cylinder?
  - *The volume formula is area of the base times height, or $Ah$.*

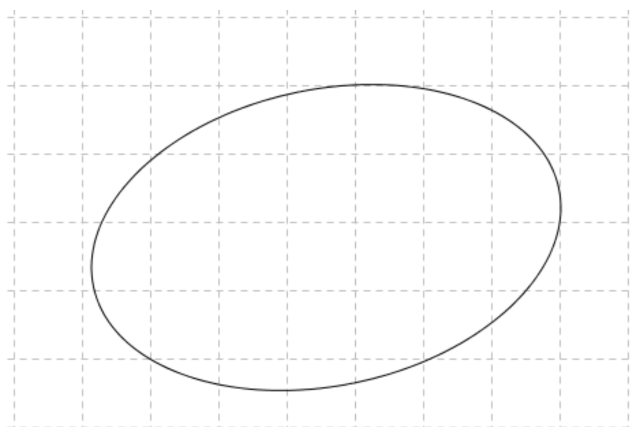
## Exit Ticket (5 minutes)

Name ______________________________ Date ________________

# Lesson 8: Definition and Properties of Volume

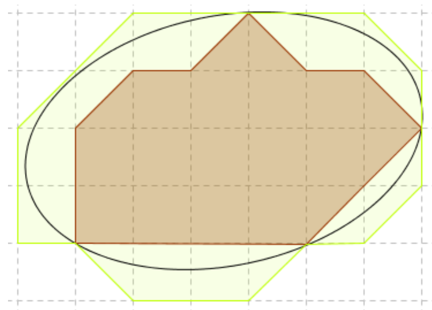
## Exit Ticket

The diagram shows the base of a cylinder. The height of the cylinder is $14$ cm. If each square in the grid is $1\text{ cm} \times 1\text{ cm}$, make an approximation of the volume of the cylinder. Explain your reasoning.



## Exit Ticket Sample Solutions

The diagram shows the base of a cylinder. The height of the cylinder is $14$ cm. If each square in the grid is $1 \text{ cm} \times 1 \text{ cm}$, make an approximation of the volume of the cylinder. Explain your reasoning.



*By counting the unit squares and triangles that make up a polygonal region lying just within the given region, a low approximation for the area of the region is $16 \text{ cm}^2$. By counting unit squares and triangles that make up a polygonal region that just encloses the given region, an upper approximation for the area of the region is $28 \text{ cm}^2$. The average of these approximations gives a closer approximation of the actual area of the base.*

$$\frac{1}{2}(16 + 28) = 22$$

*The average approximation of the area of the base of the cylinder is $22 \text{ cm}^2$.*

*The volume of the prism is equal to the product of the area of the base times the height of the prism.*

$$V = (22 \text{ cm}^2) \cdot (14 \text{ cm})$$
$$V = 308 \text{ cm}^3$$

*The volume of the cylinder is approximately $308 \text{ cm}^3$.*

## Problem Set Sample Solutions

1. Two congruent solids $S_1$ and $S_2$ have the property that $S_1 \cap S_2$ is a right triangular prism with height $\sqrt{3}$ and a base that is an equilateral triangle of side length $2$. If the volume of $S_1 \cup S_2$ is $25 \text{ units}^3$, find the volume of $S_1$.

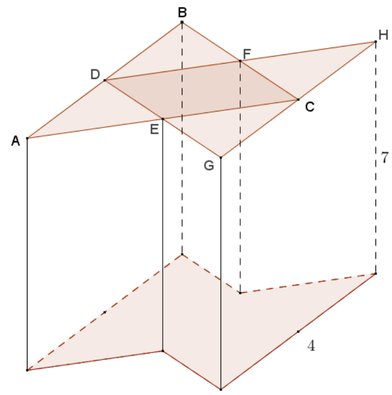
   *The area of the base of the right triangular prism is $\sqrt{3}$ and the volume of the right triangular prism is $\sqrt{3} \cdot \sqrt{3} = 3$. Let $x$ equal the volume of $S_1$ in cubic units. Then, the volume of $S_2$ in cubic units is $x$.*

   *The volume of $S_1 \cup S_2 = x + x - 3 = 25$. So $x = 14$. The volume of $S_1$ is $14 \text{ units}^3$.*

2. Find the volume of a triangle with side lengths $3$, $4$, and $5$.

   *A triangle is a planar figure. The volume of any planar figure is zero because it lies in the plane and, therefore, has no height.*

3. The base of the prism shown in the diagram consists of overlapping congruent equilateral triangles $ABC$ and $DGH$. Points $C$, $D$, $E$, and $F$ are midpoints of the sides of triangles $ABC$ and $DGH$. $GH = AB = 4$, and the height of the prism is $7$. Find the volume of the prism.




*$\overline{DC}$ connects the midpoints of $\overline{AB}$ and $\overline{GH}$ and is, therefore, the altitude of both triangles $ABC$ and $DGH$. The altitude in an equilateral triangle splits the triangle into two congruent 30-60-90 triangles. Using the relationships of the legs and hypotenuse of a 30-60-90 triangle, $DC = 2\sqrt{3}$.*

*Volume of triangular prism with base $ABC$:*

$$V = \frac{1}{2}\left(4 \cdot 2\sqrt{3}\right) \cdot 7$$
$$V = 28\sqrt{3}$$

*The volume of the triangular prism with base $DGH$ is also $28\sqrt{3}$ by the same reasoning.*

*Volume of parallelogram $CEDF$:*

$V = 2 \cdot \sqrt{3} \cdot 7$

$V = 14\sqrt{3}$

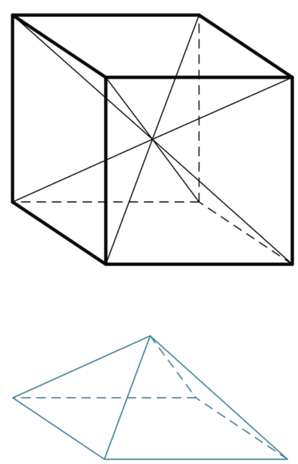
$V(A \cup B) = V(A) + V(B) - V(A \cap B)$

$V(\text{prism}) = 28\sqrt{3} + 28\sqrt{3} - 14\sqrt{3}$

$V(\text{prism}) = 42\sqrt{3}$

*The volume of the prism is $42\sqrt{3}$.*

4. Find the volume of a right rectangular pyramid whose base is a square with side length $2$ and whose height is $1$.
Hint: Six such pyramids can be fit together to make a cube with side length $2$ as shown in the diagram.



*Piecing six congruent copies of the given pyramid together forms a cube with edges of length $2$. The volume of the cube is equal to the area of the base times the height:*

$\text{Volume}_{\text{cube}} = 2^2 \cdot 2$

$\text{Volume}_{\text{cube}} = 8$

*Since there are six identical copies of the pyramid forming the cube, the volume of one pyramid is equal to $\frac{1}{6}$ of the total volume of the cube:*

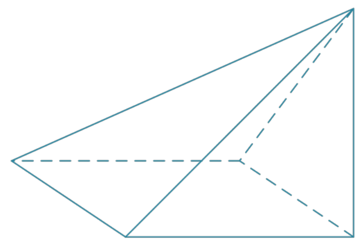
$V_p = \frac{1}{6}(8)$

$V_p = \frac{8}{6} = \frac{4}{3}$

*The volume of the given rectangular pyramid is $\frac{4}{3}$ cubic units.*

5. **Draw a rectangular prism with a square base such that the pyramid's vertex lies on a line perpendicular to the base of the prism through one of the four vertices of the square base, and the distance from the vertex to the base plane is equal to the side length of the square base.**

*Sample drawing shown:*



6. **The pyramid that you drew in Problem 5 can be pieced together with two other identical rectangular pyramids to form a cube. If the side lengths of the square base are $3$, find the volume of the pyramid.**

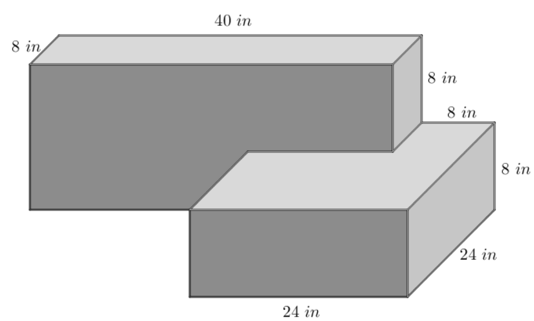
*If the sides of the square are length $3$, then the cube formed by three of the pyramids must have edges of length $3$. The volume of a cube is the cube of the length of the edges, or $s^3$. The pyramid is only $\frac{1}{3}$ of the cube, so the volume of the pyramid is $\frac{1}{3}$ of the volume of the cube:*

$$\mathrm{Volume(cube)} = \frac{1}{3} \cdot 3^3 = 9$$

*The volume of the pyramid is $9$ cubic units.*

7. **Paul is designing a mold for a concrete block to be used in a custom landscaping project. The block is shown in the diagram with its corresponding dimensions and consists of two intersecting rectangular prisms. Find the volume of mixed concrete, in cubic feet, needed to make Paul's custom block.**

*The volume is needed in cubic feet, so the dimensions of the block can be converted to feet:*




$8\text{ in.} \rightarrow \frac{2}{3}\text{ft.}$

$16\text{ in.} \rightarrow 1\frac{1}{3}\text{ ft.}$

$24\text{ in.} \rightarrow 2\text{ ft.}$

$40\text{ in.} \rightarrow 3\frac{1}{3}\text{ ft.}$

*The two rectangular prisms that form the block do not have the same height; however, they do have the same thickness of $\frac{2}{3}$ ft., and their intersection is a square prism with base side lengths of $\frac{2}{3}$ ft., so Volume Property 4 can be applied:*

$$V(A \cup B) = V(A) + V(B) - V(A \cap B)$$

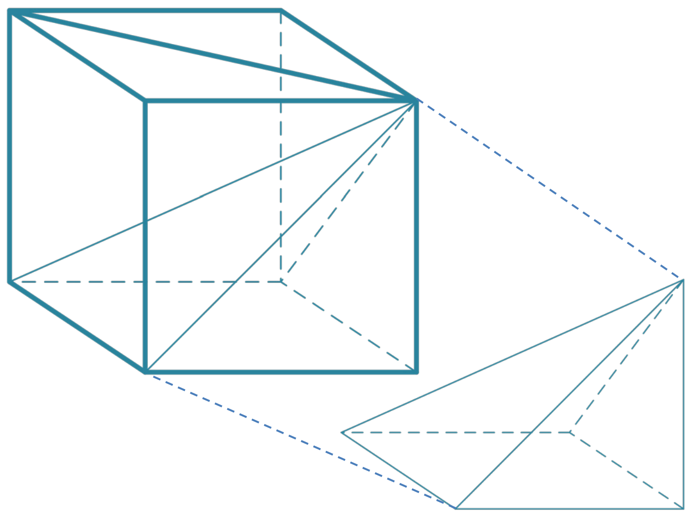
$$V(A \cup B) = \left[\left(1\frac{1}{3} \cdot 3\frac{1}{3}\right) \cdot \frac{2}{3}\right] + \left[(2 \cdot 2) \cdot \frac{2}{3}\right] - \left[\left(\frac{2}{3} \cdot 1\frac{1}{3}\right) \cdot \frac{2}{3}\right]$$

$$V(A \cup B) = \left[\frac{80}{27}\right] + \left[\frac{8}{3}\right] - \left[\frac{16}{27}\right]$$

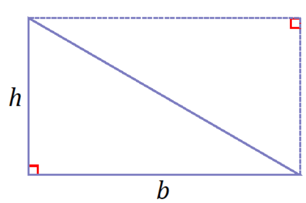
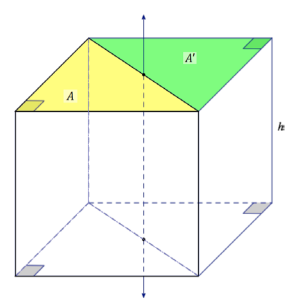
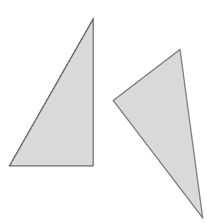
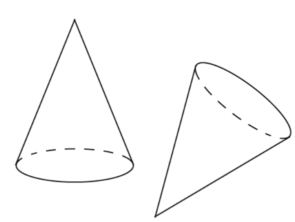
$$V(A \cup B) = \frac{136}{27} \approx 5.04\ ft^3$$

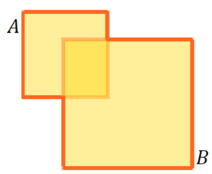
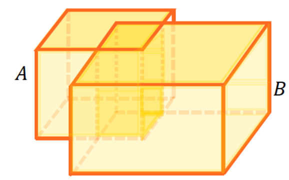
*Paul will need just over $5\text{ ft}^3$ of mixed concrete to fill the mold.*

8. **Challenge: Use card stock and tape to construct three identical polyhedron nets that together form a cube.**



### Opening

| Area Properties | Volume Properties |
|---|---|
| 1. The area of a set in two dimensions is a number greater than or equal to zero that measures the size of the set and not the shape. | 1. |
| 2. The area of a rectangle is given by the formula $\text{length} \times \text{width}$. The area of a triangle is given by the formula $\frac{1}{2} \times \text{base} \times \text{height}$. A polygonal region is the union of finitely many non-overlapping triangular regions and has area the sum of the areas of the triangles.<br><br> | 2.<br><br><br>a. |
| 3. Congruent regions have the same area.<br> | 3. Congruent solids have the same volume.<br> |

<table>
<tr>
<td>4. The area of the union of two regions is the sum of the areas minus the area of the intersection:<br>$\text{Area}(A \cup B) = \text{Area}(A) + \text{Area}(B) - \text{Area}(A \cap B)$<br></td>
<td>4.<br></td>
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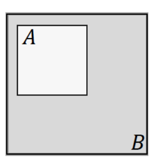
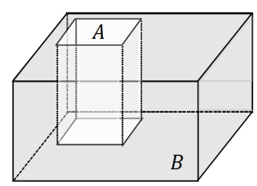
<tr>
<td>5. The area of the difference of two regions where one is contained in the other is the difference of the areas: If $A \subseteq B$, then $\text{Area}(B - A) = \text{Area}(B) - \text{Area}(A)$.<br></td>
<td>5.<br></td>
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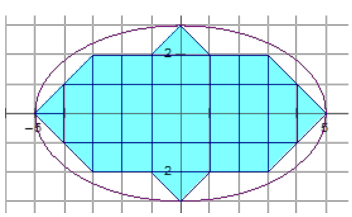
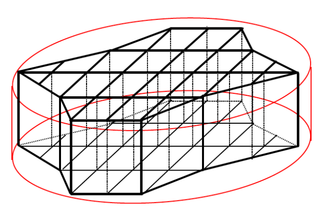
<tr>
<td>6. The area $a$ of a region $A$ can be estimated by using polygonal regions $S$ and $T$ so that $S$ is contained in $A$ and $A$ is contained in $T$.<br>Then $\text{Area}(S) \leq a \leq \text{Area}(T)$.<br></td>
<td>6.<br></td>
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# Lesson 9: Scaling Principle for Volumes

## Student Outcomes

- Students understand that given similar solids $A$ and $B$ so that the ratio of their lengths is $a$: $b$, then the ratio of their volumes is $a^3$: $b^3$.
- Students understand that if a solid with volume $V$ is scaled by factors of $r$, $s$, and $t$ in three perpendicular directions, then the volume is multiplied by a factor of $r \cdot s \cdot t$ so that the volume of the scaled solid is $(rst)V$.

## Lesson Notes

The Opening Exercise reviews the relationship between the ratio of side lengths of similar figures and the ratio of the areas of the similar figures. Some groups of students may not need the review in part (a) and can begin the Opening Exercise with part (b). In order to extend what students know about similar figures and scaled figures to three dimensions, students begin by investigating the effect that a scale factor (or scale factors) has on the volume of the figure. After each set of investigative problems is a brief discussion where students should be encouraged to share their observations. At the end of the first discussion, it should be made clear that when similar solids have side lengths with ratio $a$: $b$, then the ratio of their volumes is $a^3$: $b^3$. At the end of the second discussion, it should be made clear that when a figure is scaled by factors of $r$, $s$, and $t$ in three perpendicular directions, then the volume is multiplied by a factor of $r \cdot s \cdot t$. The foundation for understanding volume is laid in Grades 3–5. Students begin applying volume formulas for right rectangular prisms with fractional side lengths in Grade 6, Module 5 (**6.G.A.2**). Volume work is extended to three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms in Grade 7, Modules 3 and 6 (**7.G.B.6**). It is in Grade 8, Module 5 that students learn about the volume of cylinders, cones, and spheres (**8.G.C.9**). There are many exercises for students to complete independently, in pairs, or as part of a group when the class is divided. Depending on your choice, you may need to complete this lesson over two days where the second day of content would begin with Exercise 2.

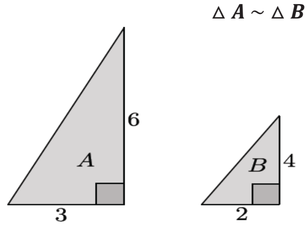
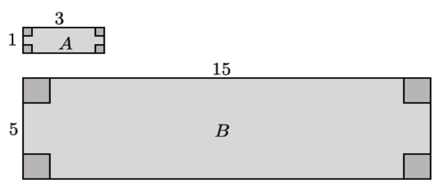
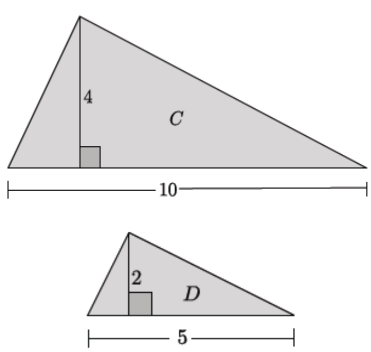
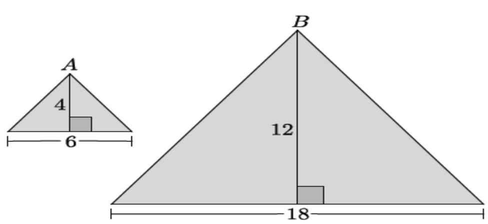
## Classwork

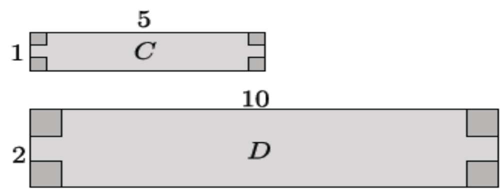
### Opening Exercise (8 minutes)

Students can complete the exercises independently or in pairs. You can divide the class and have each group complete one part of the table in part (a) and then come together as a class to share solutions and ideas about how the ratio of side lengths is related to the ratio of areas. It may be necessary to talk through the first problem with the class before letting them work independently. Some groups of students may be able to complete all of the problems independently and not need an exemplar or to work in pairs throughout the lesson. Once students have completed the exercises, have students share their conjectures about the relationship between the ratio of side lengths to the ratio of volumes as well as their reasoning. Consider having the class vote on which conjecture they believe is correct. When students have finished the Opening Exercise, show how they can write the ratio of areas using squared numbers, clearly demonstrating the relationship between side lengths of similar figures and their areas.

**Opening Exercise**

a. **For each pair of similar figures, write the ratio of side lengths $a:b$ or $c:d$ that compares one pair of corresponding sides. Then, complete the third column by writing the ratio that compares the areas of the similar figures. Simplify ratios when possible.**

| Similar Figures | Ratio of Side Lengths $a:b$ or $c:d$ | Ratio of Areas $\text{Area}(A):\text{Area}(B)$ or $\text{Area}(C):\text{Area}(D)$ |
|---|---|---|
| $\triangle A \sim \triangle B$   | $6:4$<br>$3:2$ | $9:4$<br>$3^2:2^2$ |
| Rectangle $A \sim$ Rectangle $B$   | $3:15$<br>$1:5$ | $3:75$<br>$1:25$<br>$1^2:5^2$ |
| $\triangle C \sim \triangle D$   | $4:2$<br>$10:5$<br>$2:1$ | $20:5$<br>$4:1$<br>$2^2:1^2$ |
| $\triangle A \sim \triangle B$   | $4:12$<br>$6:18$<br>$1:3$ | $12:108$<br>$1:9$<br>$1^2:3^2$ |

| | | |
|---|---|---|
| Rectangle $A \sim$ Rectangle $B$  | $1:2$<br>$5:10$ | $5:20$<br>$1:4$<br>$1^2:2^2$ |



b.

i. **State the relationship between the ratio of sides $a:b$ and the ratio of the areas $\text{Area}(A):\text{Area}(B)$.**

*When the ratio of side lengths is $a:b$, then the ratio of the areas is $a^2:b^2$.*

MP.3

ii. **Make a conjecture as to how the ratio of sides $a:b$ will be related to the ratio of volumes $\text{Volume}(S):\text{Volume}(T)$. Explain.**

*When the ratio of side lengths is $s:t$, then the ratio of the volumes will probably be $s^3:t^3$. Area is two-dimensional, and the comparison of areas was raised to the second power. Since volume is three-dimensional, I think the comparison of volumes will be raised to the third power.*

c. **What does is mean for two solids in three-dimensional space to be similar?**

*It means that a sequence of basic rigid motions and dilations maps one figure onto the other.*

*Scaffolding:*

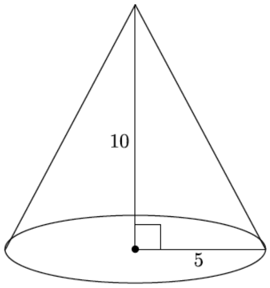
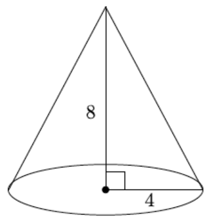
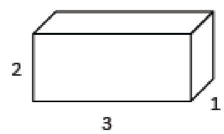
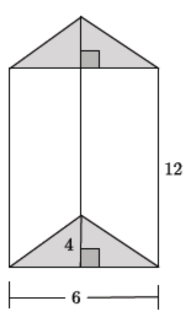
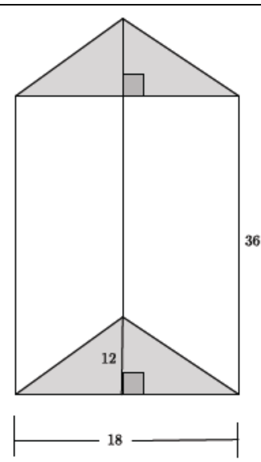
- Consider having a class discussion around part (b) to elicit the relationship asked for in part (i). Then allow students to make their conjecture for part (ii) independently.
- Encourage students to write the ratios of volume using cubed numbers (e.g., for a ratio of side lengths $2:5$, the corresponding ratio of volumes can be written $2^3:5^3$). Doing so will help students see the relationship between the ratio of side lengths and the ratio of volumes.
- Provide selected problems from previous grade levels (identified in the Lesson Notes) as additional homework problems leading up to this lesson to remind students of the volume formulas used in this portion of the lesson.

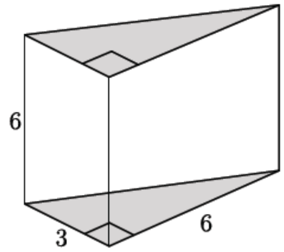
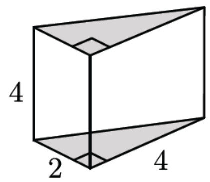
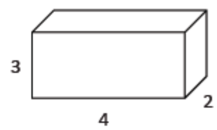
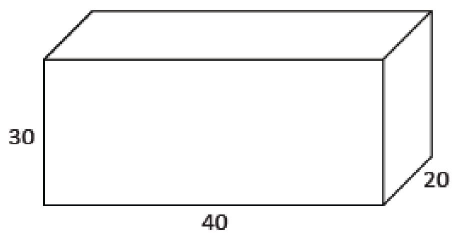
### Exercise 1 (10 minutes)

Exercise 1 provides an opportunity for students to test their conjectures about the relationship between similar solids and their related volumes. If necessary, divide the class so that they complete just one problem and then share their solutions with the class. As before, it may be necessary to talk through the first problem before students begin working independently.

**Exercises**

1. **Each pair of solids shown below is similar. Write the ratio of side lengths $a:b$ comparing one pair of corresponding sides. Then, complete the third column by writing the ratio that compares volumes of the similar figures. Simplify ratios when possible.**

| Similar Figures | Ratio of Side Lengths $a:b$ | Ratio of Volumes $\text{Volume}(A):\text{Volume}(B)$ |
|---|---|---|
|  <br>**Figure A** **Figure B** | $10:8$<br>$5:4$ | $\frac{250}{3}\pi:\frac{128}{3}\pi$<br>$250:128$<br>$125:64$<br>$5^3:4^3$ |
| <br>**Figure A**<br><br>**Figure B** | $2:10$<br>$3:15$<br>$1:5$ | $6:750$<br>$3:375$<br>$1:125$<br>$1^3:5^3$ |
|  <br>**Figure A** **Figure B** | $4:12$<br>$6:18$<br>$12:36$<br>$1:3$ | $144:3888$<br>$1:27$<br>$1^3:3^3$ |

| | | |
|---|---|---|
|   Figure A   Figure B | $6:4$<br>$3:2$ | $54:16$<br>$27:8$<br>$3^3:2^3$ |
|   Figure A   Figure B | $3:30$<br>$4:40$<br>$2:20$<br>$1:10$ | $24:24000$<br>$1:1000$<br>$1^3:10^3$ |

*Scaffolding:*

- Display the numbers in standard form in relation to their equivalent cubes for students to reference (e.g., $1 = 1^3$, $8 = 2^3$, $27 = 3^3$, $1000 = 10^3$).

### Discussion (4 minutes)

MP.8

- What do you notice about the relationships of the corresponding distances and the relationships of the volumes of the similar figures that you examined in the exercise above?
  - *The ratios of the volumes of the similar figures are the cubes of the ratios of the corresponding distances of the similar figures.*
- Suppose a similarity transformation takes a solid $S$ to a solid $T$ at scale factor $r$. How do you think the volume of $S$ compares to the volume of $T$?

Allow students time to discuss in pairs or small groups, if necessary.

  - $\text{Volume}(T) = r^3 \cdot \text{Volume}(S)$

- If the ratio of the lengths of similar solids is $a:b$, then the ratio of their volumes is $a^3:b^3$.

MP.3

- Shemar says that if the figures were right rectangular prisms, then the observation above is true, i.e., given $a:b$, then the ratio of their volumes is $a^3:b^3$, but that if the figures were triangular prisms, then the ratio of volumes would be $\frac{1}{2}a^3:\frac{1}{2}b^3$. What do you think?

MP.3

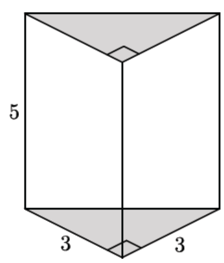
Provide students time to consider Shemar's statement. Ask students whether or not they would change their minds about the relationship between similar solids if $T$ was a right triangular prism or if $T$ was any solid. They should respond that the relationship would be the same as stated above because of what they know about equivalent ratios. Specifically, students may think that they need to include $\frac{1}{2}$ in the ratio, as Shemar did, because the comparison is of triangular prisms. However, based on what students know about equivalent ratios, they should conclude that $\frac{1}{2}a^3 : \frac{1}{2}b^3 \rightarrow 2 \cdot \frac{1}{2}a^3 : 2 \cdot \frac{1}{2}b^3 \rightarrow a^3 : b^3$.

Revisit the conjecture made in part (b) of the Opening Exercise. Acknowledge any student(s) who made an accurate conjecture about the relationship between similar figures and their related volumes.

### Exercises 2–4 (10 minutes)

In Exercises 2–4, students explore what happens to the volume of a figure when it is scaled by factors of $r$, $s$, and $t$ in three perpendicular directions. If necessary, divide the class so that they complete just one problem and then share their solutions with the class. It is not expected that all students will immediately see the relationship between the volumes. It may be necessary to do this portion of the lesson on another day.

**2. Use the triangular prism shown below to answer the questions that follow.**




**a. Calculate the volume of the triangular prism.**

$$V = \frac{1}{2}(3)(3)(5)$$
$$V = \frac{45}{2}$$
$$V = 22.5$$

**b. If one side of the triangular base is scaled by a factor of $2$, the other side of the triangular base is scaled by a factor of $4$, and the height of the prism is scaled by a factor of $3$, what are the dimensions of the scaled triangular prism?**

*The new dimensions of the base are $6$ by $12$, and the height is $15$.*

*Scaffolding:*

- Struggling students may need to work through Exercise 2 with the support and guidance of the teacher.
- Allow some groups of students to persevere in solving this problem in order to see the relationship on their own.

c. **Calculate the volume of the scaled triangular prism.**

$V = \frac{1}{2}6(12)(15)$

$V = \frac{1080}{2}$

$V = 540$

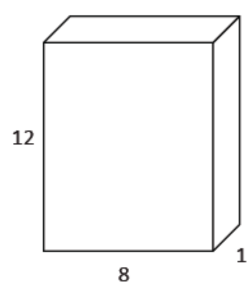
d. **Make a conjecture about the relationship between the volume of the original triangular prism and the scaled triangular prism.**

*Answers will vary. Accept any reasonable response. The correct response is that the volume of the scaled figure is equal to the volume of the original figure multiplied by the product of the scaled factors. For this specific problem,* $540 = 2(3)(4)(22.5)$.

e. **Do the volumes of the figures have the same relationship as was shown in the figures in Exercise 2? Explain.**

*No. The figure was scaled differently in each perpendicular direction, so the volumes are related by the product of the scale factors* $2 \cdot 3 \cdot 4 = 24$.

3. **Use the rectangular prism shown below to answer the questions that follow.**




*Scaffolding:*
- Consider allowing students to work in groups to complete this exercise.
- Have student groups draw the scaled figures to help them recognize that the figures are not similar but are, in fact, stretched, squeezed, or both.

a. **Calculate the volume of the rectangular prism.**

$V = 1(8)(12)$

$V = 96$

b. **If one side of the rectangular base is scaled by a factor of $\frac{1}{2}$, the other side of the rectangular base is scaled by a factor of $24$, and the height of the prism is scaled by a factor of $\frac{1}{3}$, what are the dimensions of the scaled rectangular prism?**

*Note that some students may have selected different sides of the base to scale compared to the solution below. Regardless, the result in part (c) will be the same.*

*The dimensions of the scaled rectangular prism are* $4$, $24$, *and* $4$.

*OR*

*The dimensions of the scaled rectangular prism are* $192$, $\frac{1}{2}$, *and* $4$.

c. **Calculate the volume of the scaled rectangular prism.**

$V = 4(24)(4)$
$V = 384$

d. **Make a conjecture about the relationship between the volume of the original rectangular prism and the scaled rectangular prism.**

*Answers will vary. Accept any reasonable response. The correct response is that the volume of the scaled figure is equal to the volume of the original figure multiplied by the product of the scaled factors. For this specific problem,* $384 = \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)(24)(96)$.

4. **A manufacturing company needs boxes to ship their newest widget, which measures $2 \times 4 \times 5 \text{ in}^3$. Standard size boxes, 5-inch cubes, are inexpensive but require foam packaging so the widget is not damaged in transit. Foam packaging costs $\$0.03$ per cubic inch. Specially designed boxes are more expensive but do not require foam packing. If the standard size box costs $\$0.80$ each and the specially designed box costs $\$3.00$ each, which kind of box should the company choose? Explain your answer.**

*Volume of foam packaging needed in a standard box:*

$125 - (2 \cdot 4 \cdot 5) = 85$*;* $85 \text{ in}^3$ *of foam packaging needed for standard-sized box.*

*Total cost of widget packed in a standard box:*

$85(0.03) + 0.80 = 3.35$*; total cost is* $\$3.35$

*Therefore, the specially designed packages for* $\$3.00$ *each are more cost effective.*

## Discussion (4 minutes)

Debrief the exercises using the discussion points below.

- If a solid $T$ is scaled by factors of $r$, $s$, and $t$ in three perpendicular directions, what happens to the volume?

Allow students time to discuss in pairs or small groups, if necessary.

  - *If $T'$ is the scaled solid, then* $\text{Volume}(T') = rst \cdot \text{Volume}(T)$.

Ask students whether or not they would change their mind if $T$ was a right rectangular prism, if $T$ was a right triangular prism, or if $T$ was any solid. They should respond that the relationship would be the same as stated above.

- When a solid with volume $V$ is scaled by factors of $r$, $s$, and $t$ in three perpendicular directions, then the volume of the scaled figure is multiplied by a factor of $rst$.

## Closing (4 minutes)

Have students summarize the main points of the lesson in writing, talking to a partner, or as a whole class discussion. Use the questions below, if necessary.

- When a similarity transformation takes one solid to another, how are their volumes related?
  - *For two similar figures whose corresponding lengths are in the ratio $a{:}\,b$, the ratio of their volumes is $a^3{:}\,b^3$.*

- If a solid is scaled by factors of $r$, $s$, and $t$ in three perpendicular directions, what is the effect on the volume?
  - *The volume of the scaled figure is equal to the product of the volume of the original figure and the scale factors $r$, $s$, and $t$.*

**Exit Ticket (5 minutes)**

Name ______________________________________________ Date ____________________

# Lesson 9: Scaling Principle for Volumes

## Exit Ticket

1. Two circular cylinders are similar. The ratio of the areas of their bases is $9:4$. Find the ratio of the volumes of the similar solids.

2. The volume of a rectangular pyramid is $60$. The width of the base is then scaled by a factor of $3$, the length of the base is scaled by a factor of $\frac{5}{2}$, and the height of the pyramid is scaled such that the resulting image has the same volume as the original pyramid. Find the scale factor used for the height of the pyramid.

## Exit Ticket Sample Solutions

1. **Two circular cylinders are similar. The ratio of the areas of their bases is $9:4$. Find the ratio of the volumes of the similar solids.**

   *If the solids are similar, then their bases are similar as well. The areas of similar plane figures are related by the square of the scale factor relating the figures, and if the ratios of the areas of the bases is $9:4$, then the scale factor of the two solids must be $\sqrt{\frac{9}{4}} = \frac{3}{2}$. The ratio of lengths in the two solids is, therefore, $3:2$.*

   *By the scaling principle for volumes, the ratio of the volumes of the solids is the ratio $3^3:2^3$, or $27:8$.*

2. **The volume of a rectangular pyramid is $60$. The width of the base is then scaled by a factor of $3$, the length of the base is scaled by a factor of $\frac{5}{2}$, and the height of the pyramid is scaled such that the resulting image has the same volume as the original pyramid. Find the scale factor used for the height of the pyramid.**

   *The scaling principle for volumes says that the volume of a solid scaled in three perpendicular directions is equal to the area of the original solid times the product of the scale factors used in each direction. Since the volumes of the two solids are the same, it follows that the product of the scale factors in three perpendicular directions must be $1$.*

   *Let $s$ represent the scale factor used for the height of the image:*

$$\text{Volume}(A') = \left(3 \cdot \frac{5}{2} \cdot s\right)(\text{Area}(A))$$
$$60 = \frac{15}{2} \cdot s \cdot (60)$$
$$s = \frac{2}{15}$$

   *The scale factor used to scale the height of the pyramid is $\frac{2}{15}$.*

## Problem Set Sample Solutions

1. **Coffees sold at a deli come in similar-shaped cups. A small cup has a height of $4.2"$ and a large cup has a height of $5"$. The large coffee holds $12$ fluid ounces. How much coffee is in a small cup? Round your answer to the nearest tenth of an ounce.**

   *The scale factor of the smaller cup is $\frac{4.2}{5} = 0.84$. The cups are similar so the scale factor is the same in all three perpendicular dimensions. Therefore, the volume of the small cup is equal to the volume of the large cup times $(0.84)^3$, or $0.592704$.*

   $\text{Volume} = (12)(0.592704)$

   $\text{Volume} = 7.112448$

   *The small coffee cup contains approximately $7.1$ fluid ounces.*

2. Right circular cylinder $A$ has volume $2,700$ and radius $3$. Right circular cylinder $B$ is similar to cylinder $A$ and has volume $6,400$. Find the radius of cylinder $B$.

*Let $r$ be the radius of cylinder $B$.*

$$\frac{6400}{2700} = \frac{r^3}{3^3}$$
$$r^3 = 64$$
$$r = 4$$

3. The Empire State Building is a $102$-story skyscraper. Its height is $1,250$ ft. from the ground to the roof. The length and width of the building are approximately $424$ ft. and $187$ ft., respectively. A manufacturing company plans to make a miniature version of the building and sell cases of them to souvenir shops.

   a. The miniature version is just $\frac{1}{2500}$ of the size of the original. What are the dimensions of the miniature Empire State Building?

   *The height is $0.5$ ft., the length is about $0.17$ ft., and the width is $0.07$ ft.*

   b. Determine the volume of the minature building. Explain how you determined the volume.

   *Answers will vary since the Empire State Building has irregular shape. Some students may model the shape of the building as a rectangular pyramid, rectangular prism, or combination of the two.*

   *By modeling with a rectangular prism:*

   $\text{Volume} = [(0.17)(0.07)] \cdot (0.5)$

   $\text{Volume} = 0.00595$

   *The volume of the miniature building is approximately $0.00595\ \text{ft}^3$.*

   *By modeling with a rectangular pyramid:*

   $\text{Volume} = \frac{1}{3}[(0.17)(0.07)] \cdot (0.5)$

   $\text{Volume} = 0.00198$

   *The volume of the miniature building is approximately $0.00198\ \text{ft}^3$.*

   *By modeling with the composition of a rectangular prism and a rectangular pyramid:*

   *Combination of prisms; answers will vary. The solution that follows considers the highest $0.1$ ft. of the building to be a pyramid and the lower $0.4$ ft. of the building to be rectangular.*

$$\text{Volume} = \frac{1}{3}(0.17 \times 0.07 \times 0.5) + 0.17 \times 0.07 \times 0.5$$
$$\text{Volume} = \frac{4}{3}(0.17 \times 0.07 \times 0.5)$$
$$\text{Volume} = 0.00793$$

   *The volume of the miniature building is approximately $0.00793\ \text{ft}^3$.*

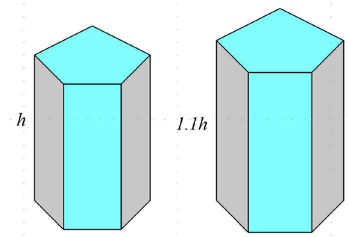
4. If a right square pyramid has a $2 \times 2$ square base and height $1$, then its volume is $\frac{4}{3}$. Use this information to find the volume of a right rectangular prism with base dimensions $a \times b$ and height $h$.

*Scale the right square pyramid by $\frac{a}{2}$ and $\frac{b}{2}$ in the directions determined by the sides of the square base and by $h$ in the direction perpendicular to the base. This turns the right square pyramid into a right rectangular pyramid with rectangular base of side lengths $a$ and $b$ and height $h$ that has volume*

$$\frac{4}{3} \cdot \frac{a}{2} \cdot \frac{b}{2} \cdot h = \frac{1}{3}abh = \frac{1}{3} area\ of\ base \times height.$$

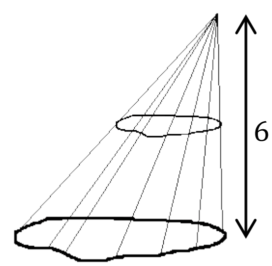
5. The following solids are similar. The volume of the first solid is $100$. Find the volume of the second solid.

$(1.1)^3 \cdot 1000 = 1331$




6. A general cone has a height of $6$. What fraction of the cone's volume is between a plane containing the base and a parallel plane halfway between the vertex of the cone and the base plane?

*The smaller top cone is similar to the whole cone and has volume $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$ of the volume of the whole cone. So, the region between the two planes is the remaining part of the volume and, therefore, has $\frac{7}{8}$ the volume of the whole cone.*




7. A company uses rectangular boxes to package small electronic components for shipping. The box that is currently used can contain $500$ of one type of component. The company wants to package twice as many pieces per box. Michael thinks that the box will hold twice as much if its dimensions are doubled. Shawn disagrees and says that Michael's idea provides a box that is much too large for $1,000$ pieces. Explain why you agree or disagree with one or either of the boys. What would you recommend to the company?

*If the box's dimensions were all doubled, the volume of the box should be enough to contain $2^3 = 8$ times the number of components that the current box holds, which is far too large for $1,000$ pieces. To double the volume of the box, the company could double the width, the height, or the length of the box, or extend all dimensions of the box by a scale factor of $\sqrt[3]{2}$.*

8. A dairy facility has bulk milk tanks that are shaped like right circular cylinders. They have replaced one of their bulk milk tanks with three smaller tanks that have the same height as the original but $\frac{1}{3}$ the radius. Do the new tanks hold the same amount of milk as the original tank? If not, explain how the volumes compare.

*If the original tank had a radius of $r$ and a height of $h$, then the volume of the original tank would be $\pi r^2 h$. The radii of the new tanks would be $\frac{1}{3}r$, and the volume of the new tanks combined would be $3\left(\pi \left(\frac{1}{3}r\right)^2 h\right) = \frac{1}{3}\pi r^2 h$. The bases of the new tanks were scaled down by a scale factor of $\frac{1}{3}$ in two directions, so the area of the base of each tank is $\frac{1}{9}$ the area of the original tank. Combining the three smaller tanks provides a base area that is $\frac{3}{9} = \frac{1}{3}$ the base area of the original tank. The ratio of the volume of the original tank to the three replacement tanks is $(\pi r^2\ h) : \frac{1}{3}(\pi r^2\ h)$ or $3:1$, which means the original tank holds three times as much milk as the replacements.*

# Lesson 10: The Volume of Prisms and Cylinders and Cavalieri's Principle

## Student Outcomes

- Students understand the principle of parallel slices in the plane and understand Cavalieri's principle as a generalization of the principle of parallel slices.
- Students use Cavalieri's principle to reason that the volume formula for a *general* cylinder is area of base $\times$ height.

## Lesson Notes

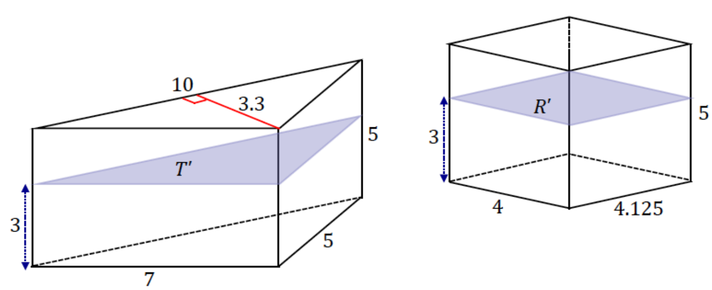
Students examine what it means to have the same area and same volume in terms of slices. This examination results in the two principles: the principle of parallel slices and Cavalieri's principle. Cavalieri's principle is used to justify why the volume of any cylinder, not just right cylinders as proposed so far in this module, is area of base $\times$ height. Neither of these principles is rigorously proven, but rather, they will be among the assumptions we use in later proofs such as the proof for the volume of a sphere.

## Classwork

### Opening Exercise (6 minutes)

**Opening Exercise**

**The bases of the following triangular prism $T$ and rectangular prism $R$ lie in the same plane. A plane that is parallel to the bases and also a distance 3 from the bottom base intersects both solids and creates cross-sections $T'$ and $R'$.**




a. **Find $\text{Area}(T')$.**

$\mathbf{Area}(T') = 16.5$

b. **Find $\text{Area}(R')$.**

$\mathbf{Area}(R') = 16.5$

**c.** **Find $\text{Vol}(T)$.**

$\text{Vol}(T) = 82.5$

**d.** **Find $\text{Vol}(R)$.**

$\text{Vol}(R) = 82.5$

**e.** **If a height other than 3 were chosen for the cross-section, would the cross-sectional area of either solid change?**

***No, the areas of the cross-sections of both solids at any given height are the same.***

## Discussion (10 minutes)

- By the end of this lesson, we are going to be able to draw a few more conclusions regarding the Opening Exercise.
- Take a look at Figure 1. Do you notice anything special regarding the red, green, and blue regions?
  - *Take any student observations; the fact that the areas are the same may or may not come up as a guess.*

**Discussion**

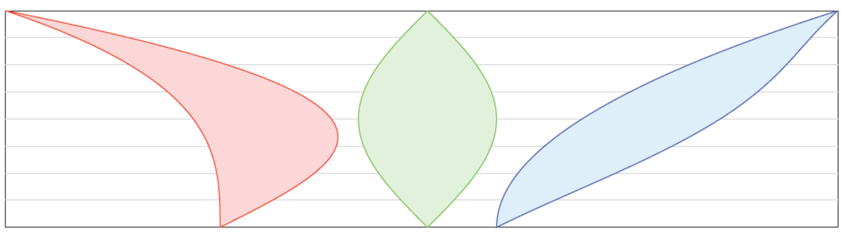


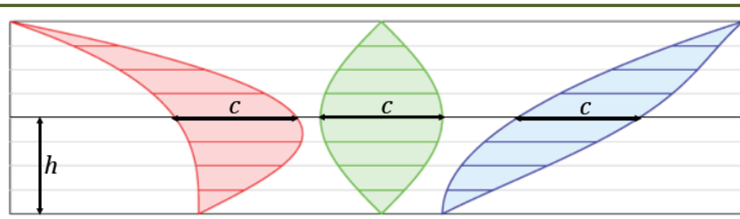
**Figure 1**

- Examine Examples 1 and 2. Each has the cross-sectional length of the red, green, and blue regions marked at a particular height. In each example, the cross-sectional length is the same.

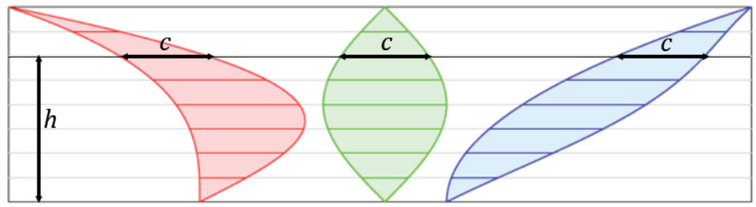
Explain what is meant by *cross-sectional length*: A cross-sectional length is the horizontal distance across a region. As an alternative, have students actually measure each cross-sectional length.

MP.3
- What would you conjecture this means for the area of the regions in relation to each other?

**Example 1**



**Example 2**




- It turns out that the area of the red, green, and blue regions is the same by the following principle:

**PRINCIPLE OF PARALLEL SLICES IN THE PLANE: If two planar figures of equal altitude have identical cross-sectional lengths at each height, then the regions of the figures have the same area.**

- The reason this principle is true is because the regions can be approximated by rectangles, and two figures with identical cross-sectional lengths can be approximated using congruent rectangles.
- That is, the same rectangles that approximate the red region can be used to approximate the green region, which can also be used to approximate the blue region as shown below.

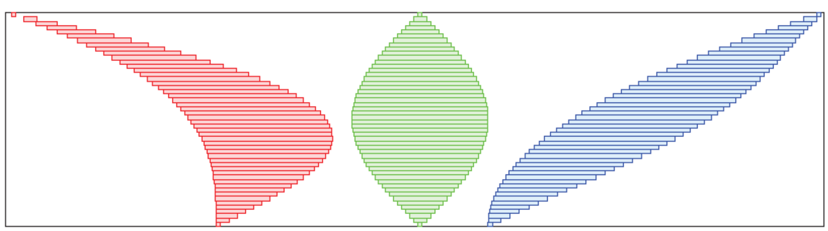


**Figure 2**

- Consider the red, green, and blue regions broken up into rectangles in this way in Figure 2. Imagine we created such a region out of little plastic (green) rectangles. Now picture pushing the top third of the rectangles over to the left without disturbing the bottom two-thirds of the rectangles. Other than moving the rectangles over, no change has been made to them. This rearranged version of the green region now looks like the red region. Since we can do this no matter how thin our slices are, we must conclude that the areas of the red and green regions are equal. We can make a similar argument in order to re-create the blue region.

*Scaffolding:*

Illustrate this point with the following hands-on approach in small groups:

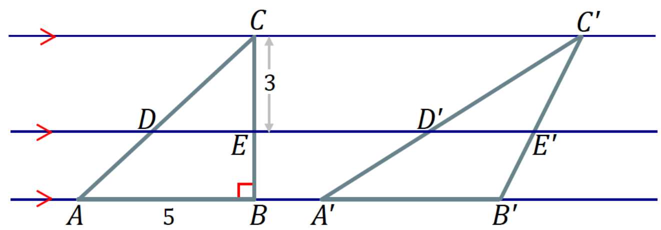
- Take two $8.5 \times 11$ in. sheets of colored paper.
- Cut each into eleven 1-inch strips.
- Have one student arrange and tape down one sheet's worth of strips to the board assembled to look like a whole sheet of paper.
- Challenge a different student to find an arrangement of the paper strips without any overlaps so that the area of the entire figure is *different* from the area of the original figure.

## Example (10 minutes)

The Example demonstrates that the converse of the principle of parallel slices is not true; that is, given two figures with the same height, and the same area, it is not necessarily true that slices at corresponding heights are equal in length.

**Example**

a. **The following triangles have equal areas: $\text{Area}(\triangle ABC) = \text{Area}(\triangle A'B'C') = 15 \text{ units}^2$. The distance between $\overleftrightarrow{DE}$ and $\overleftrightarrow{CC'}$ is $3$. Find the lengths $\overline{DE}$ and $\overline{D'E'}$.**




$$\text{Area}(\triangle ABC) = \frac{1}{2}(AB)(CB)$$
$$15 = \frac{1}{2}(5)(CB)$$
$$CB = 6$$

*Since $\overleftrightarrow{DE} \parallel \overleftrightarrow{CC'}$, the height of $\triangle A'B'C'$ must also be $6$. Then,*

$$\text{Area}(\triangle A'B'C') = \frac{1}{2}(A'B')(6)$$
$$15 = \frac{1}{2}(A'B')(6)$$
$$A'B' = 5$$

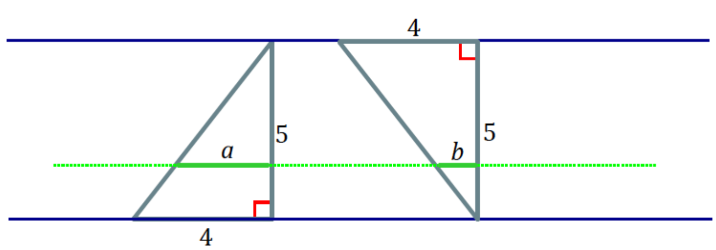
*By the AA similarity criterion, $\triangle ABC \sim \triangle DEC$ and $\triangle A'B'C' \sim \triangle D'E'C'$, and the scale factor is determined by the value of the ratio of the heights:*

$$k = \frac{CE}{CB} = \frac{3}{6} = \frac{1}{2}$$

*Then,*

$$DE = \frac{1}{2}(5) = 2.5$$
$$D'E' = \frac{1}{2}(5) = 2.5$$

b. **Joey says that if two figures have the same height and the same area, then their cross-sectional lengths at each height will be the same. Give an example to show that Joey's theory is incorrect.**




*Though both triangles have the same height and the same area, their cross-sectional lengths, represented here by $a$ and $b$, are not the same.*

**Discussion (12 minutes)**

To prime students for Cavalieri's principle, share three stacks of paper (or three decks of cards or three stacks of the same coins). Make sure that the three stacks of objects are not stacked the same. That is, let students know there are equal numbers in each stack, but arrange each stack a little differently so that they do not look like identical stacks. Then ask,

- What could you say about the volumes of each of these stacks in comparison to the others? Are the volumes the same? Are they different? How do you know?

Students are likely to say that since each stack holds the same number of individual items, the volume of the stacks should all be the same.

- Just as we have examined the parallel between two-dimensional properties of area and three-dimensional properties of volume, so we now consider what the principle of parallel slices would mean in a three-dimensional context. The idea is called Cavalieri's principle.
- In $1635$, Italian mathematician Bonaventure Cavalieri observed, much as you just have, that if you take a stack of paper and shear it, or twist it (i.e., deform) the stack in any way, the volume of paper in the stack does not change (Figure 3). Furthermore, the cross-section for each of the stacks is the same sheet of paper.

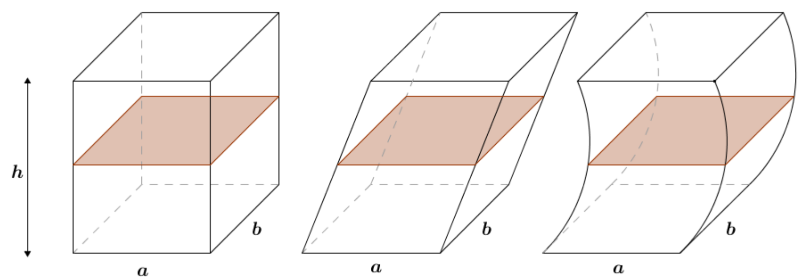



**Figure 3**

CAVALIERI'S PRINCIPLE: **Given two solids that are included between two parallel planes, if every plane parallel to the two planes intersects both solids in cross-sections of equal area, then the volumes of the two solids are equal.**

- Consider how Cavalieri's principle is related to the principle of parallel slices. Explain each principle in your own words and how they are related.

Provide time for a Quick Write with the last statement as the prompt: Consider how Cavalier's principle is related to the principle of parallel slices. Explain each principle in your own words and how they are related. Ask students to simply think for 30 seconds and write for another 90 seconds. Share out ideas as a whole group afterward.

Just as the same cross-sectional length at any given height of two regions with equal altitudes implies that the two regions are of equal area, cross-sections (i.e., regions, not lengths) of equal areas at every height of two solids of equal altitudes imply that the solids have equal volumes. We want students to notice the parallel between the two principles for two dimensions and three dimensions.

After the Quick Write, refer to Figure 4 when asking the following question. In the demonstration and regarding the anecdote on Cavalieri, students are introduced to Cavalieri's principle in the context of stacks of items that are identical. We want them to consider solids whose lateral edges are not necessarily perpendicular to the bases (see Figure 4).

MP.1

- Assuming that two solids do meet the criteria of Cavalieri's principle—that is, both solids have the same altitude and have cross-sections of equal area at every height—does it imply that both solids are of the same type? In other words, does the principle only work if we compare two triangular prisms or two cylinders? Does the principle apply if the two solids are different, e.g., one is a cylinder and one is a triangular prism, or if one solid is a right solid, while the other is an oblique solid? Why?

Allow students time in partners to discuss before sharing out responses as a whole class. Students' answers may vary. Some might believe that Cavalieri's principle only holds when the solids being compared are of the same type; others may have arguments for why the principle holds regardless of the types of solids being compared. Once students share out a few ideas, proceed with the explanation below.

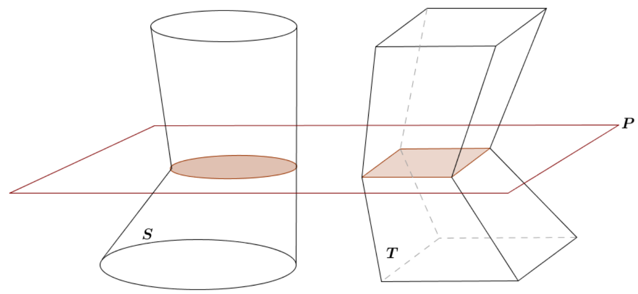


**Figure 4**

- Cavalieri's principle holds true regardless of the types of solids being compared, as long as the criteria of the principle are met. The volume of a solid can be approximated by right prisms, as we saw in Lesson 8.
- Then, if we took multiple cross-sections of each solid, all at the same heights, we would build corresponding prisms of equal volumes, and so the total volume of each solid must be the same.
- Let's look at a visual example. In Figure 5 we have two different types of solids: one right square pyramid and one oblique trapezoidal pyramid. The two solids meet the criteria of Cavalieri's principle because both are equal height, and the green cross-sections, taken at the same height, are equal in area. Does Cavalieri's principle hold? Do the two solids have the same volume?

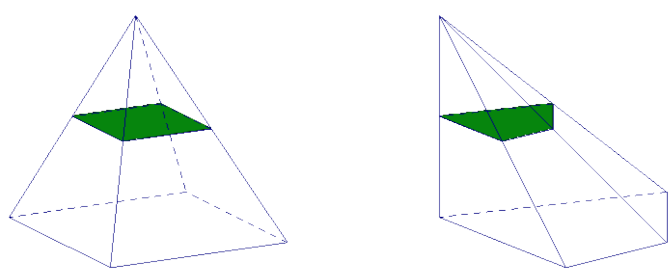


**Figure 5**

- The images in Figure 6 approximate the volumes of the solids in Figure 5. Each volume is approximated by eight right prisms. Each corresponding prism has a base equal in area and equal in height to the other. What can we conclude about the relative volumes of the two solids?
  - *Since each of the corresponding pairs of prisms has the same volume, the total volumes of each solid must be equal to each other.*

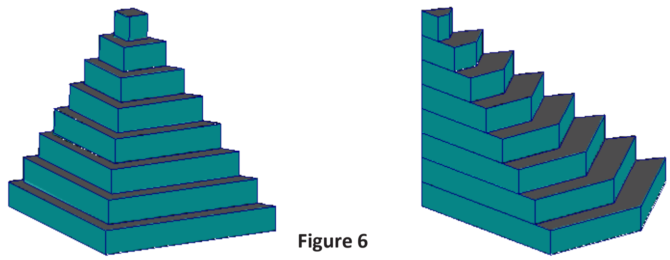


Figure 6

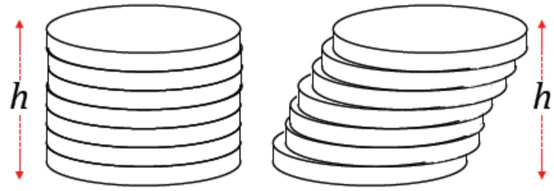
> *Scaffolding:*
> - Consider emphasizing that the volumes in Figure 6 must be the same by numbering each of the eight prisms for both solids and saying, "If prism 1 for each solid has the same base area and the same height, what do we know about the volumes of prism 1 for each solid?" (The volumes are the same.)

- Compare these images with the image of the red, green, and blue regions broken into small rectangles. These eight prisms approximate the volume of the cone in the same way that the small rectangles approximated the area of the red, green, and blue regions.
- Imagine if instead of eight prisms, there were $100$, or $1{,}000$. What happens to the approximation of the volume as the number of prisms increases?
  - *The approximation is refined as the number of cross-sections increases.*
- In fact, as we increase the number of prisms, the height of each prism used for the approximation approaches $0$, meaning that they more and more closely resemble cross-sections.

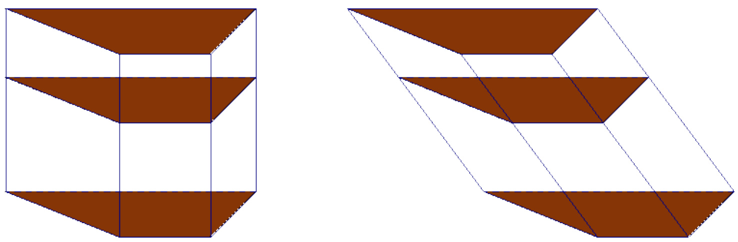
At this point, ask students to summarize what has been covered so far in this discussion. The following points succinctly capture what students should share:

- *Cavalieri's principle is for solids while the principle of parallel slices in the plane is for planar regions.*
- *Cavalieri's principle says that if two solids have the same altitude (i.e., if bases of both solids lie in the same parallel planes) and the cross-sections of both solids are equal in area at any height, then the solids are equal in volume.*
- *Furthermore, it is worth making the point that Cavalieri's principle holds for any two solids as long as its criteria are met. We should not think of the principle holding for two solids that look alike or happen to both be of the same type.*
- *Finally, the reason the principle holds for any two solids that meet its criteria is because the volume of each solid can be approximated by right prisms built over cross-sections for any given height. Since we know the area of each solid's cross-section is the same at any given height, the volume of the prisms built over each cross-section must be the same as well.*

- Recall a question we first posed in Lesson 6. In reference to two stacks of cards or coins of equal height, we asked whether the volume of the stack was different based on whether we had a right cylinder or oblique cylinder.
- We can now answer this question: Are the two volumes different? Why?
  - *By Cavalieri's principle, each cross-section between two stacks of cards or coins is the same, and the heights of each solid are the same, so the two volumes are equal.*



- Prior to this lesson, we had established the volume formula for general right prisms (this of course captures figures with polygonal, circular, and irregular bases) is $\text{area of base} \times \text{height}$.
- Now we can use Cavalieri's principle to show the volume formula of *any* cylinder, i.e., not just general right cylinders as studied so far, is $\text{area of base} \times \text{height}$. Why? Explain in terms of two cylinders that have identical bases, but one is oblique, and the other is right.
  - *If two general cylinders have the same heights and have identical bases, then the cross-section of both will be identical at any given height, regardless of whether one is oblique and the other is right. The prisms built over corresponding cross-sections will be equal in volume, so the total volumes of the two figures will be the same, and therefore we can use the volume formula* $\text{area of base} \times \text{height}$ *for any general cylinder.*



## Closing (2 minutes)

- Revisit the Opening Exercise. If you were only given the answers to parts (a) and (b) and the fact that both cylinders had the same height, what could you conclude?
  - *By Cavalieri's principle, since the two prisms were of equal height and would have cross-sections of equal areas at every height, we could conclude that the volumes of the two solids would be equal.*
- Restate the principle of parallel slices. Restate Cavalieri's principle.

**Lesson Summary**

**PRINCIPLE OF PARALLEL SLICES IN THE PLANE: If two planar figures of equal altitude have identical cross-sectional lengths at each height, then the regions of the figures have the same area.**

**CAVALIERI'S PRINCIPLE: Given two solids that are included between two parallel planes, if every plane parallel to the two planes intersects both solids in cross-sections of equal area, then the volumes of the two solids are equal.**

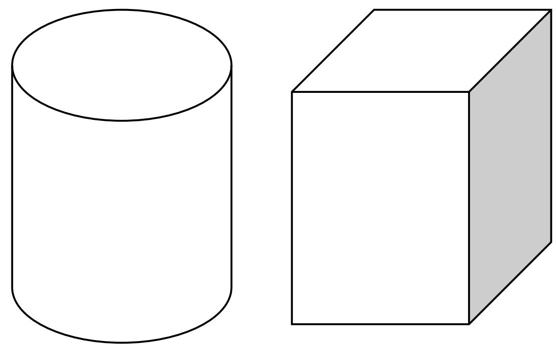
## Exit Ticket (5 minutes)

Name ____________________________________________ Date ____________________

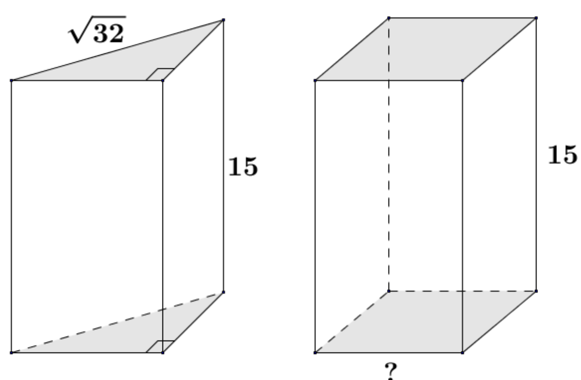
# Lesson 10: The Volume of Prisms and Cylinders and Cavalieri's Principle

## Exit Ticket

1. Morgan tells you that Cavalieri's principle cannot apply to the cylinders shown below because their bases are different. Do you agree or disagree? Explain.

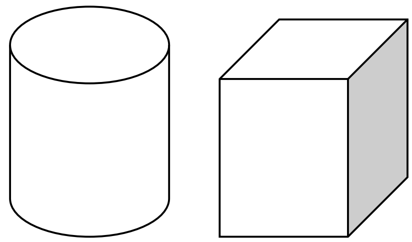


2. A triangular prism has an isosceles right triangular base with a hypotenuse of $\sqrt{32}$ and a prism height of $15$. A square prism has a height of $15$ and its volume is equal to that of the triangular prism. What are the dimensions of the square base?



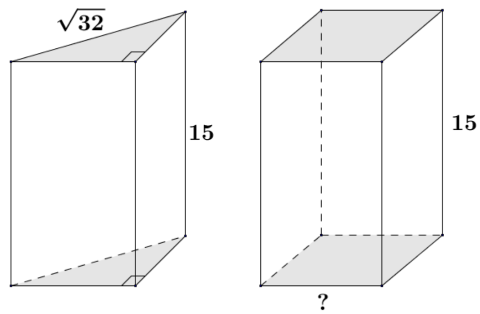
## Exit Ticket Sample Solutions

1. **Morgan tells you that Cavalieri's principle cannot apply to the cylinders shown below because their bases are different. Do you agree or disagree? Explain.**



*Even though the bases are not the same geometric figure, Cavalieri's principle may still apply because it is based on parallel slices with equal areas. If the areas of the bases are the same and the heights of the cylinders are the same, then the volumes of the cylinders are equal.*

2. **A triangular prism has an isosceles right triangular base with a hypotenuse of $\sqrt{32}$ and a prism height of $15$. A square prism has a height of $15$, and its volume is equal to that of the triangular prism. What are the dimensions of the square base?**




*If the volumes of the solids are the same and their heights are the same, then the areas of the parallel cross-sections (or bases) must be the same since the area of all parallel slices in a prism are congruent.*

$\sqrt{32} = 4\sqrt{2}$

*The legs of the isosceles right triangular base are length $4$.*

*The area of the base of the triangular prism $B_t$:*

$$B_t = \frac{1}{2} \cdot 4 \cdot 4$$
$$B_t = 8$$

*The area of the base of the triangular prism must be equal to $B_t$, and being a square, its sides, $x$, must be equal in length, making the area of the square base $x^2$:*

$$x^2 = 8$$
$$x = \sqrt{8} = 2\sqrt{2}$$

*The lengths of the sides of the square base are $2\sqrt{2}$.*

## Problem Set Sample Solutions

1. **Use the principle of parallel slices to explain the area formula for a parallelogram.**

*Every slice of a parallelogram made parallel to a side has the same length as that side, which is also true in a rectangle. Since each figure has the same height and the lengths of their slices at each level are equal in length, it follows that the figures have the same area. The area of a rectangle is the product of its length and width; therefore, the area of the parallelogram is the product of the length of a side times the distance between it and the opposite side.*

2. **Use the principle of parallel slices to show that the three triangles shown below all have the same area.**

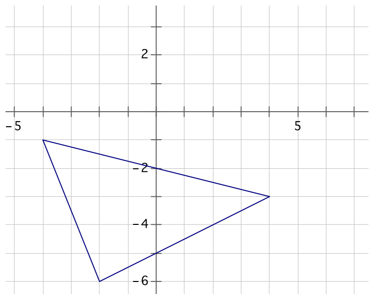



**Figure 1**

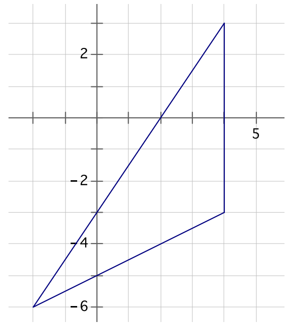



**Figure 2**

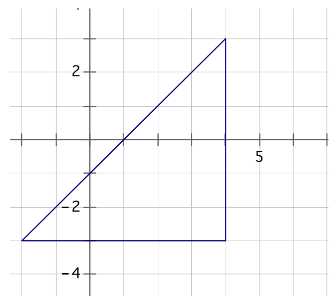
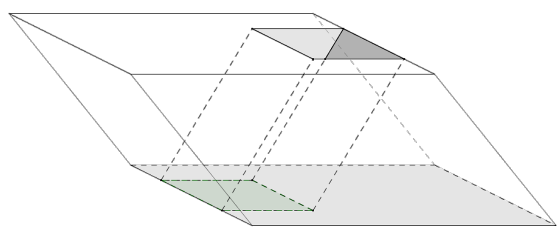



**Figure 3**

*The triangle in Figure 1 shares points at $(-2,-6)$ and $(4,-3)$ with the triangle in Figure 2, which has slope of $\frac{1}{2}$. If you slide the vertex at $(-4,-1)$ along a parallel line of slope $\frac{1}{2}$ to $(4,3)$, the height of the triangle does not change, and the triangle, therefore, has the same area. By parallel slices, the triangle in Figure 1 has the same area as the triangle in Figure 2.*

*The triangles in Figure 2 and Figure 3 share the side between points at $(4,-3)$ and $(4,3)$, which is vertical. If we slide the vertex at $(-2,-6)$ in Figure 2 vertically to $(-2,-3)$, the height of the triangle again does not change and it, therefore, has the same area. Again, by the principle of parallel slices, the areas of the triangles in Figures 2 and 3 are the same.*

3. **An oblique prism has a rectangular base that is $6 \text{ in.} \times 9 \text{ in.}$ A hole in the prism is also the shape of an oblique prism with a rectangular base that is $3$ in. wide and $6$ in. long, and the prism's height is $9$ in. (as shown in the diagram). Find the volume of the remaining solid.**



*Volume of the large prism:*

$16 \text{ in.} \cdot 9 \text{ in.} \cdot 9 \text{ in.} = 1{,}296 \text{ in}^3$

*Volume of the prism-shaped hole:*

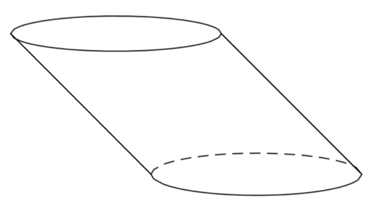
$3 \text{ in.} \cdot 6 \text{ in.} \cdot 9 \text{ in.} = 162 \text{ in}^3$

*Volume of the remaining solid:*

$1{,}296 \text{ in}^3 - 162 \text{ in}^3 = 1{,}134 \text{ in}^3$

4. **An oblique circular cylinder has height $5$ and volume $45\pi$. Find the radius of the circular base.**

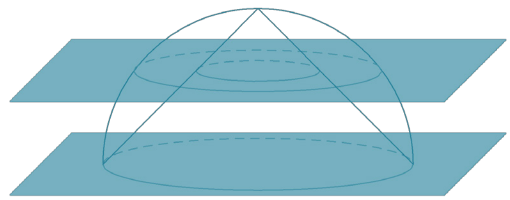
$$\text{Volume} = \pi r^2 \cdot 5$$
$$45\pi = \pi r^2 \cdot 5$$
$$r^2 = 9$$
$$r = 3$$



*The radius of the circular base is $3$.*

5. **A right circular cone and a solid hemisphere share the same base. The vertex of the cone lies on the hemisphere. Removing the cone from the solid hemisphere forms a solid. Draw a picture, and describe the cross-sections of this solid that are parallel to the base.**

   *The base plane intersects the solid in a circle. A plane through the vertex of the cone meets the solid in a single point. Cross-sections arising from intermediate planes are regions between concentric circles in the plane.*

6. **Use Cavalieri's principle to explain why a circular cylinder with a base of radius $5$ and a height of $10$ has the same volume as a square prism whose base is a square with edge length $5\sqrt{\pi}$ and whose height is also $10$.**

   *The area of the base of the circular cylinder:*

   $\text{Area} = \pi \cdot r^2$

   $\text{Area} = \pi \cdot 5^2 = 25\pi$

   The area of the base of the square prism:

   $\text{Area} = \text{length} \cdot \text{width}$

   $\text{Area} = (5\sqrt{\pi}) \cdot (5\sqrt{\pi})$

   $\text{Area} = 25\pi$

   *The bases of the two cylinders have the same areas, and their heights are given as the same. Given that both solids are cylinders, every slice made parallel to the bases of the solids will have equal area; thus, the volume of each solid is the area of its base times its height, or $250\pi$.*

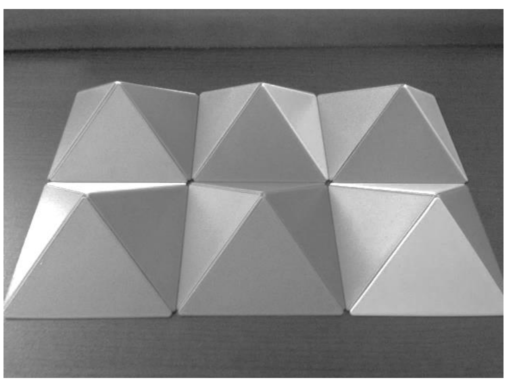
# Lesson 11: The Volume Formula of a Pyramid and Cone

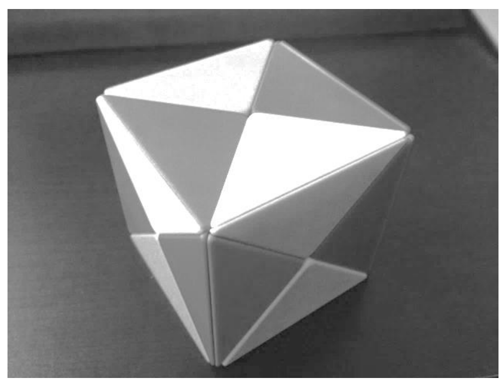
## Student Outcomes

- Students use Cavalieri's principle and the cone cross-section theorem to show that a general pyramid or cone has volume $\frac{1}{3}Bh$, where $B$ is the area of the base and $h$ is the height by comparing it with a right rectangular pyramid with base area $B$ and height $h$.

## Lesson Notes

The Exploratory Challenge is debriefed in the first discussion. The Exploratory Challenge and Discussion are the springboard for the main points of the lesson: (1) explaining why the formula for finding the volume of a cone or pyramid includes multiplying by one-third and (2) applying knowledge of the cone cross-section theorem and Cavalieri's principle from previous lessons to show why a general pyramid or cone has volume $\frac{1}{3}(\text{area of base})(\text{height})$. Provide manipulatives to students, 3 congruent pyramids and 6 congruent pyramids, to explore the volume formulas in a hands-on manner; that is, attempt to construct a cube from 3 congruent pyramids or 6 congruent pyramids such as in the image below.





## Classwork

### Exploratory Challenge (5 minutes)

**Exploratory Challenge**

**Use the provided manipulatives to aid you in answering the questions below.**

**a.**

**i. What is the formula to find the area of a triangle?**

$$A = \frac{1}{2}bh$$

ii. **Explain why the formula works.**

*The formula works because the area of a triangle is half the area of a rectangle with the same base and same height.*

b.

i. **What is the formula to find the volume of a triangular prism?**

*For base area $B$ and height of prism $h$,*

$V = Bh$.

ii. **Explain why the formula works.**

*A triangular prism is essentially a stack of congruent triangles. Taking the area of a triangle, repeatedly, is like multiplying by the height of the prism. Then the volume of the prism would be the sum of the areas of all of the congruent triangles, which is the same as the area of one triangle multiplied by the height of the prism.*

c.

i. **What is the formula to find the volume of a cone or pyramid?**

*For base area $B$ and height of prism $h$,*

$V = \frac{1}{3}Bh$.

ii. **Explain why the formula works.**

*Answers will vary. Some students may recall seeing a demonstration in Grade 8 related to the number of cones (three) it took to equal the volume of a cylinder with the same base area and height, leading to an explanation of where the one-third came from.*

## Discussion (10 minutes)

Use the following points to debrief the Exploratory Challenge. Many students will be able to explain where the $\frac{1}{2}$ in the triangle area formula comes from, so the majority of the discussion should be on where the factor $\frac{1}{3}$ comes from, with respect to pyramid and cone volume formulas. Allow ample time for students to work with the manipulatives to convince them that they cannot construct a prism (unit cube) from 3 congruent pyramids but will need 6 congruent pyramids instead.

- What is the explanation for the $\frac{1}{2}$ in the area formula for a triangle, $A = \frac{1}{2}bh$?
  - *Two congruent triangles comprise a parallelogram with base, $b$, and height, $h$. Then the area of the triangle is one-half of the area of the parallelogram (or rectangle).*
- What is the volume formula for a cone or pyramid?
  - *The volume of a cone or pyramid is found using the formula* $V = \frac{1}{3}(\text{area of base})(\text{height})$.

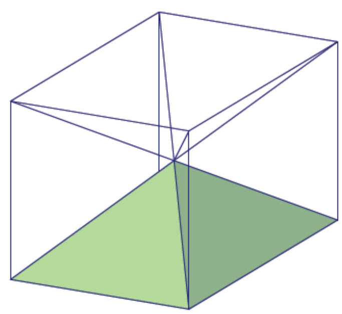
MP.3

- Where does the $\frac{1}{3}$ in that formula come from? Can we fit together three congruent pyramids or cones to form a prism or cylinder (as we did for the area of a triangle)?

MP.3 Allow time for students to attempt this with manipulatives. Encourage students to construct arguments as to why a prism can be constructed using the manipulatives, or more to the point, why they cannot do this. Some groups of students may develop an idea similar to the one noted below, i.e., using $6$ congruent pyramids instead. If so, allow students to work with the $6$ congruent pyramids. If students do not develop this idea on their own, share the point below with students, and then allow them to work with the manipulatives again.

Some students may recall seeing a demonstration in Grade 8 related to the number of cones (three) it took to equal the volume of a cylinder with the same base area and height, leading to an explanation of where the one-third came from.

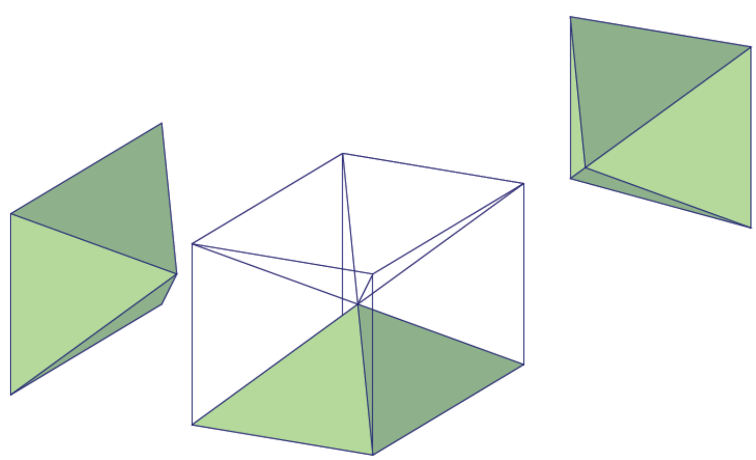
- In general, we cannot fit three congruent pyramids or cones together, but we can do something similar. Instead of fitting three congruent pyramids, consider the following: Start with a unit cube. Take a pyramid whose base is equal to $1$ (same as one face of the cube) and height equal to $\frac{1}{2}$ (half the height of the cube). If placed on the bottom of the unit cube, it would look like the green pyramid in the diagram below.



- How many total pyramids of the stated size would be needed to equal the volume of a unit cube?

MP.7 Provide time for students to discuss the answer in pairs, if needed.

   - *It would take $6$ of these pyramids to equal the volume of the cube: one at the bottom, one whose base is at the top, and four to be placed along the four sides of the cube.*
   - *Shown in the diagram below are two more pyramids in the necessary orientation to take up the volume along the sides of the cube.*



MP.7

- A cube with side length $1$ is the union of six congruent square pyramids with base dimensions of $1 \times 1$ and height of $\frac{1}{2}$. Since it takes six such pyramids, then the volume of one must be $\frac{1}{6}$. We will use this fact to help us make sense of the one-third in the volume formula for pyramids and cones.

### Discussion (15 minutes)

The discussion that follows is a continuation of the opening discussion. The first question asked of students is to make sense of the $\frac{1}{3}$ in the volume formula for pyramids and cones. It is important to provide students with time to think about how to use what was discovered in the opening discussion to make sense of the problem and construct a viable argument.

MP.3

- We would like to say that three congruent pyramids comprise the volume of a cube. If we could do that, then making sense of the one-third in the volume formula for pyramids and cones would be easy. All we know as of now is that it takes six congruent pyramids whose base must be equal to a face of the cube and whose height is one-half. How can we use what we know to make sense of the one-third in the formula?

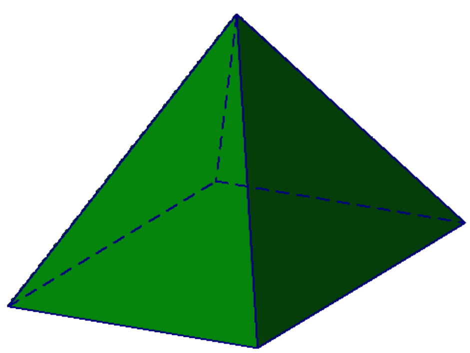
Provide time for students to make sense of the problem and discuss possible solutions in a small group. Have students share their ideas with the class so they can critique the reasoning of others.

*Scaffolding:*

Divide the class into groups. As they struggle with answering the question, consider calling one student from each group up for a huddle. In the huddle, ask students to discuss how the heights of the six pyramids compare to the height of the prism and what they could do to make the height of one of those pyramids equal to the height of the prism. Send them back to their groups to share the considerations and how they may be applied to this situation.

This reasoning uses scaling:

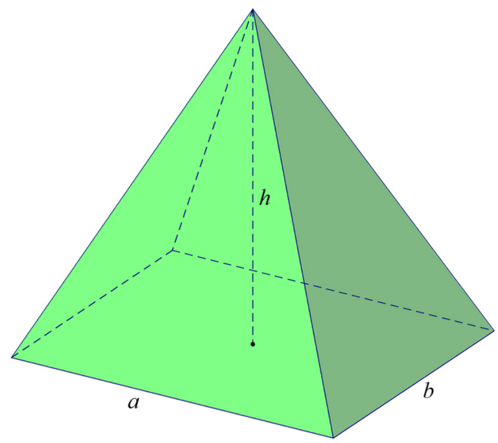
- *If we scaled one of the six pyramids by a factor of $2$ in the direction of the altitude of the cube, then the volume of the pyramid would also increase by a factor of $2$. This means that the new volume would be equal to $\frac{1}{3}$. Further, it would take $3$ such pyramids to equal the volume of the unit cube; therefore, the volume of a pyramid is $\frac{1}{3}$ of the volume of a prism with the same base and same height.*



This reasoning uses arithmetic:

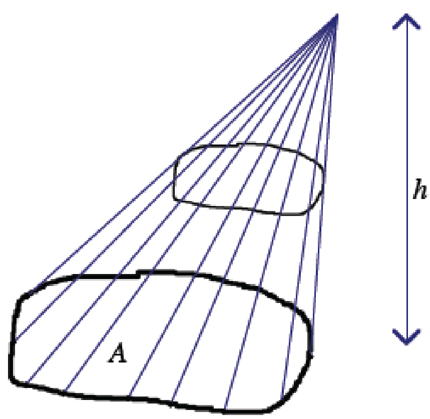
- *Since the height of the green pyramid is $\frac{1}{2}$ that of the unit cube, then we can compare the volume of half of the unit cube, $1 \times 1 \times \frac{1}{2} = \frac{1}{2}$, to the known volume of the pyramid, $\frac{1}{6}$ that of the unit cube. Since $\frac{1}{6} = \frac{1}{3} \times \frac{1}{2}$, then the volume of the pyramid is exactly $\frac{1}{3}$ the volume of a prism with the same base and same height.*

- If we scaled a square pyramid whose volume was $\frac{1}{3}$ of a unit cube by factors of $a$, $b$, and $h$ in three perpendicular directions of the sides of the square and the altitude, then the scaled pyramid would show that a right rectangular prism has volume $\frac{1}{3}abh = \frac{1}{3}(\text{area of base})(\text{height})$:

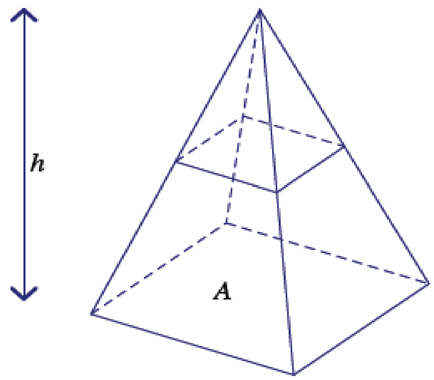



- Now let's discuss how to compute the volume of a general cone with base area $A$ and height $h$. Suppose we wanted to calculate the volume of the cone shown below. How could we do it?

Provide time for students to make sense of the problem and discuss possible solutions in a small group. Have students share their ideas with the class so they can critique the reasoning of others.

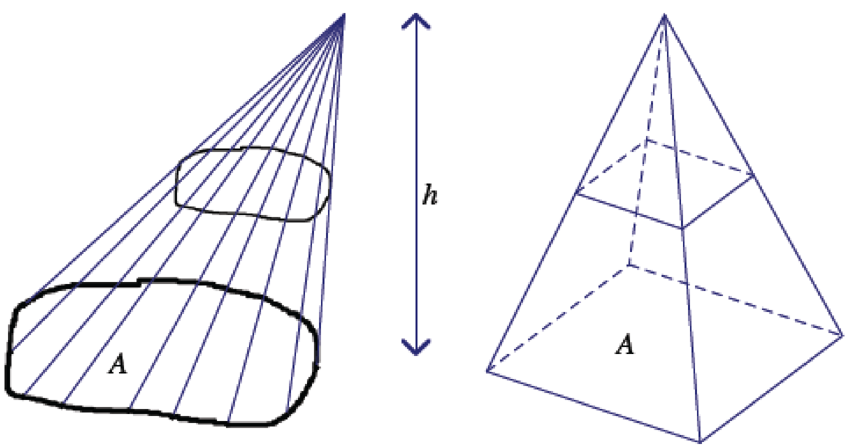



  - *Since the base, $A$, is an irregular shape, we could compare this cone to a right rectangular pyramid that has the same base area $A$ and height $h$.*




- *The cone cross-section theorem states that if two cones have the same base area and the same height, then cross-sections for the cones, the same distance from the vertex, have the same area.*

- What does Cavalieri's principle say about the volume of the general cone compared to the volume of the right rectangular pyramid?




  - *Cavalieri's principle shows that the two solids will have equal volume.*

- Given what was said about the cone cross-section theorem and Cavalieri's principle, what can we conclude about the formula to find the volume of a general cone?
  - *The formula to find the volume of a general cone is* $V = \frac{1}{3}(\text{area of base})(\text{height})$.

Before moving into the following exercises, pause for a moment to check for student understanding. Have students talk with their neighbor for a moment about what they have learned so far; then ask for students to share aloud.

### Exercises 1–4 (7 minutes)

The application of the formula in the exercises below can be assigned as part of the Problem Set or completed on another day, if necessary.

**Exercises**

1. **A cone fits inside a cylinder so that their bases are the same and their heights are the same, as shown in the diagram below. Calculate the volume that is inside the cylinder but outside of the cone. Give an exact answer.**

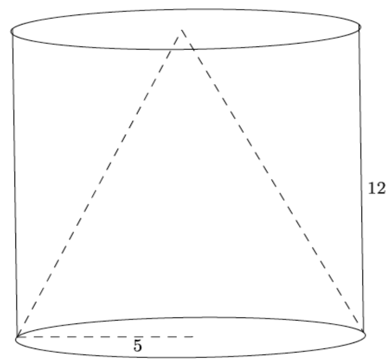
   *The volume of the cylinder is* $V = 5^2\pi(12) = 300\pi$.

   *The volume of the cone is* $V = \frac{1}{3}5^2\pi(12) = 100\pi$.

   *The volume of the space that is inside the cylinder but outside the cone is* $200\pi$.

   *Alternative solution:*

   *The space between the cylinder and cone is equal to* $\frac{2}{3}$ *the volume of the cylinder. Then the volume of the space is* $V = \frac{2}{3}\,5^2\pi(12) = 200\pi$.




2. **A square pyramid has a volume of $245\ \text{in}^3$. The height of the pyramid is $15\ \text{in}$. What is the area of the base of the pyramid? What is the length of one side of the base?**

$$V = \frac{1}{3}(\text{area of base})(\text{height})$$
$$245 = \frac{1}{3}(\text{area of base})(15)$$
$$245 = 5(\text{area of base})$$
$$49 = \text{area of base}$$

   *The area of the base is* $49\ \text{in}^2$, *and the length of one side of the base is* $7\ \text{in}$.

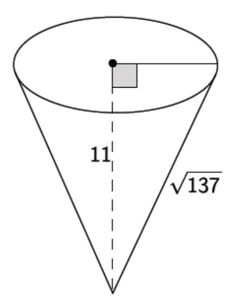
3. **Use the diagram below to answer the questions that follow.**

   a. **Determine the volume of the cone shown below. Give an exact answer.**

   *Let the length of the radius be* $r$.

$$11^2 + r^2 = \left(\sqrt{137}\right)^2$$
$$r^2 = \left(\sqrt{137}\right)^2 - 11^2$$
$$r^2 = 16$$
$$r = 4$$

   *The volume of the cone is* $V = \frac{1}{3}4^2\pi(11) = \frac{176}{3}\pi$.



b. **Find the dimensions of a cone that is similar to the one given above. Explain how you found your answers.**

*Student answers will vary. Possible student answer:*

*A cone with height $33$ and radius $12$ is similar to the one given because their corresponding sides are equal in ratio, i.e., $11:33$ and $4:12$. Therefore, there exists a similarity transformation that would map one cone onto the other.*

c. **Calculate the volume of the cone that you described in part (b) in two ways. (Hint: Use the volume formula and the scaling principle for volume.)**

*Solution based on the dimensions provided in possible student response from part (b).*

*Using the volume formula:*

$$V = \frac{1}{3}12^2\pi(33)$$
$$V = 1584\pi$$

*For two similar solids, if the ratio of corresponding sides is $a:b$, then the ratio of their volumes is $a^3:b^3$. The corresponding sides are in ratio $1:3$, so their volumes will be in the ratio $1^3:3^3 = 1:27$. Therefore, the volume of the cone in part (b) is $27$ times larger than the given one:*

$$V = 27\left(\frac{176}{3}\pi\right)$$
$$V = 1584\pi$$

4. **Gold has a density of $19.32\ \text{g/cm}^3$. If a square pyramid has a base edge length of $5\ \text{cm}$, a height of $6\ \text{cm}$, and a mass of $942\ \text{g}$, is the pyramid in fact solid gold? If it is not, what reasons could explain why it is not? Recall that density can be calculated with the formula $\text{density} = \frac{\text{mass}}{\text{volume}}$.**

$$V = \frac{1}{3}(5^2)(6)$$
$$V = 50$$

*The volume of the pyramid is $50$ g.*

$$\text{density(pyramid)} = \frac{942}{50} = 18.84$$

*Since the density of the pyramid is not $19.32\ \text{g/cm}^3$, the pyramid is not solid gold. This could be because part of it is hollow, or there is a mixture of metals that make up the pyramid.*

## Closing (3 minutes)

Ask students to summarize the main points of the lesson in writing, by sharing with a partner, or through a whole class discussion. Use the questions below, if necessary.

- Give an explanation as to where the $\frac{1}{3}$ comes from in the volume formula for general cones and pyramids.
  - *The volume formula for a general cylinder is the area of the cylinder's base times the height of the cylinder, and a cylinder can be decomposed into three cones, each with equal volume; thus, the volume of each of the cones is $\frac{1}{3}$ the volume of cylinder.*

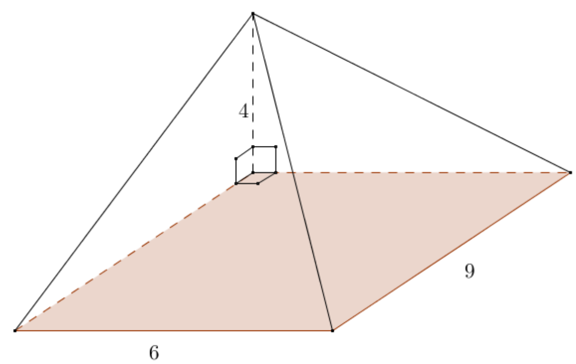
## Exit Ticket (5 minutes)

Name ___________________________________ Date ________________

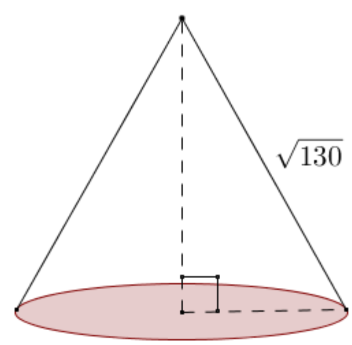
# Lesson 11: The Volume Formula of a Pyramid and Cone

## Exit Ticket

1. Find the volume of the rectangular pyramid shown.

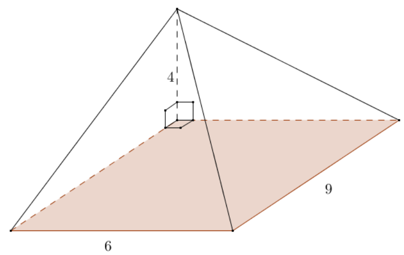



2. The right circular cone shown has a base with radius of 7. The slant height of the cone's lateral surface is $\sqrt{130}$. Find the volume of the cone.



## Exit Ticket Sample Solutions

1. **Find the volume of the rectangular pyramid shown.**



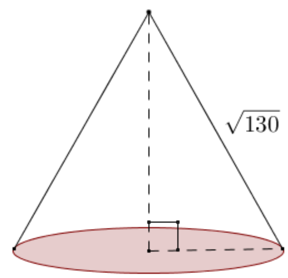

$\text{Volume} = \frac{1}{3}Bh$

$\text{Volume} = \frac{1}{3}(9 \cdot 6) \cdot 4$

$\text{Volume} = 72$

*The volume of the rectangular pyramid is* $72$*.*

2. **The right circular cone shown has a base with radius of $7$. The slant height of the cone's lateral surface is $\sqrt{130}$. Find the volume of the cone.**




*The slant surface, the radius, and the altitude, $h$, of the cone form a right triangle. Using the Pythagorean theorem,*

$$h^2 + 7^2 = \left(\sqrt{130}\right)^2$$
$$h^2 + 49 = 130$$
$$h^2 = 81$$
$$h = 9$$

$\text{Volume} = \frac{1}{3}Bh$

$\text{Volume} = \frac{1}{3}\left(\frac{22}{7} \cdot 7^2\right) \cdot 9$

$\text{Volume} = (154) \cdot 3$

$\text{Volume} = 462$

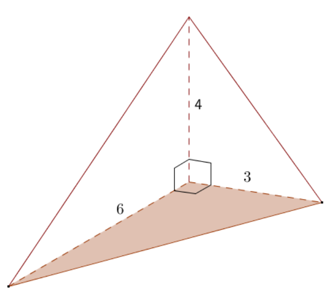
*The volume of the right circular cone is* $462$*.*

## Problem Set Sample Solutions

1. **What is the volume formula for a right circular cone with radius $r$ and height $h$?**

$\text{Area} = Bh = \frac{1}{3}(\pi r^2)h$

2. **Identify the solid shown, and find its volume.**




*The solid is a triangular pyramid.*

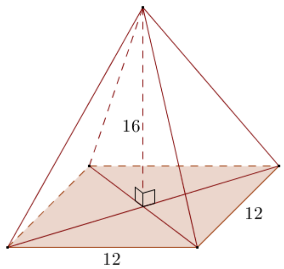
$\text{Volume} = \frac{1}{3}Bh$

$\text{Volume} = \frac{1}{3}\left(\frac{1}{2} \cdot 3 \cdot 6\right) \cdot 4$

$\text{Volume} = 12$

*The volume of the triangular pyramid is* $12$*.*

3. Find the volume of the right rectangular pyramid shown.



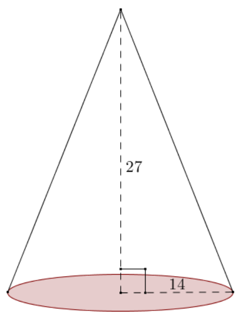

$\text{Volume} = \frac{1}{3}Bh$

$\text{Volume} = \frac{1}{3}(12 \cdot 12) \cdot 16$

$\text{Volume} = 768$

*The volume of the right rectangular pyramid is* $768$.

4. Find the volume of the circular cone in the diagram. (Use $\frac{22}{7}$ as an approximation of pi.)




$\text{Volume} = \frac{1}{3}Bh$

$\text{Volume} = \frac{1}{3}(\pi \cdot r^2)h$

$\text{Volume} = \frac{1}{3}\left(\frac{22}{7} \cdot 14^2\right) \cdot 27$

$\text{Volume} = \frac{1}{3}(616) \cdot 27$

$\text{Volume} = 5544$

*The volume of the circular cone is* $5544$.

5. Find the volume of a pyramid whose base is a square with edge length $3$ and whose height is also $3$.

$\text{Volume} = \frac{1}{3}Bh$

$\text{Volume} = \frac{1}{3}(3^2) \cdot 3 = 9$

6. Suppose you fill a conical paper cup with a height of $6''$ with water. If all the water is then poured into a cylindrical cup with the same radius and same height as the conical paper cup, to what height will the water reach in the cylindrical cup?

*A height of* $2''$

7. Sand falls from a conveyor belt and forms a pile on a flat surface. The diameter of the pile is approximately $10$ ft. and the height is approximately $6$ ft. Estimate the volume of the pile of sand. State your assumptions used in modeling.

*Assuming that the pile of sand is shaped like a cone, the radius of the pile would be* $5$ ft.*, and so the volume of the pile in cubic feet would be* $\frac{1}{3}\pi 5^2 \cdot 6 \approx 157$.

8. A pyramid has volume $24$ and height $6$. Find the area of its base.

*Let $A$ be the area of the base.*

$\text{Volume} = \frac{1}{3}Bh$

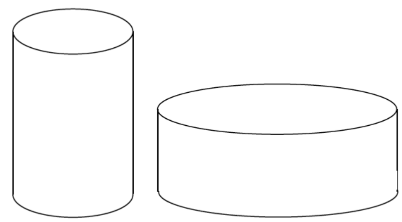
$24 = \frac{1}{3}A \cdot 6$

$A = 12$

*The area of the base is* $12$.

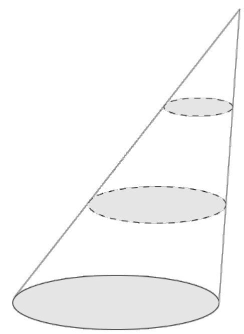
9. **Two jars of peanut butter by the same brand are sold in a grocery store. The first jar is twice the height of the second jar, but its diameter is one-half as much as the shorter jar. The taller jar costs $\$1.49$, and the shorter jar costs $\$2.95$. Which jar is the better buy?**

*The shorter jar is the better buy because it has twice the volume of the taller jar and costs five cents less than twice the cost of the taller jar.*



10. **A cone with base area $A$ and height $h$ is sliced by planes parallel to its base into three pieces of equal height. Find the volume of each section.**

*The top section is similar to the whole cone with scale factor $\frac{1}{3}$, so its volume is*



$$V_1 = \left(\frac{1}{3}\right)^3 \cdot \frac{1}{3}Ah$$
$$V_1 = \frac{1}{81}Ah.$$

*The middle section has volume equal to the difference of the volume of the top two-thirds of the cone and the top one-third of the cone.*

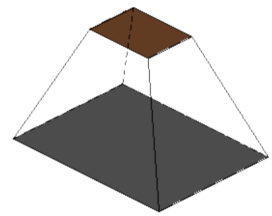
$$V_2 = \left(\frac{2}{3}\right)^3 \cdot \frac{1}{3}Ah - \left(\frac{1}{3}\right)^3 \frac{1}{3}Ah$$
$$V_2 = \left(\frac{8}{81} - \frac{1}{81}\right)Ah$$
$$V_2 = \frac{7}{81}Ah$$

*The bottom section has volume equal to the difference of the volume of the entire cone and the volume of the top two-thirds of the cone.*

$$V_3 = \frac{1}{3}Ah - \left(\frac{2}{3}\right)^3 \frac{1}{3}Ah$$
$$V_3 = \frac{1}{3}Ah - \frac{8}{81}Ah$$
$$V_3 = \frac{27}{81}Ah - \frac{8}{81}Ah$$
$$V_3 = \left(\frac{27}{81} - \frac{8}{81}\right)Ah$$
$$V_3 = \frac{19}{81}Ah$$

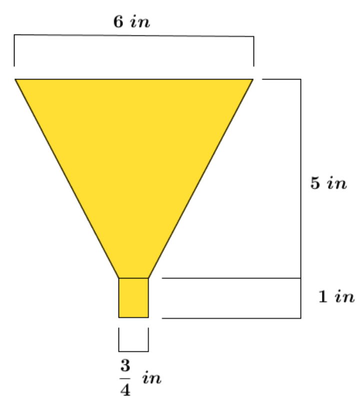
11. The frustum of a pyramid is formed by cutting off the top part by a plane parallel to the base. The base of the pyramid and the cross-section where the cut is made are called the *bases of the frustum*. The distance between the planes containing the bases is called the *height of the frustum*. Find the volume of a frustum if the bases are squares of edge lengths $2$ and $3$, and the height of the frustum is $4$.

*Let $V$ be the vertex of the pyramid and $h$ the height of the pyramid with base the smaller square of edge length $2$. Then $h+4$ is the height the pyramid. The smaller square is similar to the larger square with scale factor $\frac{2}{3}$, but the scale factor is also $\frac{h}{h+4}$. Solving $\frac{h}{h+4} = \frac{2}{3}$ gives $h = 8$. So, the volume of the pyramid whose base is the square of edge length $3$ and height $8+4 = 12$ is $\frac{1}{3}3^2 \cdot 12 = 36$. The volume of the pyramid with base the square of edge length $2$ and height $8$ is $\frac{1}{3}2^2 \cdot 8 = 10\frac{2}{3}$. The volume of the frustum is $36 - 10\frac{2}{3} = 25\frac{1}{3}$.*



12. A bulk tank contains a heavy grade of oil that is to be emptied from a valve into smaller $5.2$-quart containers via a funnel. To improve the efficiency of this transfer process, Jason wants to know the greatest rate of oil flow that he can use so that the container and funnel do not overflow. The funnel consists of a cone that empties into a circular cylinder with the dimensions as shown in the diagram. Answer each question below to help Jason determine a solution to his problem.

a. Find the volume of the funnel.




*The upper part of the funnel is a frustum, and its volume is the difference of the complete cone minus the missing portion of the cone below the frustum. The height of the entire cone is unknown; however, it can be found using similarity and scale factor. Let $x$ represent the unknown height of the cone between the frustum and the vertex. Then the height of the larger cone is $5 + x$.*

*Using corresponding diameters and heights:*

$$\frac{6}{5+x} = \frac{\frac{3}{4}}{x}$$
$$6x = \frac{3}{4}(5) + \frac{3}{4}(x)$$
$$\frac{21}{4}x = \frac{15}{4}$$
$$x = \frac{5}{7}$$

*The height of the cone between the frustum and the vertex is $\frac{5}{7}$ in., and the total height of the cone including the frustum is $5\frac{5}{7}$ in.*

*The volume of the frustum is the difference of the volumes of the total cone and the smaller cone between the frustum and the vertex.*

*Let $V_T$ represent the volume of the total cone in cubic inches, $V_S$ represent the volume of the smaller cone between the frustum and the vertex in cubic inches, and $V_F$ represent the volume of the frustum in cubic inches.*

$V_F = V_T - V_S$

$V_T = \frac{1}{3}\pi(3)^2\left(5\frac{5}{7}\right)$ $\qquad$ $V_S = \frac{1}{3}\pi\left(\frac{3}{8}\right)^2\left(\frac{5}{7}\right)$

$V_T = \pi(3)\left(\frac{40}{7}\right)$ $\qquad$ $V_S = \frac{1}{3}\pi\left(\frac{9}{64}\right)\left(\frac{5}{7}\right)$

$V_T = \frac{120}{7}\pi$ $\qquad$ $V_S = \frac{15}{448}\pi$

$V_F = \frac{120}{7}\pi - \frac{15}{448}\pi$

$V_F = \frac{1095}{64}\pi$

*Let $V_C$ represent the volume of the circular cylinder at the bottom of the funnel in cubic inches.*

$V_C = \pi\left(\frac{3}{8}\right)^2(1)$

$V_C = \frac{9}{64}\pi$

*Let $V$ represent the volume of the funnel in cubic inches.*

$V = V_F + V_C$

$V = \frac{1095}{64}\pi + \frac{9}{64}\pi$

$V = \frac{1104}{64}\pi \approx 54.2$

*The volume of the funnel is approximately* $54.2\ \text{in}^3$.

b. If $1\ \text{in}^3$ is equivalent in volume to $\frac{4}{231}$ qt., what is the volume of the funnel in quarts?

$\frac{1104}{64}\pi \cdot \left(\frac{4}{231}\right) = \frac{4416}{14784}\pi = \frac{23}{77}\pi \approx 0.938$

*The volume of the funnel is approximately* $0.938$ *quarts.*

c. If this particular grade of oil flows out of the funnel at a rate of $1.4$ quarts per minute, how much time in minutes is needed to fill the $5.2$-quart container?

$\text{Volume} = \text{rate} \times \text{time}$ $\qquad$ *Let $t$ represent the time needed to fill the container in minutes.*

$5.2 = 1.4(t)$

$t = \frac{26}{7} \approx 3.71$

*The container will fill in approximately* $3.71$ minutes ($3$ min. $43$ sec.) *at a flow rate of* $1.4$ *quarts per minute.*

d. Will the tank valve be shut off exactly when the container is full? Explain.

*If the tank valve is shut off at the same time that the container is full, the container will overflow because there is still oil in the funnel; therefore, the tank valve should be turned off before the container is filled.*

e. **How long after opening the tank valve should Jason shut the valve off?**

*The valve can be turned off when the container has enough room left for the oil that remains in the funnel.*

$$5.2 - \frac{23}{77}\pi \approx 4.262$$

*It is known that the flow rate out of the funnel is $1.4$ quarts per minute.*

*Let $t_p$ represent the time in minutes needed to fill the container and funnel with $5.2$ quarts of oil.*

$$\text{Volume} = \text{rate} \times \text{time}$$
$$5.2 - \frac{23}{77}\pi = 1.4t_p$$
$$t_p \approx 3.044$$

*The container will have enough room remaining for the oil left in the funnel at approximately $3.044$ minutes; therefore, the valve on the bulk tank can also be shut off at $3.044$ minutes.*

f. **What is the maximum constant rate of flow from the tank valve that will fill the container without overflowing either the container or the funnel?**

*The flow of oil from the bulk tank should just fill the funnel at $3.044$ minutes, when the valve is shut off.*

$\text{Volume} = \text{rate} \times \text{time}$ *Let $r$ represent the rate of oil flow from the bulk tank in quarts per minute.*

$$5.2 = r(3.044)$$
$$r \approx 1.708$$

*The flow of oil from the bulk tank can be at approximately $1.708$ quarts per minute to fill the container without overflowing the container or the funnel used.*

# Lesson 12: The Volume Formula of a Sphere

## Student Outcomes

- Students give an informal argument using Cavalieri's principle for the volume formula of a sphere and use the volume formula to derive a formula for the surface area of a sphere.

## Lesson Notes

Students will informally derive the volume formula of a sphere in Lesson 12 (**G-GMD.A.2**). To do so, they examine the relationship between a hemisphere, cone, and cylinder, each with the same radius, and for the cone and cylinder, a height equal to the radius. Students will discover that when the solids are aligned, the sum of the area of the cross-sections at a given height of the hemisphere and cone are equal to the area of the cross-section of the cylinder. They use this discovery and their understanding of Cavalieri's principle to establish a relationship between the volumes of each, ultimately leading to the volume formula of a sphere.

## Classwork

### Opening Exercise (5 minutes)

**Opening Exercise**

**Picture a marble and a beach ball. Which one would you describe as a sphere? What differences between the two could possibly impact how we describe what a sphere is?**

***Answers will vary; some students may feel that both are spheres. Some may distinguish that one is a solid while the other is hollow. Share out responses before moving to the definition of sphere.***

**Sphere**

| Definition | Characteristics |
|---|---|
| ***Sphere: Given a point $C$ in the three-dimensional space and a number $r > 0$, the sphere with center $C$ and radius $r$ is the set of all points in space that are distance $r$ from the point $C$.***<br><br>***Solid Sphere or Ball: Given a point $C$ in the three-dimensional space and a number $r > 0$, the solid sphere (or ball) with center $C$ and radius $r$ is the set of all points in space whose distance from the point $C$ is less than or equal to $r$.*** | ***Sphere: hollow***<br><br>***Solid Sphere: filled or solid*** |
| **Examples** | **Non-Examples** |
| ***Sphere: beach ball, soap bubble***<br><br>***Solid Sphere: marble, planet*** | ***Sphere: an egg-shaped figure***<br><br>***Solid Sphere: a circle*** |

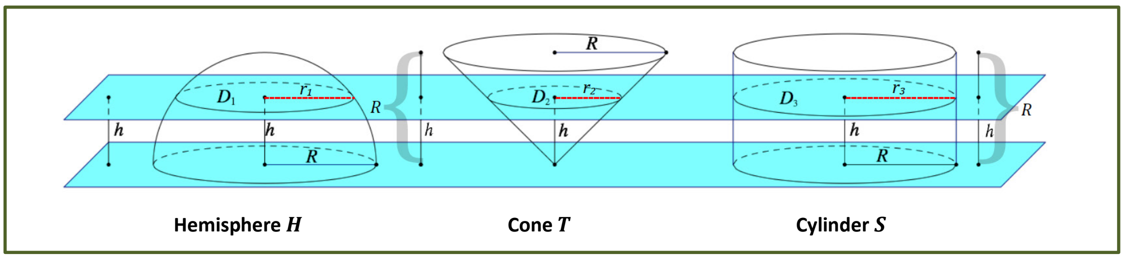
It should be noted that a sphere is just the three-dimensional analog of a circle. You may want to have students compare the definitions of the two terms.

Tell students that the term *hemisphere* refers to a half-sphere, and *solid hemisphere* refers to a solid half-sphere.

### Discussion (18 minutes)

Note that we will find the volume of a solid sphere by first finding half the volume, the volume of a solid hemisphere, and then doubling. Note that we often speak of the volume of a sphere, even though we really mean the volume of the solid sphere, just as we speak of the area of a circle when we really mean the area of a disk.

- Today, we will show that the sum of the volume of a solid hemisphere of radius $R$ and the volume of a right circular cone of radius $R$ and height $R$ is the same as the volume of a right circular cylinder of radius $R$ and height $R$. How could we use Cavalieri's principle to do this?
  - *Allow students a moment to share thoughts. Some students may formulate an idea about the relationship between the marked cross-sections based on the diagram below.*
- Consider the following solids shown: a solid hemisphere, $H$; a right circular cone, $T$; and a right circular cylinder, $S$, each with radius $R$ and height $R$ (regarding the cone and cylinder).




- The solids are aligned above a base plane that contains the bases of the hemisphere and cylinder and the vertex of the cone; the altitude of the cone is perpendicular to this plane.
- A cross-sectional plane that is distance $h$ from the base plane intersects the three solids. What is the shape of each cross-section? Sketch the cross-sections, and make a conjecture about their relative sizes (e.g., order smallest to largest and explain why).
  - *Each cross-section is in the shape of a disk. It looks like the cross-section of the cone will be the smallest, the cross-section of the cylinder will be the biggest, and the cross-section of the hemisphere will be between the sizes of the other two.*
- Let $D_1, D_2, D_3$ be the cross-sectional disks for the solid hemisphere, the cone, and the cylinder, respectively.
- Let $r_1, r_2, r_3$ be the radii of $D_1, D_2, D_3$, respectively.
- Our first task in order to accomplish our objective is to find the area of each cross-sectional disk. Since the radii are all of different lengths, we want to try and find the area of each disk in terms of $R$ and $h$, which are common between the solids.

- Examine the hemisphere more closely in Figure 1.

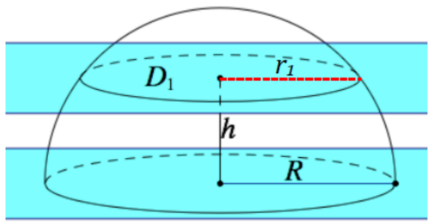
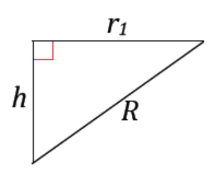



Figure 1

- What is the area formula for disk $D_1$ in terms of $r_1$?
  - $\text{Area}(D_1) = \pi {r_1}^2$
- How can we find $r_1$ in terms of $h$ and $R$?

Allow students time to piece together that the diagram they need to focus on looks like the following figure. Take student responses before confirming with the solution.




  - *By the Pythagorean theorem:*

$$r_1^2 + h^2 = R^2$$
$$r_1 = \sqrt{R^2 - h^2}$$

- Once we have $r_1$, substitute it into the area formula for disk $D_1$.
  - *The area of $D_1$:*

$$\text{Area}(D_1) = \pi r_1^2$$
$$\text{Area}(D_1) = \pi\left(\sqrt{R^2 - h^2}\right)^2$$
$$\text{Area}(D_1) = \pi R^2 - \pi h^2$$

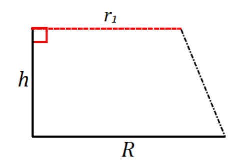
- Let us pause and summarize what we know so far. Describe what we have shown so far.
  - *We have shown that the area of the cross-sectional disk of the hemisphere is $\pi R^2 - \pi h^2$.*

Record this result in the classroom.

- Continuing on with our goal of finding the area of each disk, now find the radius $r_2$ and the area of $D_2$ in terms of $R$ and $h$. Examine the cone more closely in Figure 2.

*Scaffolding:*

- Students may have difficulty seeing that the hypotenuse of the right triangle with legs $h$ and $r_1$ is $R$. Remind students that $h$ and $r_1$ meet at $90°$.
- Students may try and form a trapezoid out of the existing segments:




- Nudge students to reexamine Figure 1 and observe that the distance from the center of the base disk to where $r_1$ meets the hemisphere is simply the radius of the hemisphere.

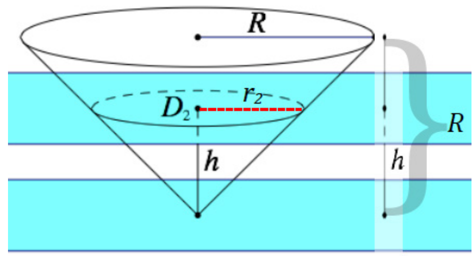
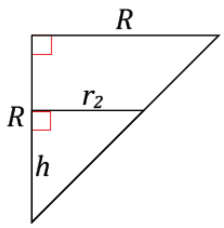
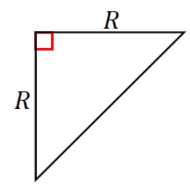



Figure 2

If students require a prompt, remind them that both the radius and the height of the cone are each length $R$.




> *Scaffolding:*
> - Consider showing students the following image as a prompt before continuing with the solution:
>
> 
> 

- *By using similar triangles:*

  $$\frac{r_2}{h} = \frac{R}{R} = 1, \textit{ or } r_2 = h$$

- *The area of $D_2$:*

  $$\text{Area}(D_2) = \pi r_2^2 = \pi h^2$$

- Let us pause again and summarize what we know about the area of the cross-section of the cone. Describe what we have shown.
  - *We have shown that the area of the cross-sectional disk of the hemisphere is $\pi R^2 - \pi h^2$.*

Record the response next to the last summary.

- Lastly, we need to find the area of disk $D_3$ in terms of $R$ and $h$. Examine the cone more closely in Figure 3.

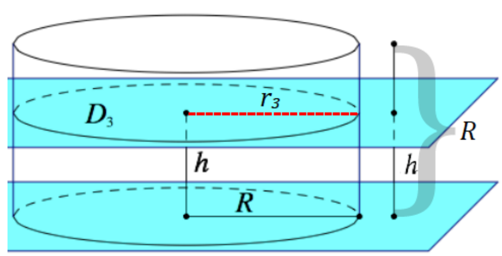



Figure 3

- In the case of this cylinder, will $h$ play a part in the area formula of disk $D_3$? Why?
  - *The radius $r_3$ is equal to $R$, so the area formula will not require $h$ this time.*
  - *The area of $D_3$:*

    $$\text{Area}(D_3) = \pi R^2$$

Write all three areas on the board as you ask:

- What do you now know about the three areas of the cross-sections?
  - $\text{Area}(D_1) = \pi R^2 - \pi h^2$
    $\text{Area}(D_2) = \pi h^2$
    $\text{Area}(D_3) = \pi R^2$

MP.7

- Do you notice a relationship between the areas of $D_1, D_2, D_3$? What is it?
  - *The area of $D_3$ is the sum of the areas of $D_1$ and $D_2$.*

$$Area(D_1) + Area(D_2) = Area(D_3)$$
$$(\pi R^2 - \pi h^2) + \pi h^2 = \pi R^2$$

- Then let us review two key facts: (1) The three solids all have the same height; (2) At any given height, the sum of the areas of the cross-sections of the hemisphere and cone are equal to the cross-section of the cylinder.

MP.1

Allow students to wrestle and share out ideas:

- How does this relate to our original objective of showing that the sum of the volume of a solid hemisphere $H$ and the volume of cone $T$ is equal to the volume of cylinder $S$?
- What does Cavalieri's principle tell us about solids with equal heights and with cross-sections with equal areas at any given height?
  - *Solids that fit that criteria must have equal volumes.*
- By Cavalieri's principle, we can conclude that the sum of the volumes of the hemisphere and cone are equal to the volume of the cylinder.
- Every plane parallel to the base plane intersects $H \cup S$ and $T$ in cross-sections of equal area. Cavalieri's principle states that if every plane parallel to the two planes intersects both solids in cross-sections of equal area, then the volumes of the two solids are equal. Therefore, the volumes of $H \cup S$ and $T$ are equal.

## Example (8 minutes)

Students now calculate the volume of a sphere with radius $R$ using the relationship they discovered between a hemisphere, cone, and cylinder of radius $R$ and height $R$.

**Example**

**Use your knowledge about the volumes of cones and cylinders to find a volume for a solid hemisphere of radius $R$.**

- We have determined that the volume of $H \cup S$ is equal to the volume of $T$. What is the volume of $S$?
  - $\text{Vol}(S) = \frac{1}{3} \times \text{area of base} \times \text{height}$
    $\text{Vol}(S) = \frac{1}{3}\pi R^3$
- What is the volume of $T$?
  - $\text{Vol}(T) = \text{area of base} \times \text{height}$
    $\text{Vol}(T) = \pi R^3$

- Set up an equation and solve for $\mathrm{Vol}(H)$.
  - $\mathrm{Vol}(H) + \frac{1}{3}\pi R^3 = \pi R^3$

    $\mathrm{Vol}(H) = \frac{2}{3}\pi R^3$
- What does $\frac{2}{3}\pi R^3$ represent?
  - *The* $\mathrm{Vol}(H)$*, or the volume of a hemisphere with radius* $R$
- Then what is the volume formula for a whole sphere?
  - *Twice the volume of a solid hemisphere or* $\frac{4}{3}\pi R^3$
- The volume formula for a sphere is $V = \frac{4}{3}\pi R^3$.

## Exercises (8 minutes)

Have students complete any two of the exercises.

**Exercises**

**1. Find the volume of a sphere with a diameter of $12$ cm to one decimal place.**

$V = \frac{4}{3}\pi(6)^3$

$V = \frac{4}{3}\pi(216)$

$V \approx 904.8$

*The volume of the sphere is approximately* $904.8\ \mathrm{cm}^3$.

**2. An ice cream cone is $11$ cm deep and $5$ cm across the opening of the cone. Two hemisphere-shaped scoops of ice cream, which also have diameters of $5$ cm, are placed on top of the cone. If the ice cream were to melt into the cone, will it overflow?**

$\mathrm{Volume(cone)} = \frac{1}{3}\pi\left(\frac{5}{2}\right)^2 \cdot (11)$

$\mathrm{Volume(cone)} \approx 72$

*The volume the cone can hold is roughly* $72\ \mathrm{cm}^3$.

$\mathrm{Volume(ice\ cream)} = \frac{4}{3}\pi\left(\frac{5}{2}\right)^3$

$\mathrm{Volume(ice\ cream)} \approx 65.4$

*The volume of ice cream is roughly* $65.4\ \mathrm{cm}^3$.

*The melted ice cream will not overflow because there is significantly less ice cream than there is space in the cone.*

3. Bouncy, rubber balls are composed of a hollow, rubber shell $0.4$" thick and an outside diameter of $1.2$". The price of the rubber needed to produce this toy is $\$0.035/\text{in}^3$.

a. What is the cost of producing $1$ case, which holds $50$ such balls? Round to the nearest cent.

*The outer shell of the ball is $0.4$" thick, so the hollow center has a diameter of $0.8$" and, therefore, a radius of $0.4$".*

*The volume of rubber in a ball is equal to the difference of the volumes of the entire sphere and the hollow center.*

$$V = \left[\frac{4}{3}\pi(0.6)^3\right] - \left[\frac{4}{3}\pi(0.4)^3\right]$$
$$V = \frac{4}{3}\pi[(0.6)^3 - (0.4)^3]$$
$$V = \frac{4}{3}\pi[0.152]$$
$$V \approx 0.6367$$

*The volume of rubber needed for each individual ball is approximately $0.6367\ \text{in}^3$. For a case of $50$ rubber balls, the volume of rubber required is $\frac{4}{3}\pi\left[0.152\right] \cdot 50 \approx 31.8$, or $31.8\ \text{in}^3$.*

*Total cost equals the cost per cubic inch times the total volume for the case.*

$$\text{Total cost} = [0.035] \cdot \left[\frac{4}{3}\pi[0.152] \cdot 50\right]$$
$$\text{Total cost} \approx 1.11$$

*The total cost for rubber to produce a case of $50$ rubber balls is approximately $\$1.11$.*

b. If each ball is sold for $\$0.10$, how much profit is earned on each ball sold?

$\text{Total sales} = (\text{selling price}) \cdot (\text{units sold})$

$\text{Total sales} = (0.1)(50)$

$\text{Total sales} = 5$

*The total sales earned for selling the case of rubber balls is $\$5$.*

$\text{Profit earned} = (\text{total sales}) - (\text{expense})$

$\text{Profit earned} = (5) - (1.11)$

$\text{Profit earned} = 3.89$

*The total profit earned after selling the case of rubber balls is $\$3.89$.*

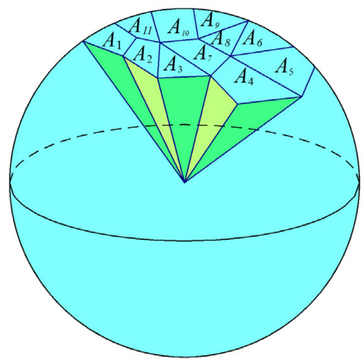
$\text{Profit per unit} = \frac{3.89}{50}$

$\text{Profit per unit} = 0.0778$

*The profit earned on the sale of each ball is approximately $\$0.08$.*

**Extension:** In the Extension, students derive the formula for the surface area of a sphere from the volume formula of the sphere.

**Extension**




- The volume formula for a solid sphere can be used to find the surface area of a sphere with radius $R$.
- Cover the sphere with non-overlapping regions with area $A_1,\ A_2,\ A_3, \dots,\ A_n$.
- For each region, draw a "cone" with the region for a base and the center of the sphere for the vertex (see image).

We say "cone" since the base is not flat, as it is a piece of a sphere. However, it is an approximation of a cone.

- The volume of the cone with base area $A_1$ and height equal to the radius $R$ is approximately

$$\text{Volume}(A_1) \approx \frac{1}{3} \cdot A_1 \cdot R.$$

MP.7

- How is the volume of the sphere related to the volume of the cones? Explain and then show what you mean in a formula.
  - *The volume of the sphere is the sum of the volumes of these "cones."*

$$\frac{4}{3}\pi R^3 \approx \frac{1}{3} \cdot (A_1 + A_2 + \cdots + A_n) \cdot R$$

- Let $S$ be the surface area of the sphere. Then the sum is $A_1 + A_2 + \cdots + A_n = S$. We can rewrite the last approximation as

$$\frac{4}{3}\pi R^3 \approx \frac{1}{3} \cdot S \cdot R.$$

Remind students that the right-hand side is only approximately equal to the left-hand side because the bases of these "cones" are slightly curved and $R$ is not the exact height.

- As the regions are made smaller, and we take more of them, the "cones" more closely approximate actual cones. Hence, the volume of the solid sphere would actually approach $\frac{1}{3} \cdot S \cdot R$ as the number of regions approaches $\infty$.
- Under this assumption we will use the equal sign instead of the approximation symbol:

$$\frac{4}{3}\pi R^3 = \frac{1}{3} \cdot S \cdot R.$$

- Solving for $S$, we get

$$S = 4\pi R^2.$$

- Thus the formula for the surface area of a sphere is

$$\text{Surface Area} = 4\pi R^2.$$

## Closing (1 minute)

- What is the relationship between a hemisphere, a cone, and a cylinder, all of which have the same radius, and the height of the cone and cylinder is equal to the radius?
  - *The area of the cross-sections taken at height $h$ of the hemisphere and cone is equal to the area of the cross-section of the cylinder taken at the same height. By Cavalieri's principle, we can conclude that the sum of the volumes of the hemisphere and cone is equal to the volume of the cylinder.*
- What is the volume formula of a sphere?
  - $V = \frac{4}{3}\pi R^3$

**Lesson Summary**

**SPHERE: Given a point $C$ in the three-dimensional space and a number $r > 0$, the *sphere with center $C$ and radius $r$* is the set of all points in space that are distance $r$ from the point $C$.**

**SOLID SPHERE OR BALL: Given a point $C$ in the three-dimensional space and a number $r > 0$, *the solid sphere (or ball) with center $C$ and radius $r$* is the set of all points in space whose distance from the point $C$ is less than or equal to $r$.**

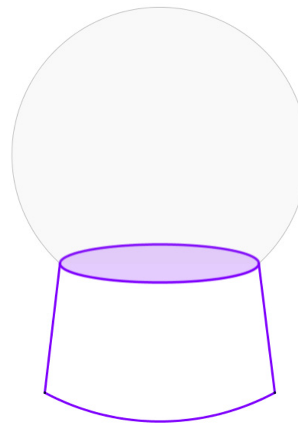
## Exit Ticket (5 minutes)

Name ______________________________ Date ______________

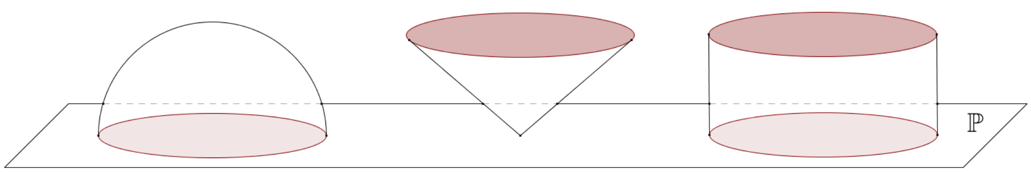
# Lesson 12: The Volume Formula of a Sphere

## Exit Ticket

1. Snow globes consist of a glass sphere that is filled with liquid and other contents. If the inside radius of the snow globe is $3 \text{ in.}$, find the approximate volume of material in cubic inches that can fit inside.



2. The diagram shows a hemisphere, a circular cone, and a circular cylinder with heights and radii both equal to $9$.

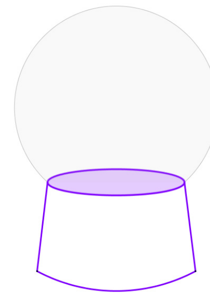



   a. Sketch parallel cross-sections of each solid at height $3$ above plane $P$.

   b. The base of the hemisphere, the vertex of the cone, and the base of the cylinder lie in base plane $P$. Sketch parallel cross-sectional disks of the figures at a distance $h$ from the base plane, and then describe how the areas of the cross-sections are related.

## Exit Ticket Sample Solutions

1. Snow globes consist of a glass sphere that is filled with liquid and other contents. If the inside radius of the snow globe is $3$ in., find the approximate volume of material in cubic inches that can fit inside.



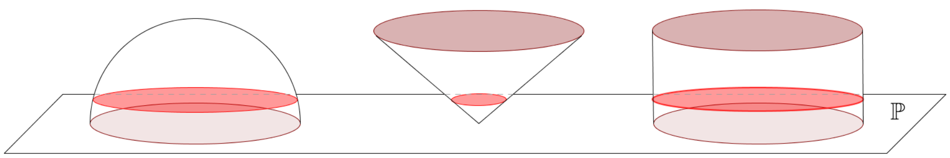
$\text{Volume} = \frac{4}{3}\pi R^3$

$\text{Volume} = \frac{4}{3}\pi \cdot 3^3$

$\text{Volume} = 36\pi \approx 113.1$

*The snow globe can contain approximately $113.1\text{ in}^3$ of material.*

2. The diagram shows a hemisphere, a circular cone, and a circular cylinder with heights and radii both equal to $9$.




a. Sketch parallel cross-sections of each solid at height $3$ above plane $P$.

*See diagram above.*

b. The base of the hemisphere, the vertex of the cone, and the base of the cylinder lie in base plane $P$. Sketch parallel cross-sectional disks of the figures at a distance $h$ from the base plane, and then describe how the areas of the cross-sections are related.

*The cross-sectional disk is $\frac{1}{3}$ of the distance from plane $P$ to the base of the cone, so its radius is also $\frac{1}{3}$ of the radius of the cone's base. Therefore, the radius of the disk is $3$, and the disk has area $9\pi$.*

*The area of the cross-sectional disk in the cylinder is the same as the area of the cylinder's base since the disk is congruent to the base. The area of the disk in the cylinder is $81\pi$.*

$\text{Area(cylinder disk)} = \text{Area(hemisphere disk)} + \text{Area(cone disk)}$

$81\pi = \text{Area(hemisphere disk)} + 9\pi$

$72\pi = \text{Area(hemisphere disk)}$

*The area of the disk in the hemisphere at height $3$ above plane $\mathbb{P}$ is $72\pi \approx 226.2$.*

## Problem Set Sample Solutions

1. A solid sphere has volume $36\pi$. Find the radius of the sphere.

$\frac{4}{3}\pi r^3 = 36\pi$

$r^3 = 27$

$r = 3$

*Therefore, the radius is $3$ units.*

2. A sphere has surface area $16\pi$. Find the radius of the sphere.

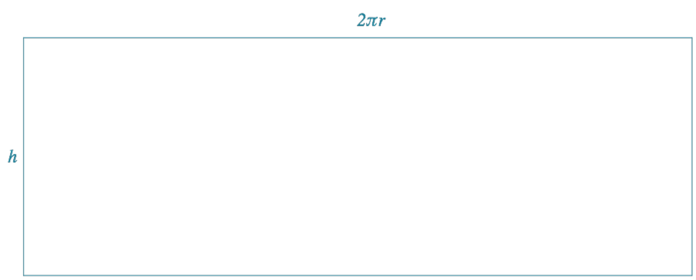
$4\pi r^2 = 16\pi$
$r^2 = 4$
$r = 2$

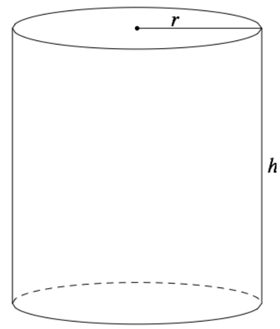
*Therefore, the radius is 2 units.*

3. Consider a right circular cylinder with radius $r$ and height $h$. The area of each base is $\pi r^2$. Think of the lateral surface area as a label on a soup can. If you make a vertical cut along the label and unroll it, the label unrolls to the shape of a rectangle.

   a. Find the dimensions of the rectangle.

   $2\pi r$ *by* $h$




   b. What is the lateral (or curved) area of the cylinder?

   $\text{Lateral Area} = \text{length} \times \text{width}$

   $\text{Lateral Area} = 2\pi rh$

4. Consider a right circular cone with radius $r$, height $h$, and slant height $l$ (see Figure 1). The area of the base is $\pi r^2$. Open the lateral area of the cone to form part of a disk (see Figure 2). The surface area is a fraction of the area of this disk.

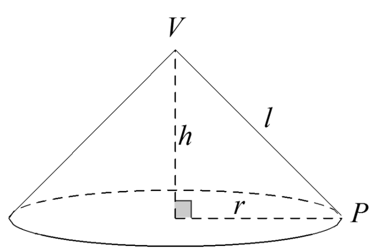



Figure 1

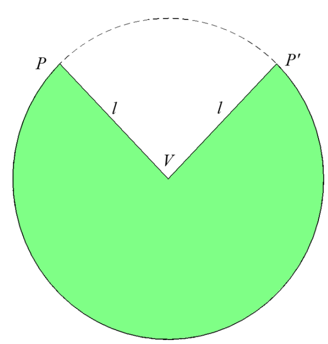



Figure 2

   a. What is the area of the entire disk in Figure 2?

   $\text{Area} = \pi l^2$

   b. What is the circumference of the disk in Figure 2?

   $\text{Circumference} = 2\pi l$

The length of the arc on this circumference (i.e., the arc that borders the green region) is the circumference of the base of the cone with radius $r$ or $2\pi r$. (Remember, the green region forms the curved portion of the cone and closes around the circle of the base.)

c. What is the ratio of the area of the disk that is shaded to the area of the whole disk?

$$\frac{2\pi r}{2\pi l} = \frac{r}{l}$$

d. What is the lateral (or curved) area of the cone?

$$\frac{r}{l} \times \pi l^2 = \pi r l$$

5. A right circular cone has radius $3$ cm and height $4$ cm. Find the lateral surface area.

*By the Pythagorean theorem, the slant height of the cone is* $5$ cm.

*The lateral surface area in* $\text{cm}^2$ *is* $\pi r l = \pi 3 \cdot 5 = 15\pi$. *The lateral surface area is* $15\,\pi\ \text{cm}^2$.

6. A semicircular disk of radius $3$ ft. is revolved about its diameter (straight side) one complete revolution. Describe the solid determined by this revolution, and then find the volume of the solid.

*The solid is a solid sphere with radius* $3$ ft. *The volume in* $\text{ft}^3$ *is* $\frac{4}{3}\pi 3^3 = 36\pi$. *The volume is* $36\pi\ \text{ft}^3$.

7. A sphere and a circular cylinder have the same radius, $r$, and the height of the cylinder is $2r$.

a. What is the ratio of the volumes of the solids?

*The volume of the sphere is* $\frac{4}{3}\pi r^3$, *and the volume of the cylinder is* $2\pi r^3$, *so the ratio of the volumes is the following:*

$$\frac{\frac{4}{3}\pi r^3}{2\pi r^3} = \frac{4}{6} = \frac{2}{3}.$$

*The ratio of the volume of the sphere to the volume of the cylinder is* $2:3$.

b. What is the ratio of the surface areas of the solids?

*The surface area of the sphere is* $4\pi r^2$, *and the surface area of the cylinder is* $2(\pi r^2) + (2\pi r \cdot 2r) = 6\pi r^2$, *so the ratio of the surface areas is the following:*

$$\frac{4\pi r^2}{6\pi r^2} = \frac{4}{6} = \frac{2}{3}.$$

*The ratio of the surface area of the sphere to the surface area of the cylinder is* $2:3$.

8. The base of a circular cone has a diameter of $10$ cm and an altitude of $10$ cm. The cone is filled with water. A sphere is lowered into the cone until it just fits. Exactly one-half of the sphere remains out of the water. Once the sphere is removed, how much water remains in the cone?

$$V_{\text{cone}} = \frac{1}{3}\pi(5)^3(10)$$

$$V_{\text{hemisphere}} = \left(\frac{1}{2}\right)\frac{4}{3}\pi(5)^3$$

$$V_{\text{cone}} - V_{\text{hemisphere}} = \frac{10}{3}\pi(5)^3 - \frac{2}{3}\pi(5)^3$$

$$V_{\text{cone}} - V_{\text{hemisphere}} = \frac{1}{3}\pi(5)^3[10-2]$$

$$V_{\text{cone}} - V_{\text{hemisphere}} = \frac{8}{3}\pi(5)^3$$

*The volume of water that remains in the cone is* $\frac{8}{3}\pi(5)^3$, *or approximately* $1,047.2\text{ cm}^3$.

9. Teri has an aquarium that is a cube with edge lengths of $24$ inches. The aquarium is $\frac{2}{3}$ full of water. She has a supply of ball bearings each having a diameter of $\frac{3}{4}$ inch.

   a. What is the maximum number of ball bearings that Teri can drop into the aquarium without the water overflowing?

$$24\text{ in.} \times 24\text{ in.} \times 24\text{ in.} = 13824\text{ in}^3$$

*The volume of the cube is* $13824\text{ in}^3$.

$$24\text{ in.} \times 24\text{ in.} \times 18\text{ in.} = 10368\text{ in}^3$$

*The volume of water in the cube is* $10368\text{ in}^3$.

$$13824\text{ in}^3 - 10368\text{ in}^3 = 3456\text{ in}^3$$

*The remaining volume in the aquarium is* $3456\text{ in}^3$.

$$V = \frac{4}{3}\pi\left(\frac{3}{8}\right)^3$$

$$V = \frac{4}{3}\pi\left(\frac{27}{512}\right)$$

$$V = \frac{9}{128}\pi$$

*The volume of one ball bearing is* $\frac{9}{128}\pi\text{ in}^3 \approx 0.221\text{ in}^3$.

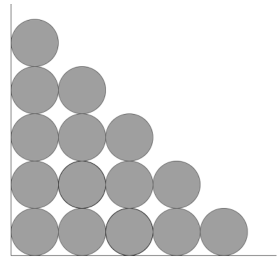
$$\frac{3456}{\frac{9}{128}\pi} = \frac{49152}{\pi} \approx 15645.6$$

*Teri can drop* $15,645$ *ball bearings into the aquarium without the water overflowing. If she drops in one more, the water (theoretically without considering a meniscus) will overflow.*

   b. Would your answer be the same if the aquarium was $\frac{2}{3}$ full of sand? Explain.

*In the original problem, the water will fill the gaps between the ball bearings as they are dropped in; however, the sand will not fill the gaps unless the mixture of sand and ball bearings is continuously stirred.*

c. If the aquarium is empty, how many ball bearings would fit on the bottom of the aquarium if you arranged them in rows and columns as shown in the picture?



*The length and width of the aquarium are $24$ inches, and $24$ inches divided into $\frac{3}{4}$ inch intervals is $32$, so each row and column would contain $32$ bearings. The total number of bearings in a single layer would be $1,024$.*

d. How many of these layers could be stacked inside the aquarium without going over the top of the aquarium? How many bearings would there be altogether?

*The aquarium is $24$ inches high as well, so there could be $32$ layers of bearings for a total of $32,768$ bearings.*

e. With the bearings still in the aquarium, how much water can be poured into the aquarium without overflowing?

$$(32768)\left(\frac{9}{128}\pi\right) \approx 7238.229474$$

*The total volume of the ball bearings is approximately $7238.229474\ \text{in}^3$.*

$$13824 - 7238.229474 = 6585.770526$$

*The space between the ball bearings has a volume of $6585.770526\ \text{in}^3$.*

$$6585.770526 \times 0.004329 = 28.50980061$$

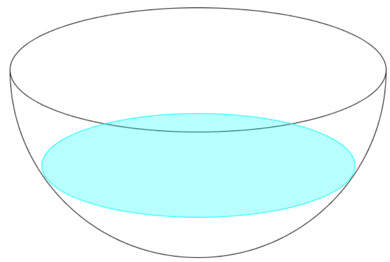
*With the bearings in the aquarium, approximately $28.5$ gallons of water could be poured in without overflowing the tank.*

f. Approximately how much of the aquarium do the ball bearings occupy?

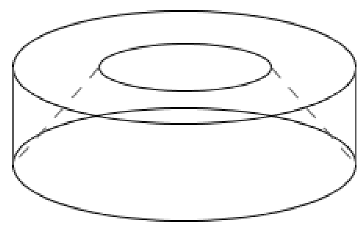
*A little more than half of the space.*

10. **Challenge: A hemispherical bowl has a radius of $2$ meters. The bowl is filled with water to a depth of $1$ meter. What is the volume of water in the bowl? (Hint: Consider a cone with the same base radius and height and the cross-section of that cone that lies $1$ meter from the vertex.)**

*The volume of a hemisphere with radius $2$ is equal to the difference of the volume of a circular cylinder with radius $2$ and height $2$ and the volume of a circular cone with base radius $2$ and height $2$. The cross-sections of the circular cone and the hemisphere taken at the same height $h$ from the vertex of the cone and the circular face of the hemisphere have a sum equal to the base of the cylinder.*



*Using Cavalieri's principle, the volume of the water that remains in the bowl can be found by calculating the volume of the circular cylinder with the circular cone removed, and below the cross-section at a height of $1$ (See diagram right).*



*The area of the base of the cone, hemisphere, and cylinder:*

$\text{Area} = \pi r^2$

$\text{Area} = \pi(2\text{ m})^2 = 4\pi\text{ m}^2$

*The height of the given cross-section is $1\text{ m} = \frac{1}{2}(2\text{ m})$, so the scale factor of the radius of the cross-sectional disk in the cone is $\frac{1}{2}$, and the area is then $\left(\frac{1}{2}\right)^2 \times (4\pi\text{ m}^2) = \pi\text{ m}^2$.*

*The total volume of the cylinder:*

$V = Bh$

$V = \pi(2\text{ m})^2 \cdot 2\text{ m}$

$V = 8\pi\text{ m}^3$

*The total volume of the cone:*

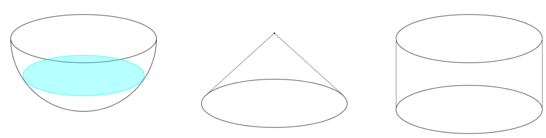
$V = \frac{1}{3}\pi r^3$

$V = \frac{1}{3}\pi(2\text{ m})^3$

$V = \frac{8}{3}\pi\text{ m}^3$

*The volume of the cone above the cross-section:*

$$V = \frac{1}{3}\pi(1\text{ m})^3 = \frac{\pi}{3}\text{ m}^3$$



*The volume of the cone below the cross-section:*

$$V = \frac{8}{3}\pi\text{ m}^3 - \frac{1}{3}\pi\text{ m}^3 = \frac{7}{3}\pi\text{ m}^3$$

*The volume of the cylinder below the cross-section is $4\pi\ m^3$.*

*The volume of the cylinder below the cross-section with the section of cone removed:*

$$V = 4\pi\text{ m}^3 - \frac{7}{3}\pi\text{ m}^3 = \frac{5}{3}\pi\text{ m}^3$$

*The volume of water left in the bowl is $\left(\frac{5}{3}\pi\right)\ m^3$, or approximately $5.2\ m^3$.*

11. Challenge: A certain device must be created to house a scientific instrument. The housing must be a spherical shell, with an outside diameter of $1\text{ m}$. It will be made of a material whose density is $14\text{ g/cm}^3$. It will house a sensor inside that weighs $1.2\text{ kg}$. The housing, with the sensor inside, must be neutrally buoyant, meaning that its density must be the same as water. Ignoring any air inside the housing, and assuming that water has a density of $1\text{ g/cm}^3$, how thick should the housing be made so that the device is neutrally buoyant? Round your answer to the nearest tenth of a centimeter.

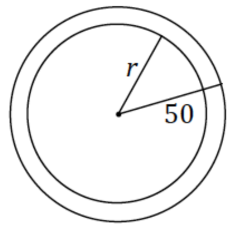
*Volume of outer sphere:*
$\frac{4}{3}\pi(50)^3$

*Volume of inner sphere:*
$\frac{4}{3}\pi(r)^3$

*Volume of shell:*
$\frac{4}{3}\pi[(50)^3 - (r)^3]$

*Mass of shell:*

$$\text{Mass} = (14)\left(\frac{4}{3}\pi[(50)^3 - (r)^3]\right)$$




*In order for the device to be neutrally buoyant:*

$$\text{density of water} = \frac{\text{mass(sensor)} + \text{mass(shell)}}{\text{volume of housing}}$$

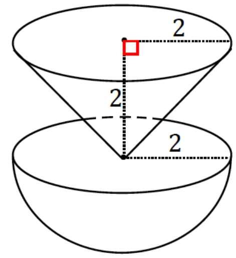
$$1 = \frac{1200 + (14)\left(\frac{4}{3}\pi[(50)^3 - (r)^3]\right)}{\frac{4}{3}\pi(50)^3}$$

$$r = \sqrt[3]{\frac{900\pi + (13)(50^3)}{14}}$$

*The thickness of the shell is* $50 - \sqrt[3]{\frac{900\pi+(13)(50^3)}{14}}$*, which is approximately* $1.2$ *cm.*

12. **Challenge: An inverted, conical tank has a circular base of radius $2$ m and a height of $2$ m and is full of water. Some of the water drains into a hemispherical tank, which also has a radius of $2$ m. Afterward, the depth of the water in the conical tank is $80$ cm. Find the depth of the water in the hemispherical tank.**

*Total volume of water:*

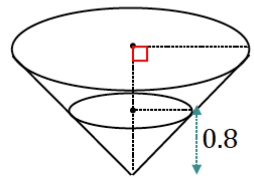



$V = \frac{1}{3}\pi(2)^2(2)$

$V = \frac{8}{3}\pi$; *the total volume of water is* $\frac{8}{3}\pi\ \mathrm{m}^3$.

*The cone formed when the water level is at* $80$ cm $(0.8$ m$)$ *is similar to the original cone; therefore, the radius of that cone is also* $80$ cm.

*Volume of water once the water level drops to* $80$ cm:




$V = \frac{1}{3}\pi(0.8)^2(0.8\ \mathrm{m})$

$V = \frac{1}{3}\pi(0.512)$

*The volume of water in the cone water once the water level drops to* $80$ cm *is* $\frac{1}{3}\pi(0.512)\ \mathrm{m}^3$.

*Volume of water that has drained into the hemispherical tank:*

$V = \left[\frac{8}{3}\pi - \frac{1}{3}\pi(0.512)\right]$

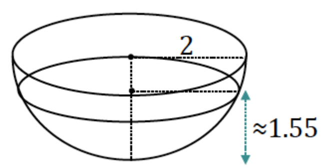
$V = \frac{1}{3}\pi[8-(0.512)]$

$V = \frac{1}{3}\pi[7.488]$

$V = 2.496\pi$

*The volume of water in the hemispherical tank is* $2.496\pi\ \mathrm{m}^3$.

*The pool of water that sits in the hemispherical tank is also hemispherical.*




$2.496\pi = \left(\frac{1}{2}\right)\frac{4}{3}\pi r^3$

$r^3 = 2.496\left(\frac{3}{2}\right)$

$r^3 = 3.744$

$r \approx 1.55$

*The depth of the water in the hemispherical tank is approximately* $1.55$ m.

# Lesson 13: How Do 3D Printers Work?

## Student Outcomes

- Visualize cross-sections of three-dimensional objects.
- Have an understanding of how a 3D printer works and its relation to Cavalieri's principle.

## Lesson Notes

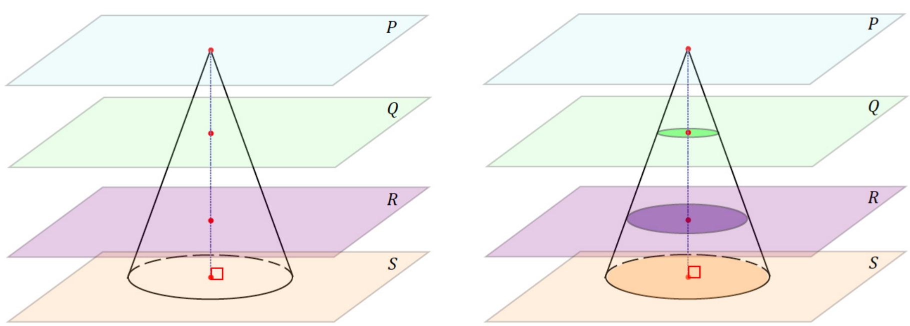
Students consider what it means to build a three-dimensional figure out of cross-sections and discuss the criteria to build a good approximation of a figure. After some practice of drawing cross-sections, students watch a 3D printer in action, and make a tie between how a 3D printer works and Cavalieri's principle.

## Classwork

### Opening Exercise (5 minutes)

**Opening Exercise**

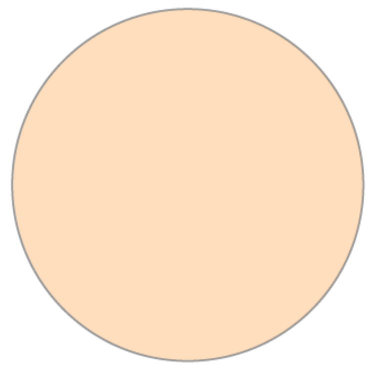
a. **Observe the following right circular cone. The base of the cone lies in plane $S$, and planes $P$, $Q$, and $R$ are all parallel to $S$. Plane $P$ contains the vertex of the cone.**




Tell students that the sketches should be relative to each other; no exact dimensions can be determined from the figure. The goal is to get students thinking about solids as a set of cross-sections.

*Scaffolding:*

- Nets for circular cones are available in Grade 8, Module 7, Lesson 19.
- Consider marking a cone at levels that model the intersections with planes $Q$ and $R$ as a visual aid to students.

| Sketch the cross-section $P'$ of the cone by plane $P$. | Sketch the cross-section $Q'$ of the cone by plane $Q$. |
|---|---|
|  |  |
| **Sketch the cross-section $R'$ of the cone by plane $R$.** | **Sketch the cross-section $S'$ of the cone by plane $S$.** |
|  |  |

b. **What happens to the cross-sections as we look at them starting with $P'$ and work toward $S'$?**

*The intersection of plane $P$ with the cone is a point, and each successive cross-section is a disk of greater radius than the previous disk.*

## Discussion (10 minutes)

Lead students through a discussion that elicits how to build a right circular cone (such as the one in the Opening Exercise) with common materials, such as foam board, styrofoam, cardboard, and card stock and what must be true about the material to get a good approximation of the cone.

Begin by showing students a sheet of Styrofoam; for example, a piece with dimensions $10 \text{ cm} \times 10 \text{ cm} \times 1 \text{ cm}$. The idea is to begin with a material that is not as thin as say, card stock, but rather a material that has some thickness to it.

- Suppose I have several pieces of this material. How can I use the idea of slices to build the cone from the Opening Exercise? What steps would I need to take?

Allow students a moment to discuss with a partner before sharing out responses.

  - *You can cut several slices of the cone.*
  - *The disk cross-sections would have to be cut and stacked and aligned over the center of each disk. Each successive disk after the base disk must have a slightly smaller radius.*

- Say the cone we are trying to approximate has a radius of 3 inches and a height of 3 inches. Assume we have as many pieces of quarter-inch thick styrofoam as we want. How many cross-sections would there be, and how would you size each cross-section?

Allow students time to discuss and make the necessary calculations. Have them justify their responses by showing the calculations or the exact measurements.

- *If each piece of styrofoam is a quarter-inch thick, we will need a total of* 12 *pieces of styrofoam.*
- *The base should have a radius of* 3 *inches, and each successive slice above the base should have a radius that is a quarter-inch less than the former slice.*

| **Slice** | 1 (base) | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Radius** | 3" | 2.75" | 2.5" | 2.25" | 2" | 1.75" | 1.5" | 1.25" | 1" | 0.75" | 0.5" | 0.25" |

Now hold up a sheet of the styrofoam and a thinner material, such as card stock, side by side.

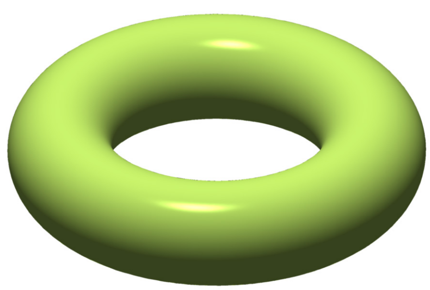
- If the same cone is built with both of these materials, which will result as a better approximation of the cone? Why?
  - *The use of card stock will result in a better approximation of the cone, since the thickness of the card stock would require more cross-sections, and sizing each successive cross-section would be more finely sized than the quarter-inch jumps in radius of the styrofoam.*
- To create a good approximation of a three-dimensional object, we must have ideal materials, ones that are very thin, and many cross-sections, in order to come as close as possible to the volume of the object.

*Scaffolding:*
- Consider having a model of this example built to share with students after completing the exercise.

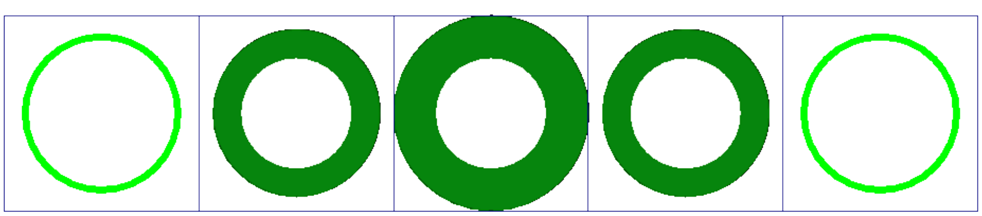
### Exercise 1 (5 minutes)

**Exercise 1**

1. **Sketch five evenly spaced, horizontal cross-sections made with the following figure.**



**http://commons.wikimedia.org/wiki/File%3ATorus_illustration.png; By Oleg Alexandrov (self-made, with MATLAB) [Public domain], via Wikimedia Commons.** ***Attribution not legally required.***

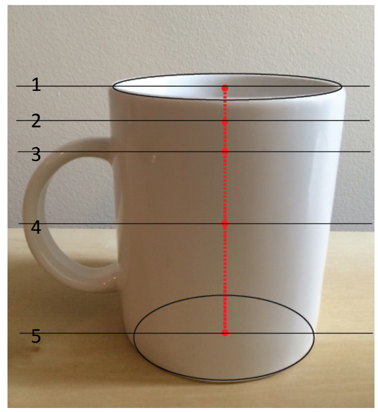


## Example 1 (7 minutes)

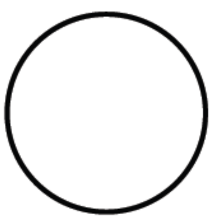
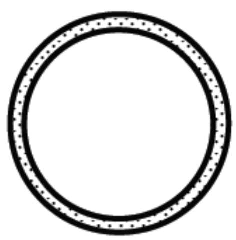
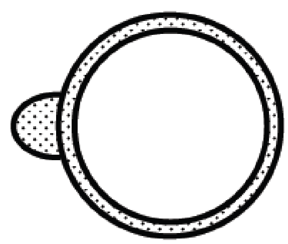
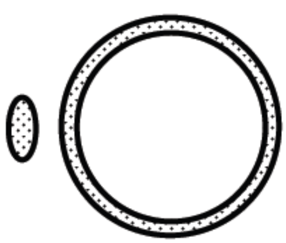
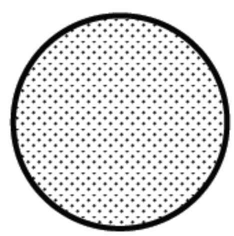
Before showing students a video clip of a 3D printer printing a coffee cup, have students sketch cross-sections of a coffee cup. Consider providing students with model coffee cups to make their sketches from. Otherwise students can use the photo below.

**Example 1**

**Let us now try drawing cross-sections of an everyday object, such as a coffee cup.**




**Sketch the cross-sections at each of the indicated heights.**

| 1 | 2 | 3 |
|---|---|---|
|  |  |  |
| **4** | **5** | |
|  |  | |

Review each of the five sketches. Have students explain why the cross-sections look the way they do. For example, students should be able to describe why sketch 4 has a gap between the body of the mug and the handle.

### Discussion (11 minutes)

We provide four video links for this lesson, one we embed in this discussion to follow the previous example on the cross-section of the coffee cup. You may want to briefly discuss what a 3D printer is first, or show the video first and then discuss what a 3D printer is based on what students see in the video (this is the treatment taken in the Discussion below).

The coffee cup video clip is just over $13$ minutes long and shows the print process of a full-sized coffee cup from start to finish. During preparation of this lesson, you will probably want to watch the video and decide which sections of it to show students. For example, after showing roughly a minute of it, you may want to skip to $1:27$ and pause to highlight the beginning of the handle of the cup.

**Video (3D Printer, Coffee Cup):** https://www.youtube.com/watch?v=29yHrWrs1ok

- What you just saw is a 3D printer at work, printing a coffee cup. How would you describe a 3D printer?
  - *Answers will vary. Students may describe in their own words how a 3D printer looks like it builds a three-dimensional object a little at a time, perhaps one layer at a time.*
- Consider what a regular printer does. It takes data from an application like Microsoft Word, and the file has the instructions on how the printer should deliver ink to paper.
- A 3D printer effectively does the same thing. Electronic data from an application such as CAD (computer-aided design) is used to design a 3D model. When the data is sent to the printer, software will create fine slices of the model, and the printer will release "ink" in the form of a plastic or a metal (or some combination of several mediums) one slice at a time, until the entire model is complete.
- The technology is relatively recent, beginning in the $1980$s, but it is showing signs of revolutionizing how we think about design. You will hear 3D printing referred to as *additive manufacturing* in the next clip.

Now show the entirety of the next clip (roughly three and a half minutes), which provides a history and the scope of possibilities with 3D printing.

**Video (3D Printer, General):** http://computer.howstuffworks.com/3-d-printing.htm

- Why is 3D printing also called additive manufacturing?
  - *Objects are created by adding only what is needed.*
- What is one benefit to producing an object in this way?
  - *There is no excess waste, since you are not carving, cutting, or melting out an object out of the original material.*
- What are some of the mentioned uses of 3D printing?
  - *It has medical uses, such as printing human tissue components to create a kidney. It has aerospace uses, NASA is using it make repairs on equipment. It is also used in jewelry and food industries.*
- Explain how the process of 3D printing invokes Cavalieri's principle.
  - *By knowing the cross-section of a solid, we can approximate the volume of the solid.*

The programming in the software that dictates how 3D printers print is rooted in algebra. We can see an example of this in the platform that moves left, right, and down to add each successive layer of the coffee cup. This movement is programmed in a coordinate system.

## Exercises 2–4

With any time remaining, have students complete a selection of the following exercises.

**Exercises 2–4**

2. **A cone with a radius of $5$ cm and height of $8$ cm is to be printed from a 3D printer. The medium that the printer will use to print (i.e., the "ink" of this 3D printer) is a type of plastic that comes in coils of tubing which has a radius of $1\frac{1}{3}$ cm. What length of tubing is needed to complete the printing of this cone?**

   *The volume of medium used (contained in a circular cylinder) must be equal to the volume of the cone being printed.*

   $$V_{\text{cone}} = \frac{1}{3}\pi(5)^2(8) \qquad V_{\text{cylinder}} = \pi\left(\frac{4}{3}\right)^2 h$$

   $$V_{\text{cone}} = \frac{200\pi}{3} \qquad V_{\text{cylinder}} = \frac{16\pi}{9}h$$

   $$\frac{200\pi}{3} = \frac{16\pi h}{9}$$

   $$h = 37.5$$

   *The cone will require $37.5$ cm of tubing.*

3. **A cylindrical dessert $8$ cm in diameter is to be created using a type of 3D printer specially designed for gourmet kitchens. The printer will "pipe" or, in other words, "print out" the delicious filling of the dessert as a solid cylinder. Each dessert requires $300\text{ cm}^3$ of filling. Approximately how many layers does each dessert have if each layer is $3$ mm thick?**

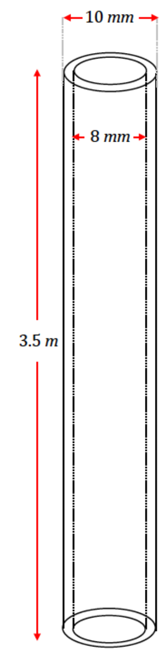
   $$\text{Volume} = \pi r^2 h$$
   $$300 = \pi(4)^2 h$$
   $$\frac{300}{16\pi} = h$$
   $$h \approx 5.968$$

   *The total height of the dessert is approximately $6$ cm. Since each layer is $3$ mm (or $0.3$ cm) thick,*

   $$\text{height} = (\text{thickness of layer}) \times (\text{number of layers})$$
   $$\frac{300}{16\pi} = (0.3)n$$
   $$n \approx 19.89.$$

   *Each dessert has about $20$ layers.*

4. The image shown to the right is of a fine tube that is printed from a 3D printer that prints replacement parts. If each layer is $2$ mm thick, and the printer prints at a rate of roughly $1$ layer in $3$ seconds, how many minutes will it take to print the tube?




*The printer prints at a rate of* $\frac{1}{3}$ *layers per second. The total height of the tube is* $3.5$ m, *or* $350$ cm. *If each layer is* $2$ mm, *or* $0.2$ cm *thick, then the tube has a total of* $1750$ *layers.*

$$\text{distance} = \text{rate} \times \text{time}$$
$$1750 = \frac{1}{3}(t)$$
$$t = 5250$$

*The time to print the tube is* $5,250$ *seconds, or* $87.5$ *minutes.*

Note: Figure not drawn to scale.

## Closing (2 minutes)

Consider watching a clip of either of the remaining videos:

- Video (3D Printer, Wedding Rings): https://www.youtube.com/watch?v=SIuDbRUUG1w
- Video (3D Printer, Food): https://www.youtube.com/watch?v=XQni3wb0tyM

## Exit Ticket (5 minutes)

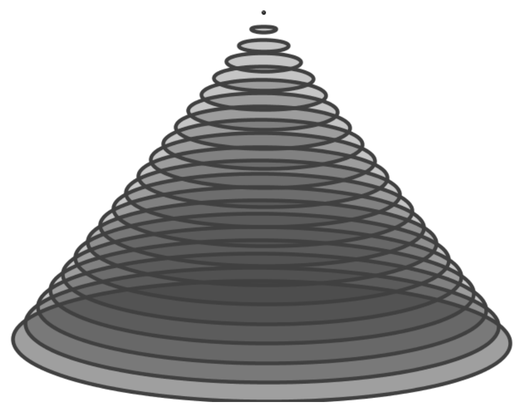
Name ______________________________ Date ______________

# Lesson 13: How Do 3D Printers Work?

## Exit Ticket

Lamar is using a 3D printer to construct a circular cone that has a base with radius $6$ in.

a. If his 3D printer prints in layers that are $0.004$ in. thick (similar to what is shown in the image below), what should be the change in radius for each layer in order to construct a cone with height $4$ in.?



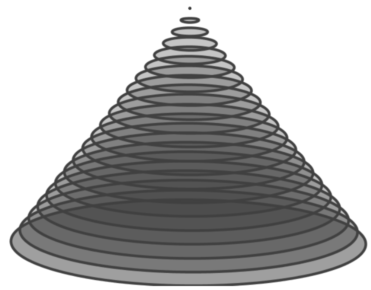
b. What is the area of the base of the $27^{\text{th}}$ layer?

c. Approximately how much printing material is required to produce the cone?

## Exit Ticket Sample Solutions

Lamar is using a 3D printer to construct a circular cone that has a base with radius $6$ in.

a. If his 3D printer prints in layers that are $0.004$ in. thick (similar to what is shown in the image below), what should be the change in radius for each layer in order to construct a cone with height $4$ in.?



$$\frac{4}{6} = \frac{3.996}{x}$$
$$x = 5.994$$
$$6 - 5.994 = 0.006$$

*The change in radius between consecutive layers is* $0.006$ *inch.*

b. What is the area of the base of the $27^{\text{th}}$ layer?

*The* $27^{th}$ *layer of the cone will have a radius reduced by* $0.006$ *inch* $26$ *times.*

$$26(0.006) = 0.156$$
$$6 - 0.156 = 5.844$$

*The radius of the* $27^{th}$ *layer is* $5.844$ in.

$$A = \pi(5.844)^2$$
$$A \approx 107.3$$

*The area of the base of the disk in the* $27^{th}$ *layer is approximately* $107.3$ in$^2$.

c. Approximately how much printing material is required to produce the cone?

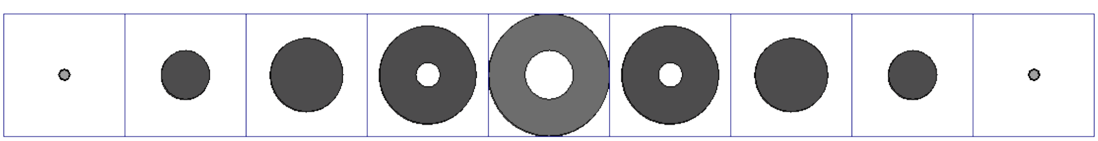
*The volume of printing material is approximately equal to the volume of a true cone with the same dimensions.*

$$V = \frac{1}{3}\pi(6)^2(4)$$
$$V = 48\pi$$

*Approximately* $150.8$ in$^3$ *of printing material is required.*

## Problem Set Sample Solutions

1. Horizontal slices of a solid are shown at various levels arranged from highest to lowest. What could the solid be?



*Answers will vary. The solid could be a ball with a hole in it.*

2. **Explain the difference in a 3D printing of the ring pictured in Figure 1 and Figure 2 if the ring is oriented in each of the following ways.**

*For the first ring, the cross-sections are circles or regions between concentric circles. For the second ring, the cross-sections are stretched cirles and then two separted regions.*

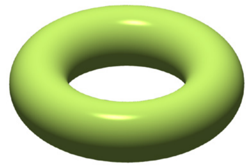


**Figure 1**

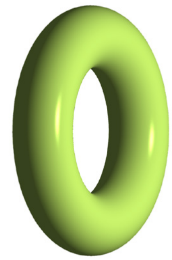


**Figure 2**

3. **Each bangle printed by a 3D printer has a mass of exactly $25$ g of metal. If the density of the metal is $14\ \text{g/cm}^3$, what length of a wire $1$ mm in radius is needed to produce each bangle? Find your answer to the tenths place.**

*Radius of filament:* $1\ \text{mm} = 0.1\ \text{cm}$

$\text{Volume}_{\text{bangle}} = \pi(0.1)^2(h) = 0.01\pi h$

$$14 = \frac{25}{0.01\pi h}$$

$$h \approx 56.8$$

*The wire must be a length of* $56.8$ cm.

4. **A certain 3D printer uses $100$ m of plastic filament that is $1.75$ mm in diameter to make a cup. If the filament has a density of $0.32\ \text{g/cm}^3$, find the mass of the cup to the tenths place.**

*Length of filament:* $100\ \text{m} = 10^4\ \text{cm}$

*Radius of filament:* $0.875\ \text{mm} = 0.0875\ \text{cm}$

$\text{Volume}_{\text{filament}} = \pi(0.0875)^2(10^4) = 76.5625\pi$

*Mass of cup:* $(76.5625\pi\ \text{cm}^3)(0.32\ \text{g/cm}^3) = 24.5\pi\ \text{g}$

*The mass of the cup is approximately* $77$ g .

5. **When producing a circular cone or a hemisphere with a 3D printer, the radius of each layer of printed material must change in order to form the correct figure. Describe how radius must change in consecutive layers of each figure.**

*The slanted side of a circular cone can be modeled with a sloped line in two dimensions, so the change in radius of consecutive layers must be constant; i.e., the radius of each consecutive layer of a circular cone decreases by a constant* $c$.

*The hemisphere does not have the same profile as the cone. In two dimensions, the profile of the hemisphere cannot be modeled by a line but, rather, an arc of a circle, or a semicircle. This means that the change in radius between consecutive layers of print material is not constant. In fact, if printing from the base of the hemisphere, the change in radius must start out very close to* $0$*, then increase as the printer approaches the top of the hemisphere, which is a point that we can think of as a circle of radius* $0$.

6. **Suppose you want to make a 3D printing of a cone. What difference does it make if the vertex is at the top or at the bottom? Assume that the 3D printer places each new layer on top of the previous layer.**

*If the vertex is at the top, new layers will always be supported by old layers. If the vertex is at the bottom, new layers will hang over previous layers.*

**7.** **Filament for 3D printing is sold in spools that contain something shaped like a wire of diameter $3\text{ mm}$. John wants to make 3D printings of a cone with radius $2\text{ cm}$ and height $3\text{ cm}$. The length of the filament is $25$ meters. About how many cones can John make?**

*The volume of the cone is* $\frac{1}{3}\pi(2)^2 \cdot 3\text{ cm}^3 = 4\pi\text{ cm}^3$.

*The volume of the filament is* $2500\text{ cm} \cdot \pi \cdot 0.15^2\text{ cm}^2 = 56.25\pi\text{ cm}^3$.

*Since* $56.25\pi\text{ cm}^3 \div 4\pi\text{ cm}^3 = 14.0625$*, John can make* $14$ *cones as long as each 3D-printed cone has volume no more than* $4\pi\text{ cm}^3$.

**8.** **John has been printing solid cones but would like to be able to produce more cones per each length of filament than you calculated in Problem 7. Without changing the outside dimensions of his cones, what is one way that he could make a length of filament last longer? Sketch a diagram of your idea and determine how much filament John would save per piece. Then determine how many cones John could produce from a single length of filament based on your design.**

*Students' answers will vary. One possible solution would be to print only the outer shell of the cone, leaving a hollow center in the shape of a scaled-down cone. Something that students may consider in their solution is the thickness of the wall of the shell and the integrity of the final product. This will provide a variety of answers.*

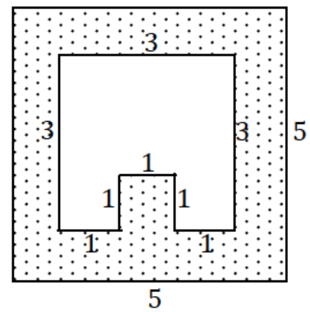
**9.** **A 3D printer uses one spool of filament to produce $20$ congruent solids. Suppose you want to produce similar solids that are $10\%$ longer in each dimension. How many such figures could one spool of filament produce?**

*Each new solid would have volume* $1.331$ *times the volume of an original solid. Let* $x$ *be the number of new figures produced. Then* $1.331x = 20$ *and* $x = \frac{20}{1.331} \approx 15.03$.

*It should be possible to print* $15$ *of the larger solids.*

10. **A fabrication company 3D-prints parts shaped like a pyramid with base as shown in the following figure. Each pyramid has a height of $3$ cm. The printer uses a wire with a density of $12\ \text{g/cm}^3$, at a cost of $\$0.07/\text{g}$.**

**It costs $\$500$ to set up for a production run, no matter how many parts they make. If they can only charge $\$15$ per part, how many do they need to make in a production run to turn a profit?**




*Volume of a single part:*

$V = \frac{1}{3}Bh$

$V = \frac{1}{3}(17)(3)$

$V = 17$*; the volume of each part is* $17\ \text{cm}^3$.

*Mass of a single part:*

$$\text{density} = \frac{\text{mass}}{\text{volume}} \quad \textit{or} \quad \text{mass} = (\text{density})(\text{volume})$$

$$\text{mass} = (12)(17)$$

$$\text{mass} = 204$$

*The mass of a single part is* $204$ g.

*Cost of a single part:*

$\text{cost} = (204)(0.07)$

$\text{cost} = (204)(0.07)$

$\text{cost} = 14.28$

*The cost of a single part is* $\$14.28$.

*The sum of the production run cost and the cost to make the total number of parts must be less than the product of the price and the total number of parts. Let $n$ be the total number of parts:*

$$500 + 14.28n < 15n$$
$$500 < 0.72n$$
$$n > 694.\overline{4}$$

*Therefore, in order to turn a profit,* $695$ *parts must be made in a production run.*

Name ______________________________ Date ________________

1.
  a. State the volume formula for a cylinder. Explain why the volume formula works.

  b. The volume formula for a pyramid is $\frac{1}{3}Bh$, where $B$ is the area of the base and $h$ is the height of the solid. Explain where the $\frac{1}{3}$ comes from in the formula.

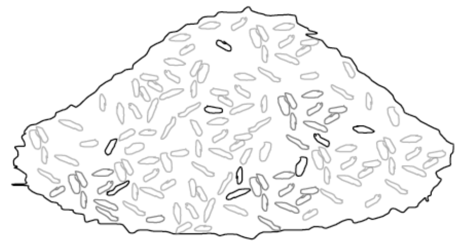
c. Give an explanation of how to use the volume formula of a pyramid to show that the volume formula of a circular cone is $\frac{1}{3}\pi r^2 h$, where $r$ is the radius of the cone and $h$ is the height of the cone.

2. A circular cylinder has a radius between $5.50$ and $6.00$ centimeters and a volume of $225$ cubic centimeters. Write an inequality that represents the range of possible heights the cylinder can have to meet this criterion to the nearest hundredth of a centimeter.

3. A machine part is manufactured from a block of iron with circular cylindrical slots. The block of iron has a width of $14$ in., a height of $16$ in., and a length of $20$ in. The number of cylinders drilled out of the block is determined by the weight of the leftover block, which must be less than $1{,}000$ lb.

    a. If iron has a weight of roughly $491\ \text{lb}/\text{ft}^3$, how many cylinders with the same height as the block and with radius $2$ in. must be drilled out of the block in order for the remaining solid to weigh less than $1{,}000$ lb.?

    b. If iron ore costs $\$115$ per ton ($1\ \text{ton} = 2200\ \text{lb.}$) and the price of each part is based solely on its weight of iron, how many parts can be purchased with $\$1{,}500$? Explain your answer.

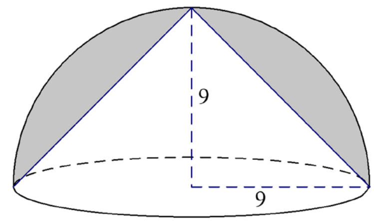


4. Rice falling from an open bag piles up into a figure conical in shape with an approximate radius of $5\text{ cm}$.

   a. If the angle formed by the slant of the pile with the base is roughly $30°$, write an expression that represents the volume of rice in the pile.

   b. If there are approximately $20$ grains of rice in a cubic centimeter, approximately how many grains of rice are in a $4.5$-kilogram bag of rice?

5. In a solid hemisphere, a cone is removed as shown. Calculate the volume of the resulting solid. In addition to your solution, provide an explanation of the strategy you used in your solution.




6. Describe the shape of the cross-section of each of the following objects.

*Right circular cone:*

a. Cut by a plane through the vertex and perpendicular to the base

*Square pyramid:*

b. Cut by a plane through the vertex and perpendicular to the base

c. Cut by a vertical plane that is parallel to an edge of the base but not passing through the vertex

*Sphere with radius $r$:*

d. Describe the radius of the circular cross-section created by a plane through the center of the sphere.

e. Describe the radius of the circular cross-section cut by a plane that does not pass through the center of the sphere.

*Triangular Prism:*

f. Cut by a plane parallel to a base

g. Cut by a plane parallel to a face

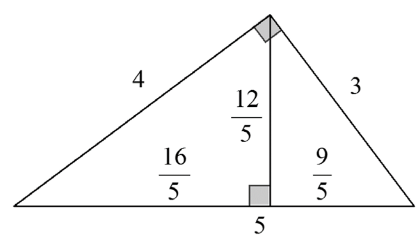
7.

a. A $3 \times 5$ rectangle is revolved about one of its sides of length $5$ to create a solid of revolution. Find the volume of the solid.

b. A 3-4-5 right triangle is revolved about a leg of length $4$ to create a solid of revolution. Describe the solid.

c. A 3-4-5 right triangle is revolved about its legs to create two solids. Find each of the volumes of the two solids created.

d. Show that the volume of the solid created by revolving a 3-4-5 triangle about its hypotenuse is $\frac{48}{5}\pi$.



| A Progression Toward Mastery | | | | | |
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| Assessment Task Item | | **STEP 1** Missing or incorrect answer and little evidence of reasoning or application of mathematics to solve the problem. | **STEP 2** Missing or incorrect answer but evidence of some reasoning or application of mathematics to solve the problem. | **STEP 3** A correct answer with some evidence of reasoning or application of mathematics to solve the problem, OR an incorrect answer with substantial evidence of solid reasoning or application of mathematics to solve the problem. | **STEP 4** A correct answer supported by substantial evidence of solid reasoning or application of mathematics to solve the problem. |
| 1 | a<br>G-GMD.A.1 | Student incorrectly states the volume formula. Response is not coherent or does not provide reasoning as to why the formula works or response not attempted. | Student correctly states the volume formula but does not explain why it works.<br>OR<br>Student incorrectly states the volume formula, but provides reasoning as to why the formula works. | Student correctly states the volume formula and provides some evidence of understanding why the formula works. | Student correctly states the volume formula and provides substantial reasoning as to why the formula works. |
| | b<br>G-GMD.A.1 | Student demonstrates little or no understanding of the one-third factor in the volume formula. Student may have left item blank. | Student demonstrates some understanding of the one-third factor in the volume formula. | Student demonstrates understanding of the one-third factor in the volume formula. | Student demonstrates clear and coherent reasoning of understanding of the one-third factor in the volume formula. More than one explanation may have been given. |
| | c<br>G-GMD.A.1 | Student demonstrates little or no understanding of how the volume of a pyramid can be used to establish the volume formula for a circular cone. | Student demonstrates an understanding of how the volume of a pyramid can approximate the volume of a circular cone, but the response lacks specificity regarding how the polygonal base approximates the circular base and fails to mention the volume formula of a pyramid. | Student demonstrates an understanding of how the volume of a pyramid can approximate the volume of a circular cone, including the formula of the volume of a pyramid, but the response lacks specificity regarding how the polygonal base approximates the circular base. | Student demonstrates clear and coherent reasoning of understanding of how the volume of a pyramid can approximate the volume of a circular cone. |

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| **2** | G-GMD.A.3 | Student does not provide a coherent response.<br>OR<br>Student does not attempt to provide a response. | Student provides a response that contains two computational errors or two conceptual errors, or one computational and one conceptual error. | Student provides a response that contains one computational error or one conceptual error, such as failing to find one extreme of the range of heights. | Student correctly identifies range of possible heights rounded to the nearest hundredth of a centimeter in an inequality. |
| **3** | **a**<br>G-MG.A.2 | Student provides an incorrect answer and does not show work. | Student provides a response that contains two computational errors or two conceptual errors, or one computational and one conceptual error. | Student provides a response that contains one computational error, such as an incorrect value of the weight of the iron block, or one conceptual error, such as incorrect conversion of units. | Student correctly determines that $5$ cylinders must be drilled out to reduce the weight of the solid less than $1{,}000$ lb. and provides a complete solution. |
| | **b**<br>G-MG.A.3 | Student provides an incorrect answer and does not show work. | Student provides a response that contains two computational errors or two conceptual errors, or one computational and one conceptual error. | Student provides a response that contains one computational error, such as an incorrect cost per pound of iron, or one conceptual error, such as incorrect conversion of units. | Student correctly determines the number of parts that can be purchased with $\$1{,}500$ and shows a complete solution. |
| **4** | **a**<br>G-MG.A.1 | Student demonstrates little or no understanding of how the volume of a pyramid can be used to establish the volume formula for a circular cone. | Student demonstrates an understanding of how the volume of a pyramid can approximate the volume of a circular cone, but the response lacks specificity regarding how the polygonal base approximates the circular base and fails to mention the volume formula of a pyramid. | Student demonstrates an understanding of how the volume of a pyramid can approximate the volume of a circular cone, including the formula of the volume of a pyramid, but the response lacks specificity regarding how the polygonal base approximates the circular base. | Student demonstrates clear and coherent reasoning of understanding of how the volume of a pyramid can approximate the volume of a circular cone. |
| | **b**<br>G-MG.A.2 | Student demonstrates little or no understanding of how to determine the number of grains of rice in the pile. | Student provides a response that contains one computational and one conceptual error, such as incorrect conversion of units. | Student provides a response that contains one computational error or one conceptual error, such as incorrect conversion of units. | Student correctly determines the number of grains of rice in the pile and shows a complete solution. |

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| **5** | **G-GMD.A.3** | Student makes a computational error leading to an incorrect answer; additionally, the reasoning is not correct. | Student provides a response that contains a computational error leading to an incorrect answer but uses correct reasoning. | Student provides a correct solution but does not include an explanation as to how they arrived at their answer. | Student provides a correct solution which includes a clear explanation of how the problem is approached and solved. |
| **6** | **a–g**<br><br>**G-GMD.B.4** | Student provides a correct response to one or less parts. | Student provides a correct response to three parts. | Student provides a correct response to five parts. | Student provides a correct response to all seven parts. |
| **7** | **a**<br><br>**G-GMD.B.4** | Student shows little or no attempt to answer the question. | Student provides a response that contains one conceptual error, such as calculating the volume of a rectangular prism. | Student provides a response that contains one computational error, such as using an incorrect height or radius leading to an answer of $75\pi$. | Student provides a correct solution and final answer of $45\pi$. |
| | **b**<br><br>**G-GMD.B.4** | Student identifies the shape as a triangle or other two-dimensional figure or left the item blank. | Student misidentifies the shape formed but correctly states it as a three-dimensional solid. | Student concludes that the shape formed by the revolution is a right circular cone or cone. | Student provides a complete sentence identifying the solid formed as a right circular cone or cone. |
| | **c**<br><br>**G-GMD.B.4** | Student provides incorrect answers for both solids, and it is not evident that formulas were used. | Student used the correct formulas for volume but makes computational errors leading to incorrect volumes for both solids. | Student correctly calculates the volume of one of the solids but may have made a computational error leading to an incorrect volume for the second solid. | Student correctly provides the volume of both solids (i.e., $12\pi$ and $16\pi$). |
| | **d**<br><br>**G-GMD.B.4** | Student made little or no attempt to answer the question. | Student provides a solution with the correct steps but makes one conceptual error leading to an incorrect answer, such as using an incorrect formula. | Student provides a solution with the correct steps but makes one computational error leading to an incorrect answer. | Student correctly identifies the volume as $\frac{48}{5}\pi$ and includes a complete solution. |

Name ________________________________ Date ________________

1.

a. State the volume formula for a cylinder. Explain why the volume formula works.

$V = Bh$, where $B$ is the area of the base.
Since a cylinder is comprised of congruent disks (circles), then the height of the cylinder is produced by the "stack" of disks. Congruent figures have equal area, so multiplying $B \times h$ gives the volume of a cylinder.

b. The volume formula for a pyramid is $\frac{1}{3}Bh$, where $B$ is the area of the base and $h$ is the height of the solid. Explain where the $\frac{1}{3}$ comes from in the formula.

If you consider a unit cube and a pyramid whose base is equal to $1 \times 1$ and whose height is $\frac{1}{2}$, it would take 6 such pyramids to equal the volume of the cube. If you consider just half the volume of the cube, then we have a solid with volume $\frac{1}{6}$, the pyramid, when compared $\frac{1}{6} = ? \times \frac{1}{2}$, the missing factor is $\frac{1}{3}$. So the volume of the pyramid is $\frac{1}{3}$ of the volume of the rectangular prism that is the same base and same height.

c. Give an explanation of how to use the volume formula of a pyramid to show that the volume formula of a circular cone is $\frac{1}{3}\pi r^2 h$, where $r$ is the radius of the cone and $h$ is the height of the cone.

We can approximate the cone with a pyramid, so that the base of the pyramid is an n-sided polygon that approximates the base of the circular cone, and the height is the same as that of the cone. The volume of the pyramid can be expressed as $\frac{1}{3}B_n h$ where $B_n$ is the area of the polygonal base, and h is the height. As the number of sides approaches infinity, the value of $B_n$ approaches $\pi r^2$, where r is the radius of the circle within which the polygonal base is inscribed.

2. A circular cylinder has a radius between 5.50 and 6.00 centimeters and a volume of 225 cubic centimeters. Write an inequality that represents the range of possible heights the cylinder can have to meet this criterion to the nearest hundredth of a centimeter.

$5.50 < r < 6.00$, $V = 225 \text{ cm}^3$

$V = \pi r^2 h$ $\quad$ $V = \pi r^2 h$

$225 = \pi (5.5)^2 h$ $\quad$ $225 = \pi (6.00)^2 h$

$2.37 \approx h$ $\quad$ $1.99 \approx h$

$2.37 < h < 1.99$

3. A machine part is manufactured from a block of iron with circular cylindrical slots. The block of iron has a width of $14$ in., a height of $16$ in., and a length of $20$ in. The number of cylinders drilled out of the block is determined by the weight of the leftover block, which must be less than $1{,}000$ lb.

   a. If iron has a weight of roughly $491$ lb/ft$^3$, how many cylinders with the same height as the block and with radius $2$ in. must be drilled out of the block in order for the remaining solid to weigh less than $1{,}000$ lb.?

$$\text{WEIGHT OF BLOCK} = \left(\frac{14 \times 16 \times 20 \text{ in}^3}{12^3 \frac{\text{in}^3}{\text{ft}^3}}\right)\left(491 \frac{\text{lbs}}{\text{ft}^3}\right) \approx 1273 \text{ lbs}$$

$$\text{WEIGHT OF EACH DRILLED OUT CYLINDER} = \left(\frac{64\pi \text{ in}^3}{12^3 \frac{\text{in}^3}{\text{ft}^3}}\right)\left(491 \frac{\text{lbs}}{\text{ft}^3}\right) \approx 57 \text{ lbs}$$

$$\text{NUMBER OF CYLINDERS NEEDED TO BE DRILLED OUT} = \frac{(1273 - 1000)}{57} \approx 4.8$$

5 CYLINDERS MUST BE DRILLED OUT TO REDUCE THE WEIGHT TO NO MORE THAN 1000 lbs.

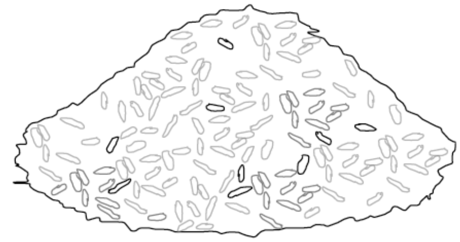
   b. If iron ore costs $\$115$ per ton ($1$ ton $= 2200$ lb.) and the price of each part is based solely on its weight of iron, how many parts can be purchased with $\$1{,}500$? Explain your answer.

$$\text{REMAINING SOLID} = [1273 - 5(57)] \text{ lbs} = 988 \text{ lbs.}$$

$$\left(\frac{\$115}{1 \text{ ton}}\right)\left(\frac{1 \text{ ton}}{2200 \text{ lbs}}\right)\left(\frac{988 \text{ lbs}}{1 \text{ block}}\right) \approx 29 \text{ blocks}$$

Approximately 29 blocks can be purchased with \$1500.

4. Rice falling from an open bag piles up into a figure conical in shape with an approximate radius of 5 centimeters.



   a. If the angle formed by the slant of the pile with the base is roughly 30˚, write an expression that represents the volume of rice in the pile.

$$\tan 30 = \frac{h}{5}$$

$$5 \tan 30 = h$$

$$\text{VOLUME} = \frac{\pi 5^2 (5 \tan 30)}{3}$$

$$= \frac{125\pi \tan 30}{3}$$

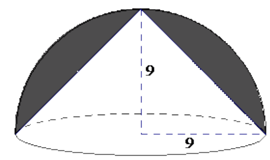
   b. If there are approximately 20 grains of rice in a cubic centimeter, approximately how many grains of rice are in the pile? Round to the nearest whole grain.

$$\left(\frac{125\pi \tan 30 \text{ cm}^3}{3}\right)\left(\frac{20 \text{ grains}}{1 \text{ cm}^3}\right)$$

$$= \left(\frac{2500\pi \tan 30}{3}\right) \text{ grains of rice}$$

$$\approx 1{,}511 \text{ grains of rice}$$

5. In a solid hemisphere, a cone is removed as shown. Calculate the volume of the resulting solid. In addition to your solution, provide an explanation of the strategy you used in your solution.




Let $V_1$ be the volume of the hemisphere.

$V_1 = \frac{1}{2}\left(\frac{4}{3}\right)\pi \cdot 9^3$

$= 486\pi$

Let $V_2$ be the volume of the cone.

$V_2 = \frac{1}{3}\pi \cdot 9^3$

$= 243\pi$

Then the volume of the resulting solid is found by

$486\pi - 243\pi = 243\pi$

6. Describe the shape of the cross-section of each of the following objects.

Right circular cone:

a. Cut by a plane through the vertex and perpendicular to the base

An isosceles triangle

Square pyramid:

b. Cut by a plane through the vertex and perpendicular to the base

An isosceles triangle

c. Cut by a vertical plane that is parallel to an edge of the base but not passing through the vertex

A trapezoid

Sphere with radius $r$:

d. Describe the radius of the circular cross-section created by a plane through the center of the sphere

The radius of the cross-section will be equal in length to the radius of the sphere.

e. Describe the radius of the circular cross-section cut by a plane that does not pass through the center of the sphere

The radius of the sphere is longer than the radius of the circular cross-section.

Triangular Prism:

f. Cut by a plane parallel to a base

A triangle

g. Cut by a plane parallel to a face

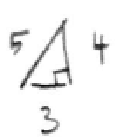
A rectangle

7.

a. A $3 \times 5$ rectangle is revolved about one of its sides of length $5$ to create a solid of revolution. Find the volume of the solid.

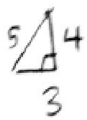

$$V = \pi 3^2(5)$$
$$= 45\pi$$

b. A 3-4-5 right triangle is revolved about a leg of length 4 to create a solid of revolution. Describe the solid.




THE SOLID WILL BE A RIGHT CIRCULAR CONE WITH RADIUS 3 AND HEIGHT 4.

c. A 3-4-5 right triangle is revolved about its legs to create two solids. Find each of the volumes of the two solids created.



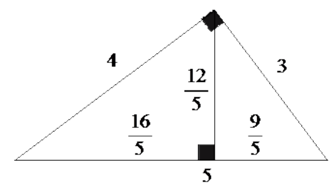

$V = \frac{1}{3}\pi 3^2(4)$

$= 12\pi$


$V = \frac{1}{3}\pi 4^2(3)$

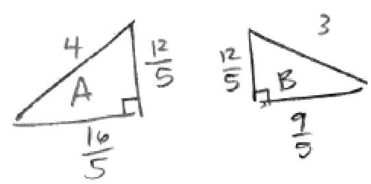
$= 16\pi$

d. Show that the volume of the solid created by revolving a 3-4-5 triangle about its hypotenuse is $\frac{48}{5}\pi$.




$V_A = \frac{1}{3}\pi\left(\frac{16}{5}\right)\left(\frac{12}{5}\right)^2$

$V_B = \frac{1}{3}\pi\left(\frac{12}{5}\right)^2\left(\frac{9}{5}\right)$




$V_A + V_B = \frac{1}{3}\pi\frac{16}{5}\left(\frac{12}{5}\right)^2 + \frac{1}{3}\pi\left(\frac{12}{5}\right)^2\left(\frac{9}{5}\right)$

$= \left[\frac{1}{3}\pi\left(\frac{12}{5}\right)^2\right]\left(\frac{16}{5}+\frac{9}{5}\right)$

$= \frac{1}{3}\pi\left(\frac{12}{5}\right)^2(5)$

$= \frac{48}{5}\pi$