

Lesson 1: Scale Drawings

Classwork

Opening Exercise

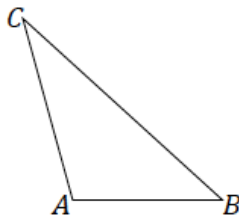


Above is a picture of a bicycle. Which of the images below appears to be a well-scaled image of the original? Why?

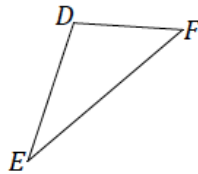


Example 1

Use construction tools to create a scale drawing of $\triangle ABC$ with a scale factor of $r = 2$.

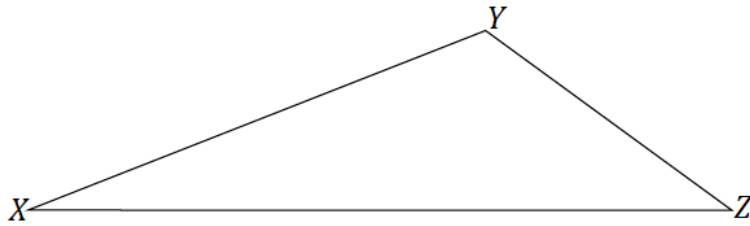
**Exercise 1**

Use construction tools to create a scale drawing of $\triangle DEF$ with a scale factor of $r = 3$. What properties does your scale drawing share with the original figure? Explain how you know.

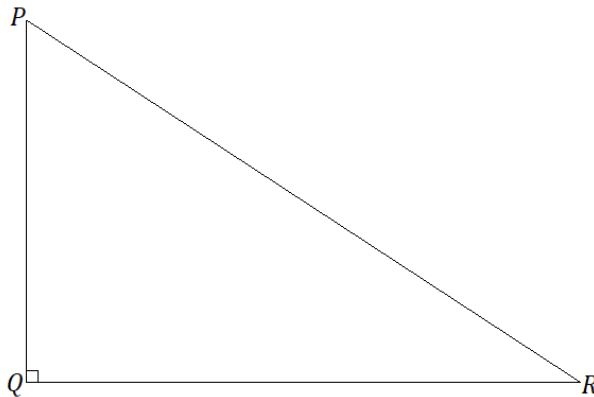


Example 2

Use construction tools to create a scale drawing of $\triangle XYZ$ with a scale factor of $r = \frac{1}{2}$.

**Exercises 2–4**

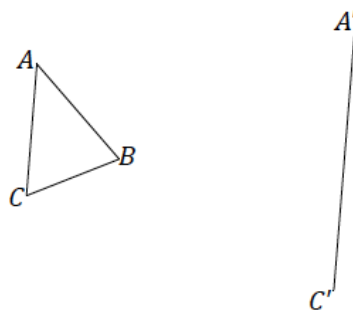
2. Use construction tools to create a scale drawing of $\triangle PQR$ with a scale factor of $r = \frac{1}{4}$. What properties do the scale drawing and the original figure share? Explain how you know.



3. Triangle EFG is provided below, and one angle of scale drawing $\triangle E'F'G'$ is also provided. Use construction tools to complete the scale drawing so that the scale factor is $r = 3$. What properties do the scale drawing and the original figure share? Explain how you know.



4. Triangle ABC is provided below, and one side of scale drawing $\triangle A'B'C'$ is also provided. Use construction tools to complete the scale drawing and determine the scale factor.

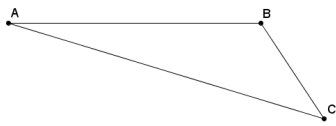


Lesson Summary

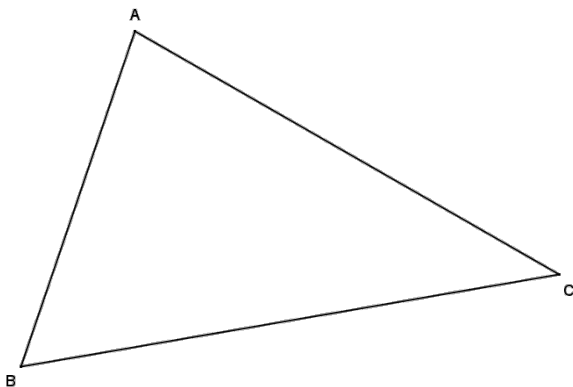
There are two properties of a scale drawing of a figure: corresponding angles are equal in measurement, and corresponding lengths are proportional in measurement.

Problem Set

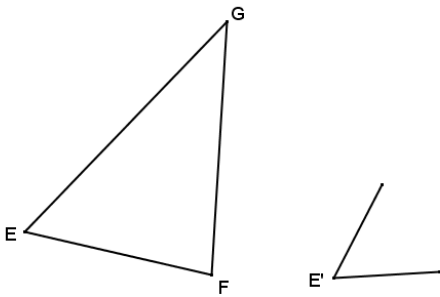
1. Use construction tools to create a scale drawing of $\triangle ABC$ with a scale factor of $r = 3$.



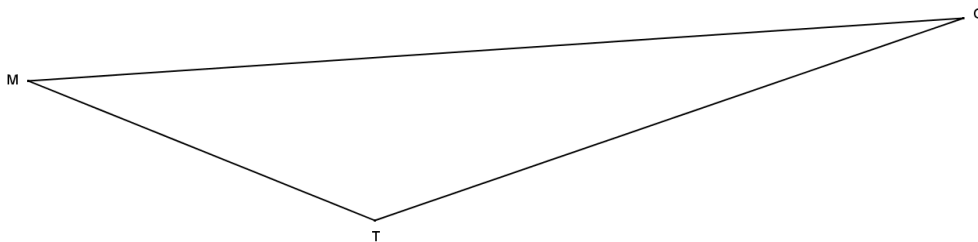
2. Use construction tools to create a scale drawing of $\triangle ABC$ with a scale factor of $r = \frac{1}{2}$.



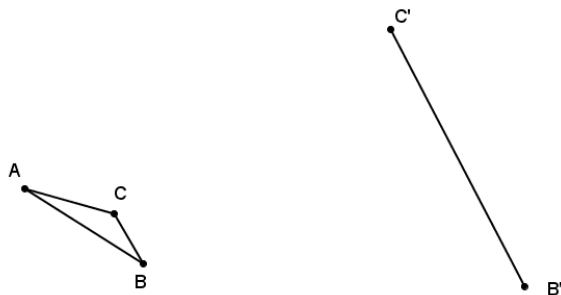
3. Triangle EFG is provided below, and one angle of scale drawing $\triangle E'F'G'$ is also provided. Use construction tools to complete a scale drawing so that the scale factor is $r = 2$.



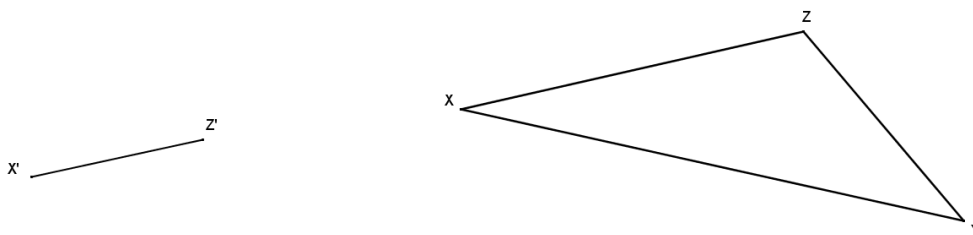
4. Triangle MTC is provided below, and one angle of scale drawing $\triangle M'T'C'$ is also provided. Use construction tools to complete a scale drawing so that the scale factor is $\frac{1}{4}$.



5. Triangle ABC is provided below, and one side of scale drawing $\triangle A'B'C'$ is also provided. Use construction tools to complete the scale drawing and determine the scale factor.



6. Triangle XYZ is provided below, and one side of scale drawing $\triangle X'Y'Z'$ is also provided. Use construction tools to complete the scale drawing and determine the scale factor.



7. Quadrilateral $GHIJ$ is a scale drawing of quadrilateral $ABCD$ with scale factor r . Describe each of the following statements as always true, sometimes true, or never true, and justify your answer.
- $AB = GH$
 - $m\angle ABC = m\angle GHI$
 - $\frac{AB}{GH} = \frac{BC}{HI}$
 - $\text{Perimeter}(GHIJ) = r \cdot \text{Perimeter}(ABCD)$
 - $\text{Area}(GHIJ) = r \cdot \text{Area}(ABCD)$ where $r \neq 1$
 - $r < 0$

Lesson 2: Making Scale Drawings Using the Ratio Method

Classwork

Opening Exercise

Based on what you recall from Grade 8, describe what a *dilation* is.

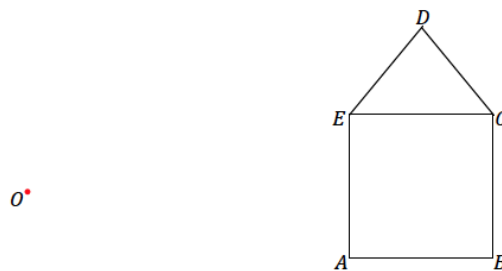
Example 1

Create a scale drawing of the figure below using the ratio method about center O and scale factor $r = \frac{1}{2}$.

Step 1. Draw a ray beginning at O through each vertex of the figure.

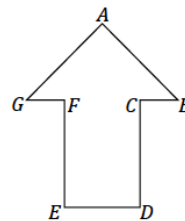
Step 2. Dilate each vertex along the appropriate ray by scale factor $r = \frac{1}{2}$. Use the ruler to find the midpoint between O and D and then each of the other vertices. Label each respective midpoint with prime notation, i.e., D' .

Step 3. Join vertices in the way they are joined in the original figure, e.g., segment $A'B'$ corresponds to segment AB .



Exercise 1

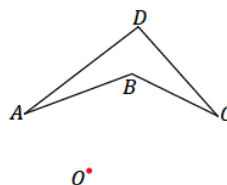
1. Create a scale drawing of the figure below using the ratio method about center O and scale factor $r = \frac{3}{4}$. Verify that the resulting figure is in fact a scale drawing by showing that corresponding side lengths are in constant proportion and the corresponding angles are equal in measurement.



O

Example 2

- a. Create a scale drawing of the figure below using the ratio method about center O and scale factor $r = 3$.
 - Step 1. Draw a ray beginning at O through each vertex of the figure.
 - Step 2. Use your ruler to determine the location of A' on \overrightarrow{OA} ; A' should be three times as far from O as A . Determine the locations of B' and C' in the same way along the respective rays.
 - Step 3. Draw the corresponding line segments, e.g., segment $A'B'$ corresponds to segment AB .

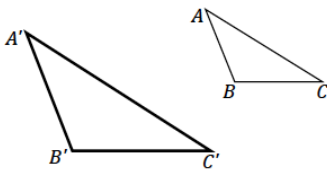


O

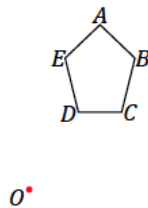
- b. Locate a point X so that it lies between endpoints A and B on segment AB of the original figure in part (a). Use the ratio method to locate X' on the scale drawing in part (a).
- c. Imagine a dilation of the same figure as in parts (a) and (b). What if the ray from the center passed through two distinct points, such as B and D below? What does that imply about the locations of B' and D' ?

Exercises 2–6

2. $\triangle A'B'C'$ is a scale drawing of $\triangle ABC$ drawn by using the ratio method. Use your ruler to determine the location of the center O used for the scale drawing.



3. Use the figure below with center O and a scale factor of $r = \frac{5}{2}$ to create a scale drawing. Verify that the resulting figure is in fact a scale drawing by showing that corresponding side lengths are in constant proportion and that the corresponding angles are equal in measurement.



4. Summarize the steps to create a scale drawing by the ratio method. Be sure to describe all necessary parameters to use the ratio method.
5. A clothing company wants to print the face of the Statue of Liberty on a T-shirt. The length of the face from the top of the forehead to the chin is 17 feet and the width of the face is 10 ft. Given that a medium sized T-shirt has a length of 29 in and a width of 20 in, what dimensions of the face are needed to produce a scaled version that will fit on the T-shirt?
- a. What shape would you use to model the face of the statue?

- b. Knowing that the maximum width of the T-shirt is 20 in, what scale factor is needed to make the width of the face fit on the shirt?
- c. What scale factor should be used to scale the length of the face? Explain.
- d. Using the scale factor identified in part (c), what is the scaled length of the face? Will it fit on the shirt?
- e. Identify the scale factor you would use to ensure that the face of the statue was in proportion and would fit on the T-shirt. Identify the dimensions of the face that will be printed on the shirt.
- f. The T-shirt company wants the width of the face to be no smaller than 10 inches. What scale factors could be used to create a scaled version of the face that meets this requirement?
- g. If it costs the company \$0.005 for each square inch of print on a shirt, what is the maximum and minimum costs for printing the face of the statue of liberty on one T-shirt?

6. Create your own scale drawing using the ratio method. In the space below:

- a. Draw an original figure.
- b. Locate and label a center of dilation O .
- c. Choose a scale factor r .
- d. Describe your dilation using appropriate notation.
- e. Complete a scale drawing using the ratio method.

Show all measurements and calculations to confirm that the new figure is a scale drawing. The work here will be your answer key.

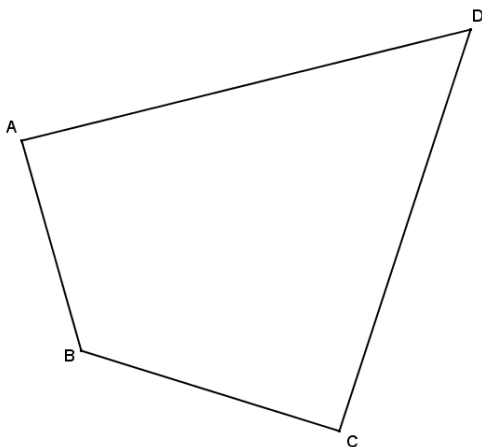
Next, trace your original figure onto a fresh piece of paper. Trade the traced figure with a partner. Provide your partner with the dilation information. Each partner should complete the other's scale drawing. When finished, check all work for accuracy against your answer key.

Lesson Summary

1. To create a scale drawing using the ratio method, each vertex of the original figure is dilated about the center O by scale factor r . Once all the vertices are dilated, they are joined to each other in the same way as in the original figure.
2. The scale factor tells us whether the scale drawing is being enlarged ($r > 1$) or reduced ($0 < r < 1$).

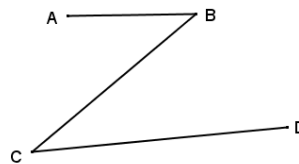
Problem Set

1. Use the ratio method to create a scale drawing about center O with a scale factor of $r = \frac{1}{4}$. Use a ruler and protractor to verify that the resulting figure is in fact a scale drawing by showing that corresponding side lengths are in constant proportion and the corresponding angles are equal in measurement.



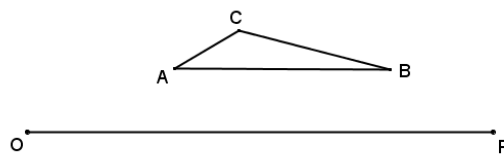
• O

2. Use the ratio method to create a scale drawing about center O with a scale factor of $r = 2$. Verify that the resulting figure is in fact a scale drawing by showing that corresponding side lengths are in constant proportion and that the corresponding angles are equal in measurement.



• O

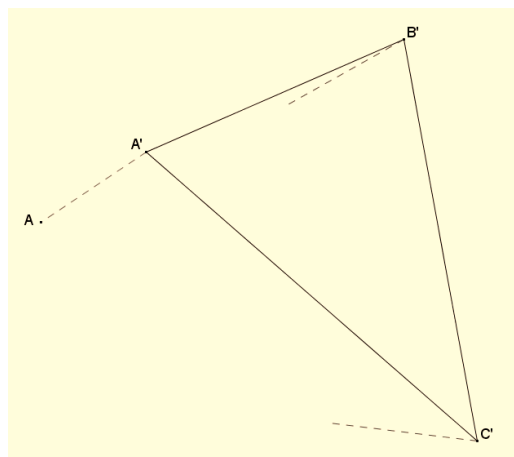
3. Use the ratio method to create two scale drawings: $D_{O,2}$ and $D_{P,2}$. Label the scale drawing with respect to center O as $\triangle A'B'C'$ and the scale drawing with respect to center P as $\triangle A''B''C''$.



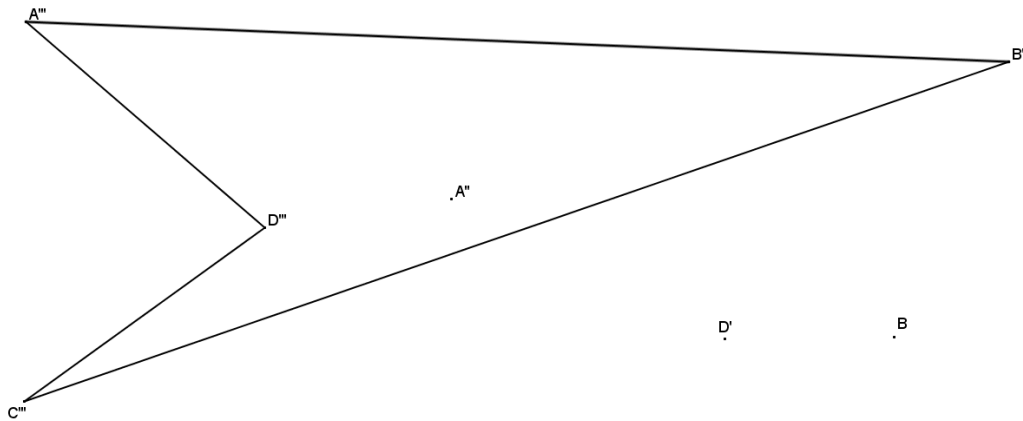
What do you notice about the two scale drawings?

What rigid motion can be used to map $\triangle A'B'C'$ onto $\triangle A''B''C''$?

4. Sara found a drawing of a triangle that appears to be a scale drawing. Much of the drawing has faded, but she can see the drawing and construction lines in the diagram below. If we assume the ratio method was used to construct $\triangle A'B'C'$ as a scale model of $\triangle ABC$, can you find the center O , the scale factor r , and locate $\triangle ABC$?

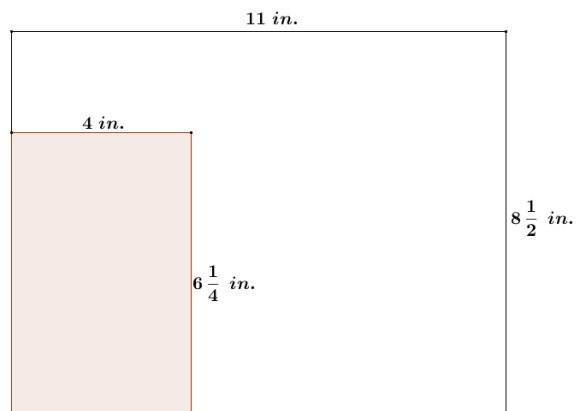


5. Quadrilateral $A'''B'''C'''D'''$ is one of a sequence of three scale drawings of quadrilateral $ABCD$ that were all constructed using the ratio method from center O . Find the center O , each scale drawing in the sequence and the scale factor for each scale drawing. The other scale drawings are quadrilaterals $A'B'C'D'$ and $A''B''C''D''$.



6. Maggie has a rectangle drawn in the corner of a $8\frac{1}{2}$ inch by 11 inch sheet of printer paper as shown in the diagram. To cut out the rectangle, Maggie must make two cuts. She wants to scale the rectangle so that she can cut it out using only one cut with a paper cutter.

- What are the dimensions of Maggie’s scaled rectangle and what is its scale factor from the original rectangle?
- After making the cut for the scaled rectangle, is there enough material left to cut another identical rectangle? If so, what is the area of scrap per sheet of paper?

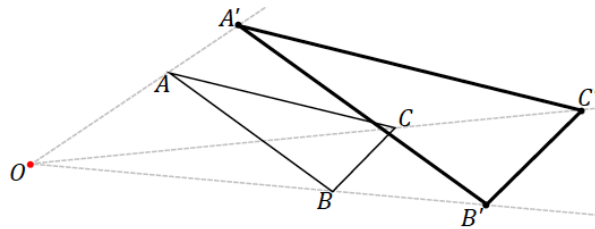


Lesson 3: Making Scale Drawings Using the Parallel Method

Classwork

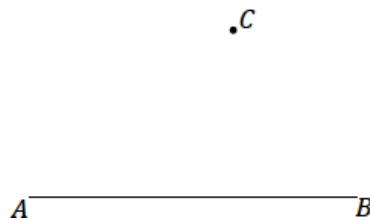
Opening Exercise

Dani dilated $\triangle ABC$ from center O , resulting in $\triangle A'B'C'$. She says that she completed the drawing using parallel lines. How could she have done this? Explain.

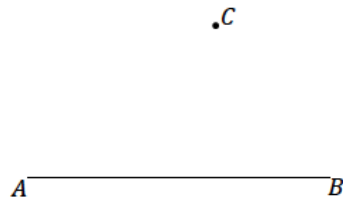


Example 1

- a. Use a ruler and setsquare to draw a line through C parallel to AB . What ensures that the line drawn is parallel to AB ?



- b. Use a ruler and setsquare to draw a parallelogram $ABCD$ around AB and point C .



Example 2

Use the figure below with center O and a scale factor of $r = 2$ and the following steps to create a scale drawing using the parallel method.

Step 1. Draw a ray beginning at O through each vertex of the figure.

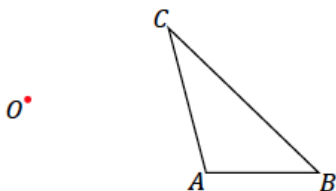
Step 2. Select one vertex of the scale drawing to locate; we have selected A' . Locate A' on ray \overrightarrow{OA} so that $OA' = 2OA$.

Step 3. Align the setsquare and ruler as in the image below; one leg of the setsquare should line up with side AB , and the perpendicular leg should be flush against the ruler.

Step 4. Slide the setsquare along the ruler until the edge of the setsquare passes through A' . Then, along the perpendicular leg of the setsquare, draw the segment through A' that is parallel to AB until it intersects with \overrightarrow{OB} , and label this point B' .

Step 5. Continue to create parallel segments to determine each successive vertex point. In this particular case, the setsquare has been aligned with AC . This is done because, in trying to create a parallel segment from BC , the parallel segment was not “reaching” B' . This could be remedied with a larger setsquare and longer ruler, but it is easily avoided by working on the segment parallel to AC instead.

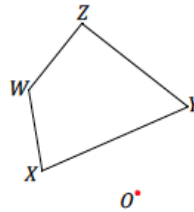
Step 6. Use your ruler to join the final two unconnected vertices.



Exercises 1–3

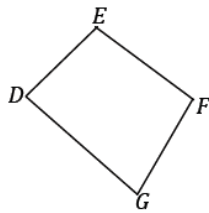
1. With a ruler and setsquare, use the parallel method to create a scale drawing of $WXYZ$ by the parallel method. W' has already been located for you. Determine the scale factor of the scale drawing. Verify that the resulting figure is in fact a scale drawing by showing that corresponding side lengths are in constant proportion and that corresponding angles are equal in measurement.

W'

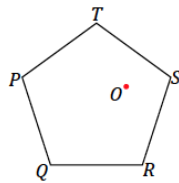


2. With a ruler and setsquare, use the parallel method to create a scale drawing of $DEFG$ about center O with scale factor $r = \frac{1}{2}$. Verify that the resulting figure is in fact a scale drawing by showing that corresponding side lengths are in constant proportion and that the corresponding angles are equal in measurement.

O

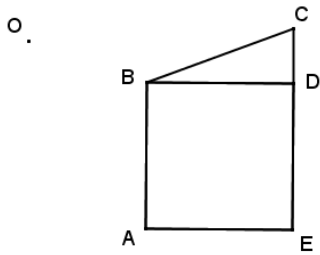


3. With a ruler and setsquare, use the parallel method to create a scale drawing of pentagon $PQRST$ about center O with scale factor $\frac{5}{2}$. Verify that the resulting figure is in fact a scale drawing by showing that corresponding side lengths are in constant proportion and that corresponding angles are equal in measurement.



Problem Set

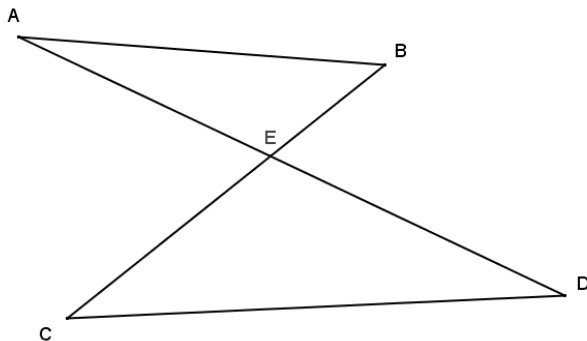
- With a ruler and setsquare, use the parallel method to create a scale drawing of the figure about center O . One vertex of the scale drawing has been provided for you.



$.A'$

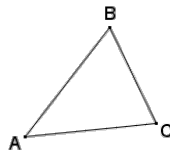
Determine the scale factor. Verify that the resulting figure is in fact a scale drawing by showing that corresponding side lengths are in constant proportion and that the corresponding angles are equal in measurement.

- Determine the scale factor. Verify that the resulting figure is in fact a scale drawing by showing that corresponding side lengths are in constant proportion and that the corresponding angles are equal in measurement.



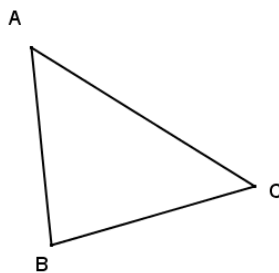
$.O$

3. With a ruler and setsquare, use the parallel method to create the following scale drawings about center O : (1) first use a scale a factor of 2 to create $\triangle A'B'C'$, (2) then, with respect to $\triangle A'B'C'$, use a scale factor of $\frac{2}{3}$ to create scale drawing $\triangle A''B''C''$. Calculate the scale factor for $\triangle A''B''C''$ as a scale drawing of $\triangle ABC$. Use angle and side length measurements and the appropriate proportions to verify your answer.



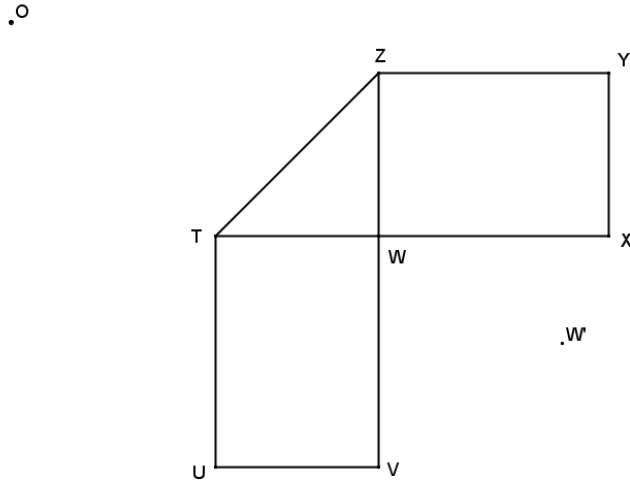
O ·

4. Follow the direction in each part below to create three scale drawings of $\triangle ABC$ using the parallel method.
- With the center at vertex A , make a scale drawing of $\triangle ABC$ with a scale factor of $\frac{3}{2}$.
 - With the center at vertex B , make a scale drawing of $\triangle ABC$ with a scale factor of $\frac{3}{2}$.
 - With the center at vertex C , make a scale drawing of $\triangle ABC$ with a scale factor of $\frac{3}{2}$.



- d. What conclusions can be drawn about all three scale drawings from parts (a)–(c)?

5. Use the parallel method to make a scale drawing of the line segments in the following figure using the given W' , the image of vertex W , from center O . Determine the scale factor.



Use your diagram from Problem 1 to answer this question.

6. If we switch perspective and consider the original drawing $ABCDE$ to be a scale drawing of the constructed image $A'B'C'D'E'$, what would the scale factor be?

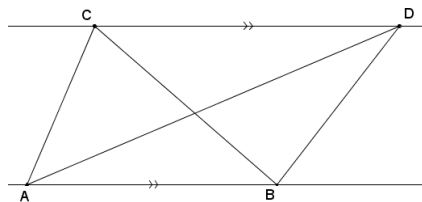
Lesson 4: Comparing the Ratio Method with the Parallel Method

Classwork

Today, our goal is to show that the parallel method and the ratio method are equivalent; that is, given a figure in the plane and a scale factor $r > 0$, the scale drawing produced by the parallel method is congruent to the scale drawing produced by the ratio method. We start with two easy exercises about the areas of two triangles whose bases lie on the same line, which will help show that the two methods are equivalent.

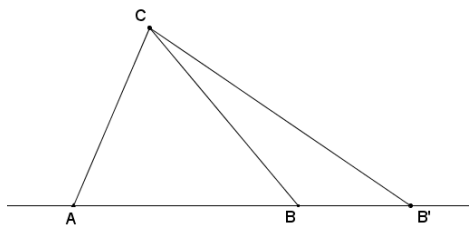
Opening Exercises 1–2

- Suppose two triangles, $\triangle ABC$ and $\triangle ABD$, share the same base \overline{AB} such that points C and D lie on a line parallel to line \overline{AB} . Show that their areas are equal, i.e., $\text{Area}(\triangle ABC) = \text{Area}(\triangle ABD)$. (Hint: Why are the altitudes of each triangle equal in length?)



- Suppose two triangles have different length bases, \overline{AB} and $\overline{AB'}$, that lie on the same line. Furthermore, suppose they both have the same vertex C opposite these bases. Show that value of the ratio of their areas is equal to the value of the ratio of the lengths of their bases, i.e.,

$$\frac{\text{Area}(\triangle ABC)}{\text{Area}(\triangle AB'C)} = \frac{AB}{AB'}$$



Discussion

To show that the parallel and ratio methods are equivalent, we need only look at one of the simplest versions of a scale drawing: scaling segments. First, we need to show that the scale drawing of a segment generated by the parallel method is the same segment that the ratio method would have generated and vice versa. (i.e., that the scaled segment generated by the ratio method is the same segment generated by the parallel method.) That is,

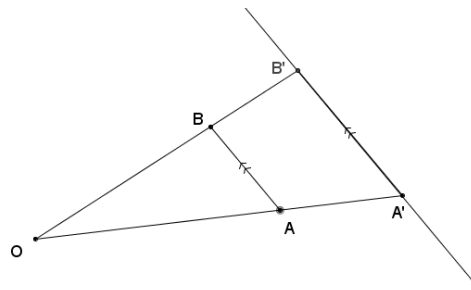
The parallel method \Rightarrow The ratio method,

and

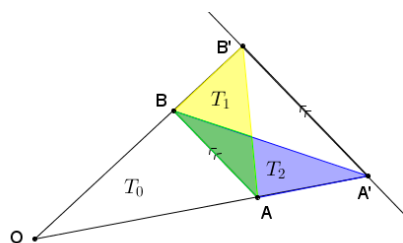
The ratio method \Rightarrow The parallel method.

The first implication above can be stated as the following theorem:

PARALLEL \Rightarrow RATIO THEOREM: Given a line segment \overline{AB} and point O not on the line \overleftrightarrow{AB} , construct a scale drawing of \overline{AB} with scale factor $r > 0$ using the parallel method: Let $A' = D_{O,r}(A)$, and ℓ be the line parallel to \overleftrightarrow{AB} that passes through A' . Let B' be the point where ray \overrightarrow{OB} intersects ℓ . Then B' is the same point found by the ratio method; that is, $B' = D_{O,r}(B)$.



PROOF: We prove the case when $r > 1$; the case when $0 < r < 1$ is the same but with a different picture. Construct two line segments $\overline{BA'}$ and $\overline{AB'}$ to form two triangles $\triangle BAB'$ and $\triangle BAA'$, labeled as T_1 and T_2 , respectively, in the picture below.



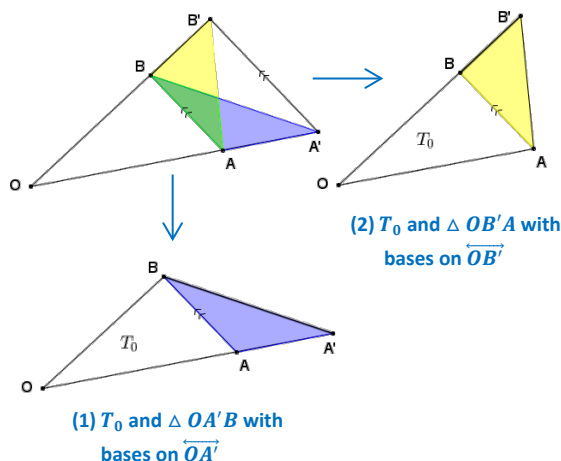
The areas of these two triangles are equal,

$$\text{Area}(T_1) = \text{Area}(T_2),$$

by Exercise 1 (why?). Label $\triangle OAB$ by T_0 . Then $\text{Area}(\triangle OA'B) = \text{Area}(\triangle OB'A)$ because areas add:

$$\begin{aligned} \text{Area}(\triangle OA'B) &= \text{Area}(T_0) + \text{Area}(T_2) \\ &= \text{Area}(T_0) + \text{Area}(T_1) \\ &= \text{Area}(\triangle OB'A). \end{aligned}$$

Next, we apply Exercise 2 to two sets of triangles: (1) T_0 and $\triangle OA'B$ and (2) T_0 and $\triangle OB'A$.



Therefore,

$$\frac{\text{Area}(\triangle OA'B)}{\text{Area}(T_0)} = \frac{OA'}{OA}, \text{ and}$$

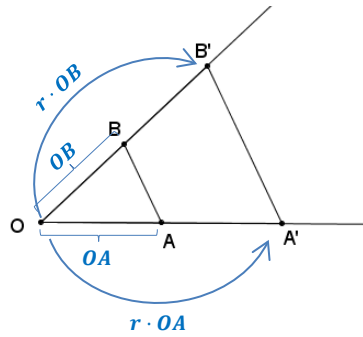
$$\frac{\text{Area}(\triangle OB'A)}{\text{Area}(T_0)} = \frac{OB'}{OB}.$$

Since $\text{Area}(\triangle OA'B) = \text{Area}(\triangle OB'A)$, we can equate the fractions: $\frac{OA'}{OA} = \frac{OB'}{OB}$. Since r is the scale factor used in dilating \overline{OA} to $\overline{OA'}$, we know that $\frac{OA'}{OA} = r$; therefore, $\frac{OB'}{OB} = r$, or $OB' = r \cdot OB$. This last equality implies that B' is the dilation of B from O by scale factor r , which is what we wanted to prove.

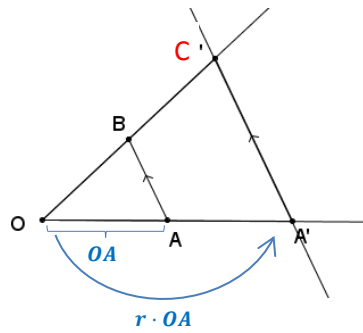
Next, we prove the reverse implication to show that both methods are equivalent to each other.

RATIO \Rightarrow PARALLEL THEOREM: Given a line segment \overline{AB} and point O not on the line \overleftrightarrow{AB} , construct a scale drawing $\overline{A'B'}$ of \overline{AB} with scale factor $r > 0$ using the ratio method (Find $A' = D_{O,r}(A)$ and $B' = D_{O,r}(B)$, and draw $\overline{A'B'}$). Then B' is the same as the point found using the parallel method.

PROOF: Since both the ratio method and the parallel method start with the same first step of setting $A' = D_{O,r}(A)$, the only difference between the two methods is in how the second point is found. If we use the parallel method, we construct the line ℓ parallel to \overleftrightarrow{AB} that passes through A' and label the point where ℓ intersects \overline{OB} by C . Then B' is the same as the point found using the parallel method if we can show that $C = B'$.



The ratio method

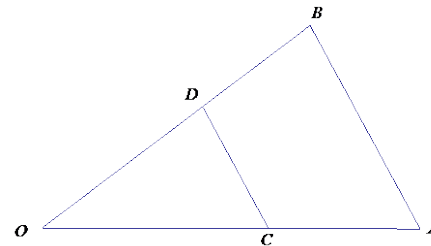


The parallel method

By the parallel \Rightarrow ratio theorem, we know that $C = D_{O,r}(B)$, i.e., that C is the point on ray \overrightarrow{OB} such that $OC = r \cdot OB$. But B' is also the point on ray \overrightarrow{OB} such that $OB' = r \cdot OB$. Hence, they must be the same point.

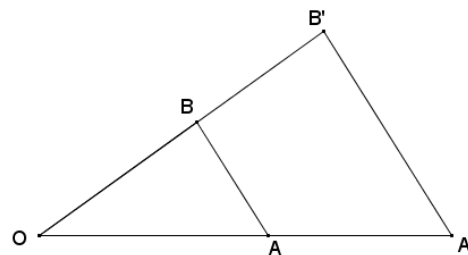
The fact that the ratio and parallel methods are equivalent is often stated as the triangle side splitter theorem. To understand the triangle side splitter theorem, we need a definition:

SIDE SPLITTER: A line segment CD is said to *split the sides of $\triangle OAB$ proportionally* if C is a point on \overline{OA} , D is a point on \overline{OB} , and $\frac{OA}{OC} = \frac{OB}{OD}$ (or equivalently, $\frac{OC}{OA} = \frac{OD}{OB}$). We call line segment CD a *side splitter*.



TRIANGLE SIDE SPLITTER THEOREM: A line segment splits two sides of a triangle proportionally if and only if it is parallel to the third side.

Restatement of the triangle side splitter theorem:



Lesson Summary

THE TRIANGLE SIDE SPLITTER THEOREM: A line segment splits two sides of a triangle proportionally if and only if it is parallel to the third side.

Problem Set

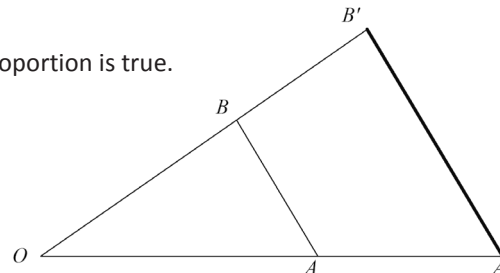
1. Use the diagram to answer each part below.

a. Measure the segments in the figure below to verify that the proportion is true.

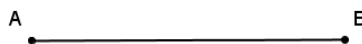
$$\frac{OA'}{OA} = \frac{OB'}{OB}$$

b. Is the proportion $\frac{OA}{OA'} = \frac{OB}{OB'}$ also true? Explain algebraically.

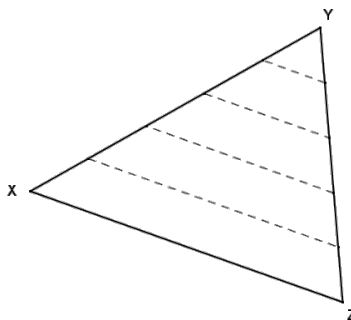
c. Is the proportion $\frac{AA'}{OA'} = \frac{BB'}{OB'}$ also true? Explain algebraically.



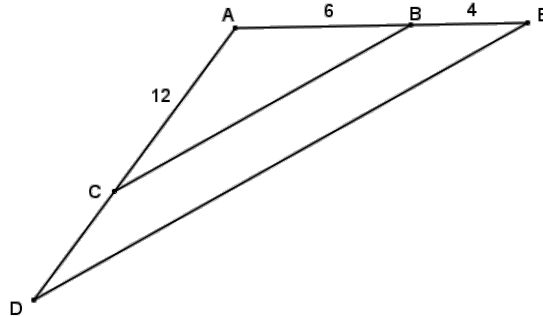
2. Given the diagram below, $AB = 30$, line ℓ is parallel to \overline{AB} , and the distance from \overline{AB} to ℓ is 25. Locate point C on line ℓ such that $\triangle ABC$ has the greatest area. Defend your answer.



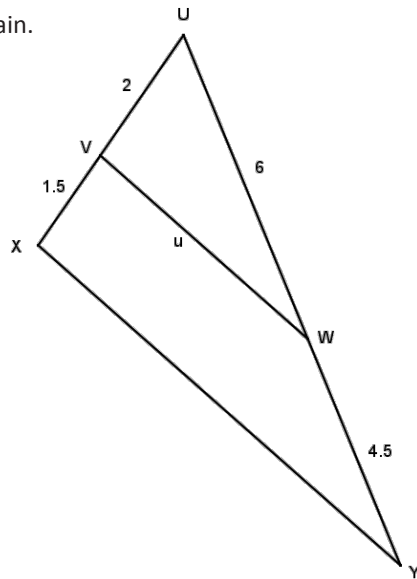
3. Given $\triangle XYZ$, \overline{XY} and \overline{YZ} are partitioned into equal length segments by the endpoints of the dashed segments as shown. What can be concluded about the diagram?



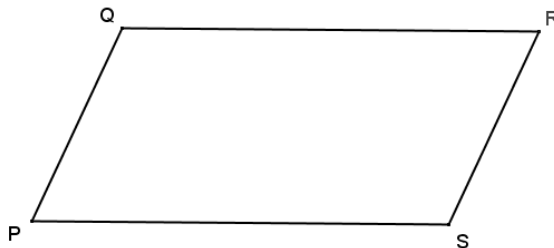
4. Given the diagram, $AC = 12$, $AB = 6$, $BE = 4$, $\angle ACB = x^\circ$, and $\angle D = x^\circ$, find CD .



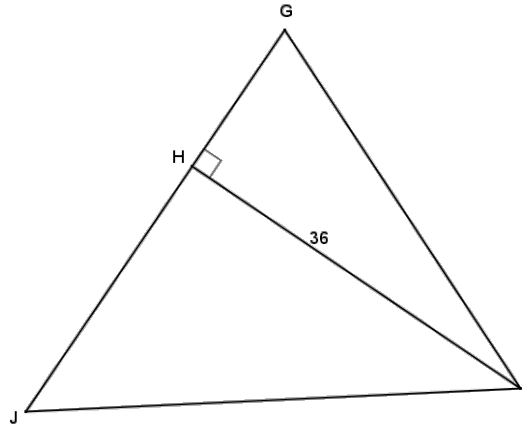
5. What conclusions can be drawn from the diagram shown to the right? Explain.



6. Parallelogram $PQRS$ is shown. Two triangles are formed by a diagonal within the parallelogram. Identify those triangles and explain why they are guaranteed to have the same areas.



7. In the diagram to the right, $HI = 36$ and $GJ = 42$. If the ratio of the areas of the triangles is $\frac{\text{Area } \triangle GHI}{\text{Area } \triangle JHI} = \frac{5}{9}$, find JH , GH , GI , and JI .



Lesson 5: Scale Factors

Classwork

Opening Exercise

Quick Write: Describe how a figure is transformed under a dilation with a scale factor $r = 1$, $r > 1$, and $0 < r < 1$.

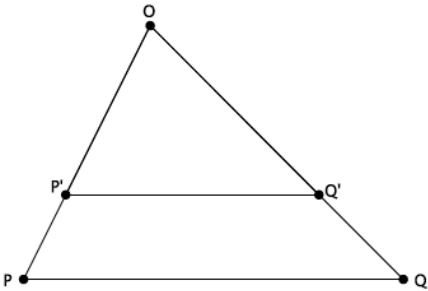
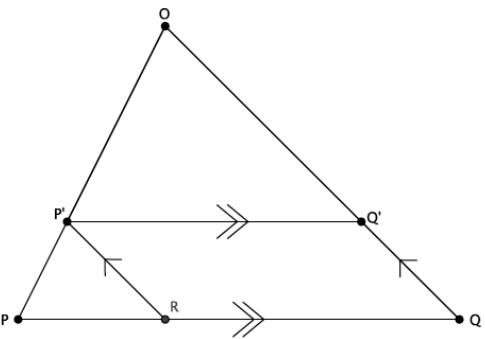
Discussion

DILATION THEOREM: If a dilation with center O and scale factor r sends point P to P' and Q to Q' , then $|P'Q'| = r|PQ|$. Furthermore, if $r \neq 1$ and O, P , and Q are the vertices of a triangle, then $\overline{PQ} \parallel \overline{P'Q'}$.

Now consider the dilation theorem when O, P , and Q are the vertices of $\triangle OPQ$. Since P' and Q' come from a dilation with scale factor r and center O , we have $\frac{OP'}{OP} = \frac{OQ'}{OQ} = r$.

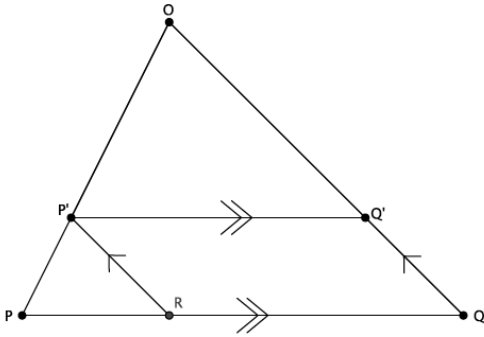
There are two cases that arise, recall what you wrote in your Quick Write. We must consider the case when $r > 1$ and when $0 < r < 1$. Let's begin with the latter.

Dilation Theorem Proof, Case 1

Statements	Reasons/Explanations
	
<p>1. A dilation with center O and scale factor r sends point P to P' and Q to Q'.</p>	<p>1.</p>
<p>2. $\frac{OP'}{OP} = \frac{OQ'}{OQ} = r$</p>	<p>2.</p>
<p>3. $\overrightarrow{PQ} \parallel \overrightarrow{P'Q'}$</p>	<p>3.</p>
<p>4. A dilation with center P and scale factor $\frac{PP'}{PO}$ sends point O to P' and point Q to R. Draw $\overline{P'R}$.</p> 	<p>4.</p>

5. $\overline{P'R} \parallel \overline{OQ'}$

6. $RP'Q'Q$ is a parallelogram.



7. $RQ = P'Q'$

8. $\frac{RQ}{PQ} = \frac{P'O}{PO}$

9. $\frac{RQ}{PQ} = r$

10. $RQ = r \cdot PQ$

11. $P'Q' = r \cdot PQ$

5.

6.

7.

8.

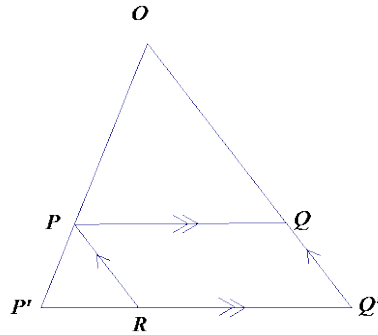
9.

10.

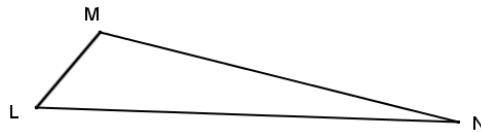
11.

Exercises 1–4

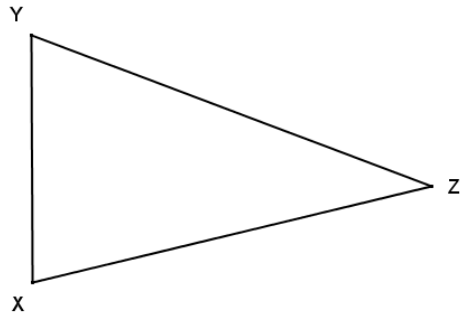
1. Prove Case 2: If $O, P,$ and Q are the vertices of a triangle and $r > 1,$ show that (a) $\overrightarrow{PQ} \parallel \overrightarrow{P'Q'}$ and (b) $P'Q' = rPQ.$ Use the diagram below when writing your proof.



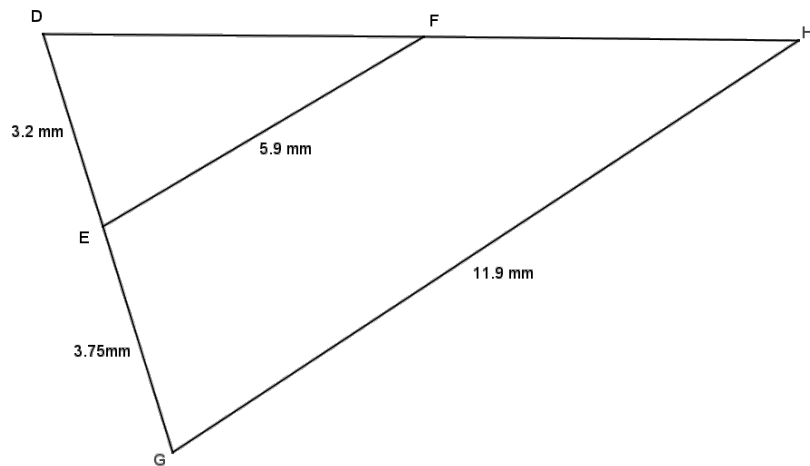
- 2.
- a. Produce a scale drawing of $\triangle LMN$ using either the ratio or parallel method with point M as the center and a scale factor of $\frac{3}{2}.$



- b. Use the dilation theorem to predict the length of $L'N'$, then measure its length directly using a ruler.
- c. Does the dilation theorem appear to hold true?
3. Produce a scale drawing of $\triangle XYZ$ with point X as the center and a scale factor of $\frac{1}{4}$. Use the dilation theorem to predict $Y'Z'$, and then measure its length directly using a ruler. Does the dilation theorem appear to hold true



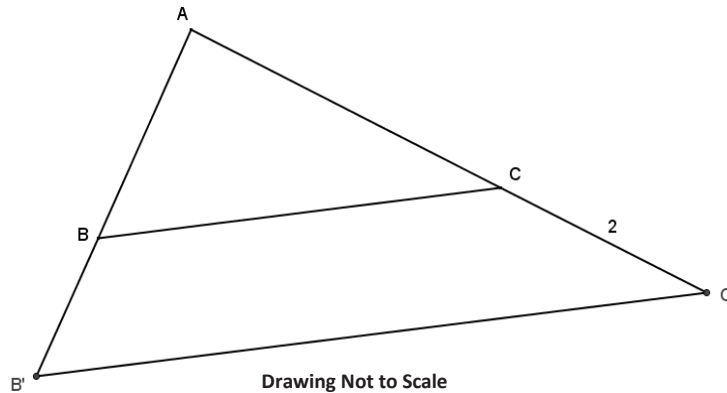
4. Given the diagram below, determine if $\triangle DEF$ is a scale drawing of $\triangle DGH$. Explain why or why not.



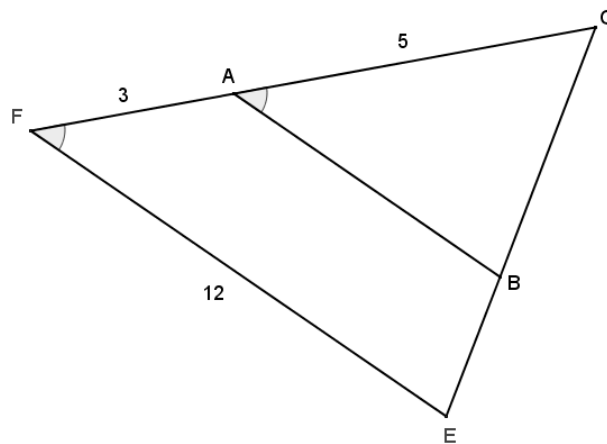
Problem Set

1. $\triangle AB'C'$ is a dilation of $\triangle ABC$ from vertex A , and $CC' = 2$. Use the given information in each part and the diagram to find $B'C'$.

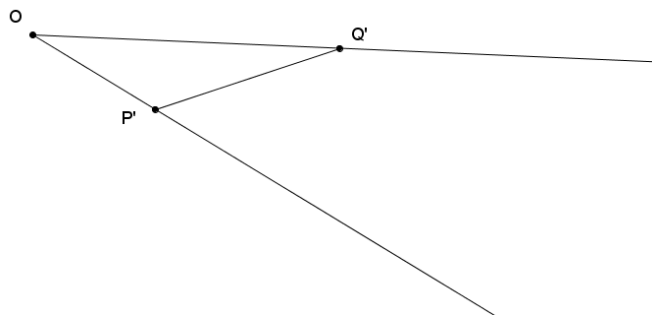
- a. $AB = 9, AC = 4,$ and $BC = 7$
- b. $AB = 4, AC = 9,$ and $BC = 7$
- c. $AB = 7, AC = 9,$ and $BC = 4$
- d. $AB = 7, AC = 4,$ and $BC = 9$
- e. $AB = 4, AC = 7,$ and $BC = 9$
- f. $AB = 9, AC = 7,$ and $BC = 4$



2. Given the diagram, $\angle CAB \cong \angle CFE$. Find AB .

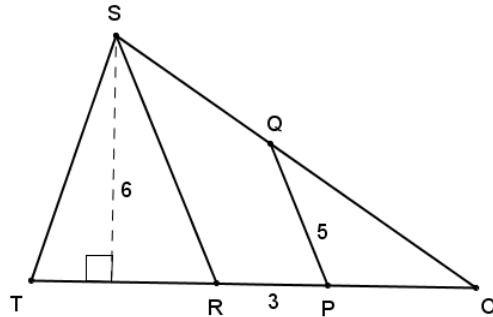


3. Use the diagram to answer each part below.

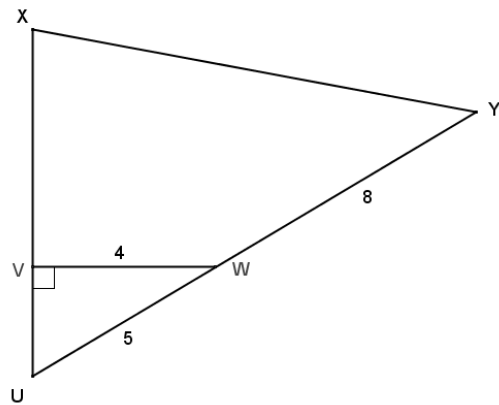


- a. $\triangle OP'Q'$ is the dilated image of $\triangle OPQ$ from point O with a scale factor of $r > 1$. Draw a possible \overline{PQ} .
- b. $\triangle OP''Q''$ is the dilated image of $\triangle OPQ$ from point O with a scale factor $k > r$. Draw a possible $\overline{P''Q''}$.

4. Given the diagram to the right, $\overline{RS} \parallel \overline{PQ}$, $\text{Area}(\triangle RST) = 15 \text{ units}^2$, and $\text{Area}(\triangle OSR) = 21 \text{ units}^2$, find RS .



5. Using the information given in the diagram and $UX = 9$, find Z on \overline{XU} such that \overline{YZ} is an altitude. Then find YZ and XZ .

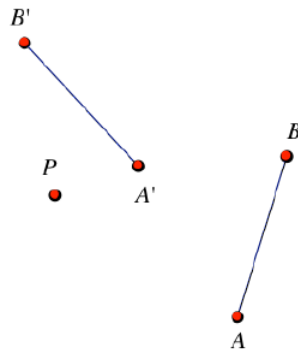


Lesson 6: Dilations as Transformations of the Plane

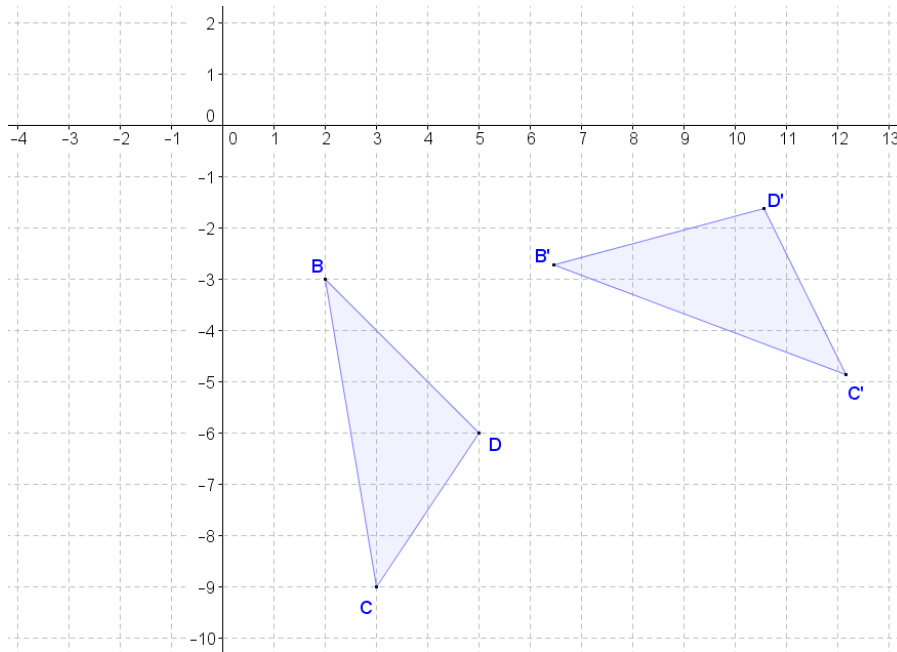
Classwork

Exercises 1–6

1. Find the center and the angle of the rotation that takes AB to $A'B'$. Find the image P' of point P under this rotation.



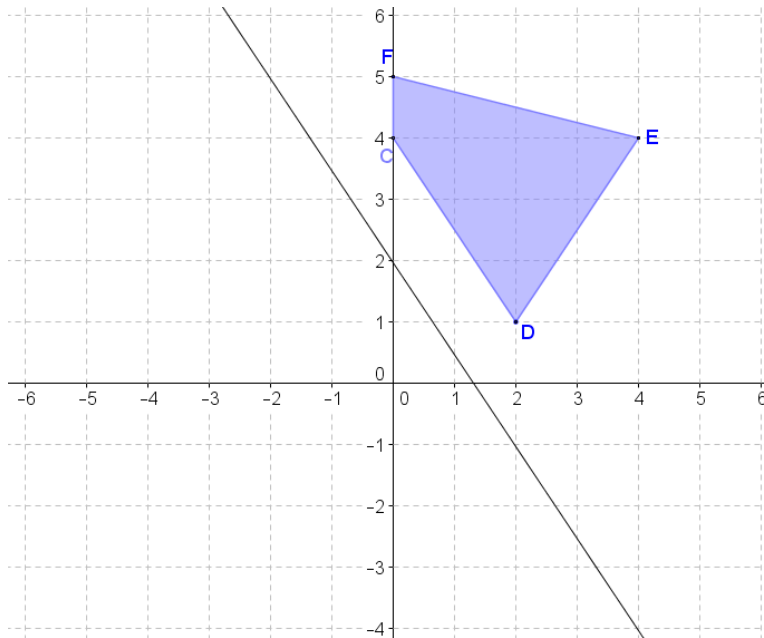
2. In the diagram below, $\triangle B'C'D'$ is the image of $\triangle BCD$ after a rotation about a point A . What are the coordinates of A , and what is the degree measure of the rotation?



3. Find the line of reflection for the reflection that takes point A to point A' . Find the image P' under this reflection.



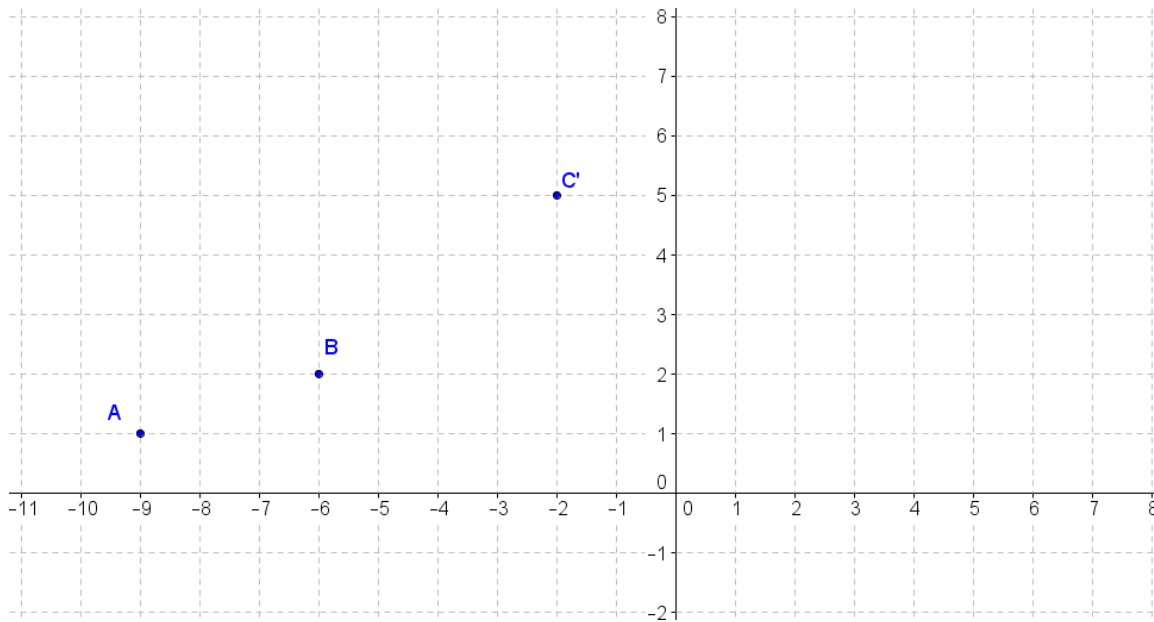
4. Quinn tells you that the vertices of the image of quadrilateral $CDEF$ reflected over the line representing the equation $y = -\frac{3}{2}x + 2$ are the following: $C'(-2,3)$, $D'(0,0)$, $E'(-3,-3)$, and $F'(-3,3)$. Do you agree or disagree with Quinn? Explain.



5. A translation takes A to A' . Find the image P' and pre-image P'' of point P under this translation. Find a vector that describes the translation.



6. The point C' is the image of point C under a translation of the plane along a vector.

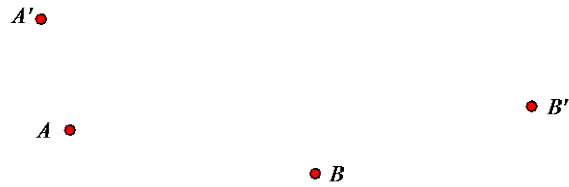


- a. Find the coordinates of C' if the vector used for the translation is \overrightarrow{BA} .

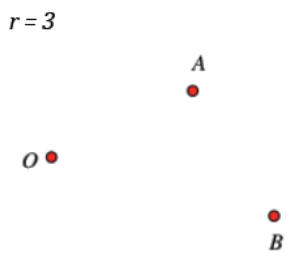
- b. Find the coordinates of C' if the vector used for the translation is \overrightarrow{AB} .

Exercises 7–9

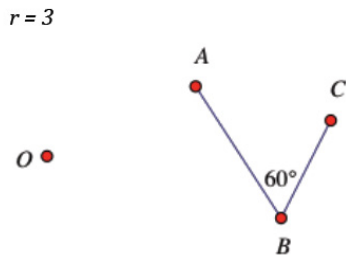
7. A dilation with center O and scale factor r takes A to A' and B to B' . Find the center O and estimate the scale factor r .



8. Given a center O , scale factor r , and points A and B , find the points $A' = D_{O,r}(A)$ and $B' = D_{O,r}(B)$. Compare length AB with length $A'B'$ by division; in other words, find $\frac{A'B'}{AB}$. How does this number compare to r ?



9. Given a center O , scale factor r , and points A , B , and C , find the points $A' = D_{O,r}(A)$, $B' = D_{O,r}(B)$, and $C' = D_{O,r}(C)$. Compare $m\angle ABC$ with $m\angle A'B'C'$. What do you find?



Lesson Summary

- There are two major classes of transformations; those that are distance-preserving (translations, reflections, rotations) and those that are not (dilations).
- Like rigid motions, dilations involve a rule assignment for each point in the plane and also have inverse functions that return each dilated point back to itself.

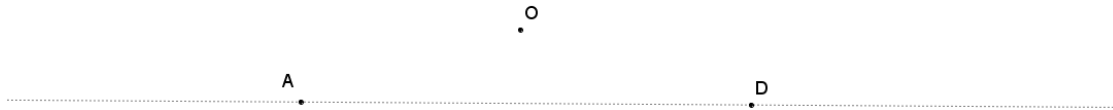
Problem Set

1. In the diagram below, A' is the image of A under a single transformation of the plane. Use the given diagram to show your solutions to parts (a)–(d).

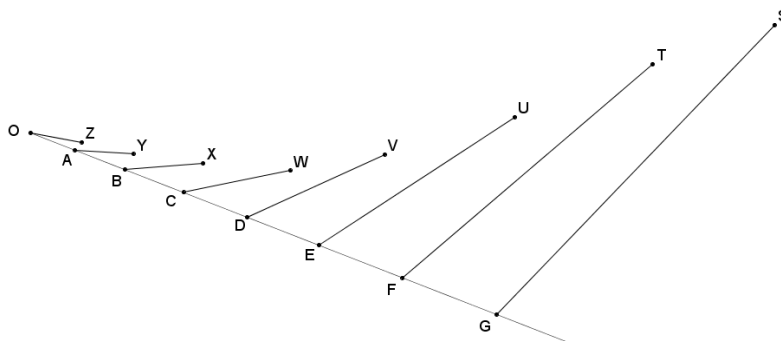


- Describe the translation that maps $A \rightarrow A'$, and then use the translation to locate P' , the image of P .
- Describe the reflection that maps $A \rightarrow A'$, and then use the reflection to locate P' , the image of P .
- Describe a rotation that maps $A \rightarrow A'$, and then use your rotation to locate P' , the image of P .
- Describe a dilation that maps $A \rightarrow A'$, and then use your dilation to locate P' , the image of P .

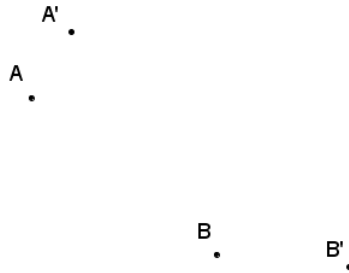
2. On the diagram below, O is a center of dilation and \overleftrightarrow{AD} is a line not through O . Choose two points B and C on \overleftrightarrow{AD} between A and D .



- Dilate $A, B, C,$ and D from O using scale factor $r = \frac{1}{2}$. Label the images $A', B', C',$ and D' , respectively.
 - Dilate $A, B, C,$ and D from O using scale factor $r = 2$. Label the images $A'', B'', C'',$ and D'' , respectively.
 - Dilate $A, B, C,$ and D from O using scale factor $r = 3$. Label the images $A''', B''', C''',$ and D''' , respectively.
 - Draw a conclusion about the effect of a dilation on a line segment based on the diagram that you drew. Explain.
3. Write the inverse transformation for each of the following so that the composition of the transformation with its inverse will map a point to itself on the plane.
- $T_{\overline{AB}}$
 - $r_{\overline{AB}}$
 - $R_{C,45}$
 - $D_{O,r}$
4. Given $U(1,3), V(-4,-4),$ and $W(-3,6)$ on the coordinate plane, perform a dilation of $\triangle UVW$ from center $O(0,0)$ with a scale factor of $\frac{3}{2}$. Determine the coordinates of images of points $U, V,$ and $W,$ and describe how the coordinates of the image points are related to the coordinates of the pre-image points.
5. Points $B, C, D, E, F,$ and G are dilated images of A from center O with scale factors 2, 3, 4, 5, 6, and 7, respectively. Are points $Y, X, W, V, U, T,$ and S all dilated images of Z under the same respective scale factors? Explain why or why not.



6. Find the center and scale factor that takes A to A' and B to B' , if a dilation exists.



7. Find the center and scale factor that takes A to A' and B to B' , if a dilation exists.



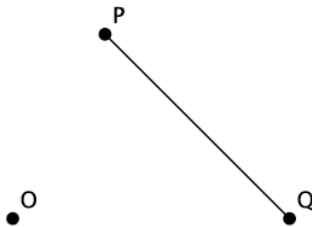
Lesson 7: How Do Dilations Map Segments?

Classwork

Opening Exercise

- Is a dilated segment still a segment? If the segment is transformed under a dilation, explain how.

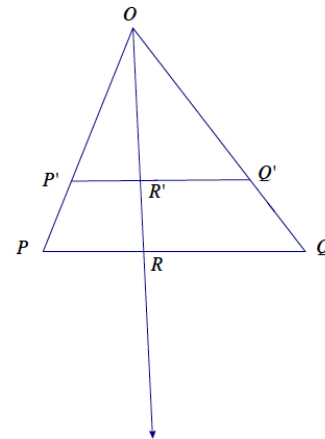
- Dilate the segment PQ by a scale factor of 2 from center O .



Example 3

Case 3. Consider the case where \overleftrightarrow{PQ} does not contain the center O of the dilation and the scale factor r of the dilation is not equal to 1; then we have the situation where the key points $O, P,$ and Q form $\triangle OPQ$. The scale factor not being equal to 1 means that we must consider scale factors such that $0 < r < 1$ and $r > 1$. However, the proofs for each are similar, so we will focus on the case when $0 < r < 1$.

When we dilate points P and Q from center O by scale factor $0 < r < 1$, as shown, what do we know about points P' and Q' ?



We will use the fact that the line segment $P'Q'$ splits the sides of $\triangle OPQ$ proportionally and that the lines containing \overleftrightarrow{PQ} and $\overleftrightarrow{P'Q'}$ are parallel to prove that a dilation maps segments to segments. Because we already know what happens when points P and Q are dilated, consider another point R that is on the segment \overleftrightarrow{PQ} . After dilating R from center O by scale factor r to get the point R' , does R' lie on the segment $\overleftrightarrow{P'Q'}$?

Putting together the preliminary dilation theorem for segments with the dilation theorem, we get:

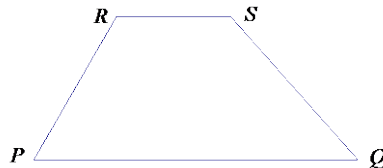
DILATION THEOREM FOR SEGMENTS: A dilation $D_{O,r}$ maps a line segment PQ to a line segment $P'Q'$ sending the endpoints to the endpoints so that $P'Q' = rPQ$. Whenever the center O does not lie in line PQ and $r \neq 1$, we conclude $\overleftrightarrow{PQ} \parallel \overleftrightarrow{P'Q'}$. Whenever the center O lies in \overleftrightarrow{PQ} or if $r = 1$, we conclude $\overleftrightarrow{PQ} = \overleftrightarrow{P'Q'}$.

As an aside, observe that dilation maps parallel line segments to parallel line segments. Further, a dilation maps a directed line segment to a directed line segment that points in the same direction.

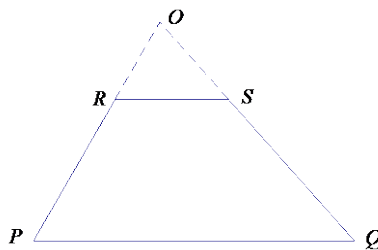
Example 4

Now look at the converse of the dilation theorem for segments: If \overline{PQ} and \overline{RS} are line segments of different lengths in the plane, then there is a dilation that maps one to the other if and only if $\overline{PQ} = \overline{RS}$ or $\overline{PQ} \parallel \overline{RS}$.

Based on Examples 2 and 3, we already know that a dilation maps a segment PQ to another line segment, say \overline{RS} , so that $\overline{PQ} = \overline{RS}$ (Example 2) or $\overline{PQ} \parallel \overline{RS}$ (Example 3). If $\overline{PQ} \parallel \overline{RS}$, then, because \overline{PQ} and \overline{RS} are different lengths in the plane, they are bases of a trapezoid, as shown.



Since \overline{PQ} and \overline{RS} are segments of different lengths, then the non-base sides of the trapezoid are not parallel, and the lines containing them will meet at a point O as shown.



Recall that we want to show that a dilation will map \overline{PQ} to \overline{RS} . Explain how to show it.

The case when the segments \overline{PQ} and \overline{RS} are such that $\overline{PQ} = \overline{RS}$ is left as an exercise.

Exercises 1–2

In the following exercises, you will consider the case where the segment and its dilated image belong to the same line; that is, when \overline{PQ} and \overline{RS} are such that $\overline{PQ} = \overline{RS}$.

1. Consider points P , Q , R , and S on a line, where $P = R$, as shown below. Show there is a dilation that maps \overline{PQ} to \overline{RS} . Where is the center of the dilation?



2. Consider points P , Q , R , and S on a line as shown below where $PQ \neq RS$. Show there is a dilation that maps \overline{PQ} to \overline{RS} . Where is the center of the dilation?

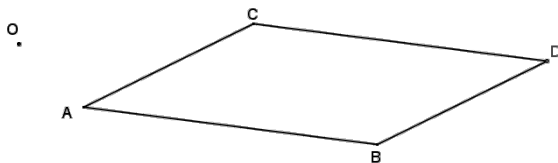


Lesson Summary

- When a segment is dilated by a scale factor of $r = 1$, then the segment and its image would be the same length.
- When the points P and Q are on a line containing the center, then the dilated points P' and Q' will also be collinear with the center producing an image of the segment that is a segment.
- When the points P and Q are not collinear with the center, and the segment is dilated by a scale factor of $r \neq 1$, then the point P' lies on the ray OP' with $OP' = r \cdot OP$ and Q' lies on ray OQ with $OQ' = r \cdot OQ$.

Problem Set

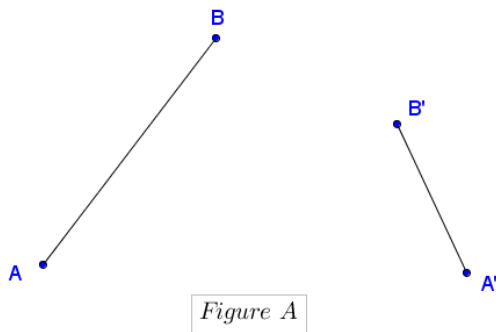
1. Draw the dilation of parallelogram $ABCD$ from center O using the scale factor $r = 2$, and then answer the questions that follow.



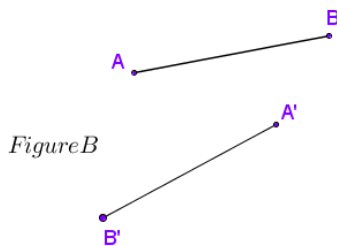
- a. Is the image $A'B'C'D'$ also a parallelogram? Explain.
 - b. What do parallel lines seem to map to under a dilation?
2. Given parallelogram $ABCD$ with $A(-8,1)$, $B(2,-4)$, $C(-3,-6)$, and $D(-13,-1)$, perform a dilation of the plane centered at the origin using the following scale factors.
 - a. Scale factor $\frac{1}{2}$
 - b. Scale factor 2
 - c. Are the images of parallel line segments under a dilation also parallel? Use your graphs to support your answer.

3. In Lesson 7, Example 3, we proved that a line segment \overline{PQ} , where O , P , and Q are the vertices of a triangle, maps to a line segment $\overline{P'Q'}$ under a dilation with a scale factor $r < 1$. Using a similar proof, prove that for O not on \overline{PQ} , a dilation with center O and scale factor $r > 1$ maps a point R on \overline{PQ} to a point R' on line \overline{PQ} .
4. On the plane, $\overline{AB} \parallel \overline{A'B'}$ and $AB \neq A'B'$. Describe a dilation mapping \overline{AB} to $\overline{A'B'}$. (Hint: There are 2 cases to consider.)
5. Only one of Figures A , B , or C below contains a dilation that maps A to A' and B to B' . Explain for each figure why the dilation does or does not exist. For each figure, assume that $AB \neq A'B'$.

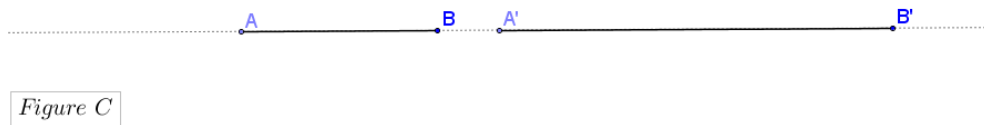
a.



b.



c.

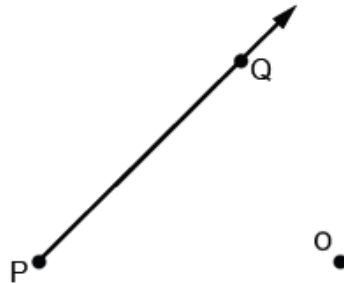


Lesson 8: How Do Dilations Map Rays, Lines, and Circles?

Classwork

Opening Exercise

- a. Is a dilated ray still a ray? If the ray is transformed under a dilation, explain how.
- b. Dilate the ray \overrightarrow{PQ} by a scale factor of 2 from center O .



- i. Is the figure \overrightarrow{PQ} a ray?

- ii. How, if at all, has the segment \overrightarrow{PQ} been transformed?
- iii. Will a ray always be mapped to a ray? Explain how you know.

Example 1

Will a dilation about center O and scale factor $r = 1$ map \overrightarrow{PQ} to $\overrightarrow{P'Q'}$? Explain.

Example 2

The line that contains \overrightarrow{PQ} does not contain point O . Will a dilation about center O and scale factor $r \neq 1$ map \overrightarrow{PQ} to $\overrightarrow{P'Q'}$?

Lesson Summary

- **DILATION THEOREM FOR RAYS:** A dilation maps a ray to a ray sending the endpoint to the endpoint.
- **DILATION THEOREM FOR LINES:** A dilation maps a line to a line. If the center O of the dilation lies on the line or if the scale factor r of the dilation is equal to 1, then the dilation maps the line to the same line. Otherwise, the dilation maps the line to a parallel line.
- **DILATION THEOREM FOR CIRCLES:** A dilation maps a circle to a circle, and maps the center to the center.

Problem Set

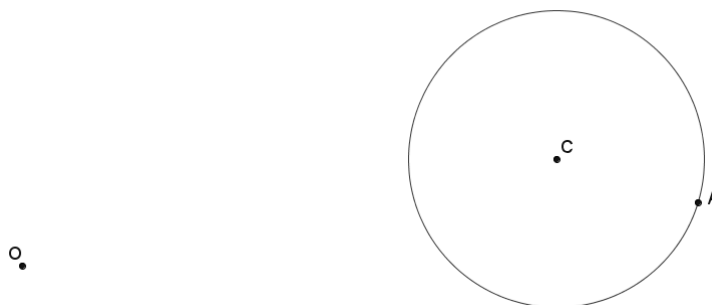
1. In Lesson 8, Example 2, you proved that a dilation with a scale factor $r > 1$ maps a ray PQ to a ray $P'Q'$. Prove the remaining case that a dilation with scale factor $0 < r < 1$ maps a ray PQ to a ray $P'Q'$.

Given the dilation $D_{O,r}$, with $0 < r < 1$ maps P to P' and Q to Q' , prove that $D_{O,r}$ maps \overrightarrow{PQ} to $\overrightarrow{P'Q'}$.

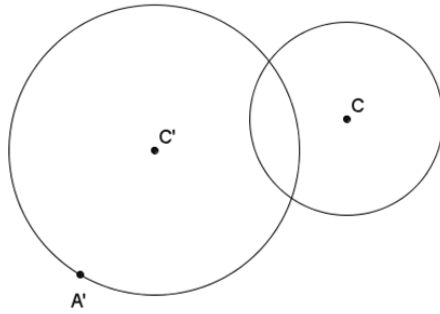
2. In the diagram below, $\overrightarrow{A'B'}$ is the image of \overrightarrow{AB} under a dilation from point O with an unknown scale factor, A maps to A' and B maps to B' . Use direct measurement to determine the scale factor r , and then find the center of dilation O .



3. Draw a line AB and dilate points A and B from center O where O is not on \overline{AB} . Use your diagram to explain why a line maps to a line under a dilation with scale factor r .
4. Let \overline{AB} be a line segment, and let m be a line that is the perpendicular bisector of \overline{AB} . If a dilation with scale factor r maps \overline{AB} to $\overline{A'B'}$ (sending A to A' and B to B') and also maps line m to line m' , show that m' is the perpendicular bisector of $\overline{A'B'}$.
5. Dilate circle C with radius CA from center O with a scale factor $r = \frac{1}{2}$.



6. In the picture below, the larger circle is a dilation of the smaller circle. Find the center of dilation O .



Exercises 5–6

5. Recall what you learned about parallel lines cut by a transversal, specifically about the angles that are formed.
6. A dilation from center O by scale factor r maps $\angle BAC$ to $\angle B'A'C'$. Show that $m\angle BAC = m\angle B'A'C'$.

Discussion

The dilation theorem for angles is as follows:

THEOREM: A dilation from center O and scale factor r maps an angle to an angle of equal measure.

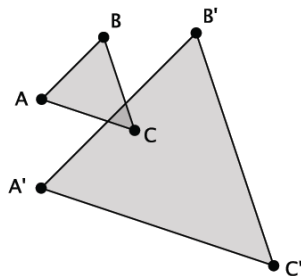
We have shown this when the angle and its image intersect at a single point, and that point of intersection is not the vertex of the angle.

Lesson Summary

- Dilations map angles to angles of equal measure.
- Dilations map polygonal figures to polygonal figures whose angles are equal in measure to the corresponding angles of the original figure and whose side lengths are equal to the corresponding side lengths multiplied by the scale factor.

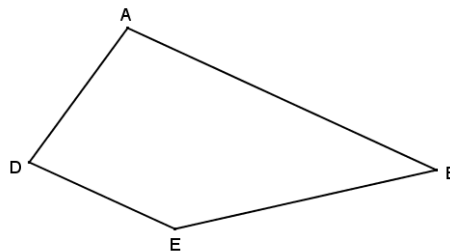
Problem Set

1. Shown below is $\triangle ABC$ and its image $\triangle A'B'C'$ after it has been dilated from center O by scale factor $r = \frac{5}{2}$. Prove that the dilation maps $\triangle ABC$ to $\triangle A'B'C'$ so that $m\angle A = m\angle A'$, $m\angle B = m\angle B'$, and $m\angle C = m\angle C'$.

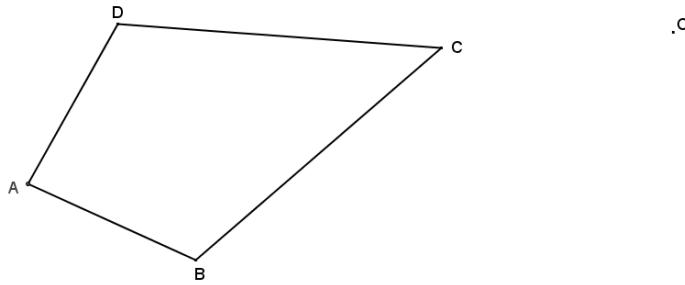


2. Explain the effect of a dilation with scale factor r on the length of the base and height of a triangle. How is the area of the dilated image related to the area of the pre-image?
3. Dilate trapezoid $ABDE$ from center O using a scale factor of $r = \frac{1}{2}$.

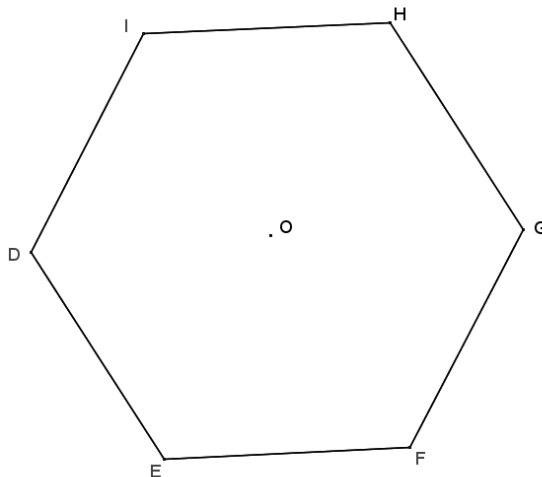
O.



4. Dilate kite $ABCD$ from center O using a scale factor $r = 1\frac{1}{2}$.



5. Dilate hexagon $DEFGHI$ from center O using a scale factor of $r = \frac{1}{4}$.



6. Examine the dilations that you constructed in Problems 2–5, and describe how each image compares to its pre-image under the given dilation. Pay particular attention to the sizes of corresponding angles and the lengths of corresponding sides.

Lesson 10: Dividing the King's Foot into 12 Equal Pieces

Classwork

Opening Exercise

Use a compass to mark off equally spaced points C , D , E , and F so that AB , BC , CD , DE , and EF are equal in length. Describe the steps you took.



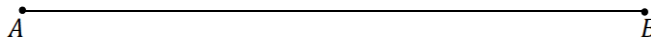
Exploratory Challenge 1

Divide segment AB into three segments of equal lengths.

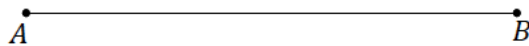


Exercise 1

Divide segment AB into five segments of equal lengths.

**Exploratory Challenge 2**

Divide segment AB into four segments of equal length.

**Exercise 2**

On a piece of poster paper, draw a segment AB with a measurement of 1 foot. Use the dilation method to divide \overline{AB} into twelve equal-length segments, or into 12 inches.

Lesson Summary

SIDE SPLITTER METHOD: If \overline{AB} is a line segment, construct a ray AA_1 and mark off n equally spaced points using a compass of fixed radius to get points $A = A_0, A_1, A_2, \dots, A_n$. Construct $\overline{A_nB}$ that is a side of $\triangle AA_nB$. Through each point A_1, A_2, \dots, A_{n-1} , construct line segments $\overline{A_iB_i}$ parallel to $\overline{A_nB}$ that connect two sides of $\triangle AA_nB$.

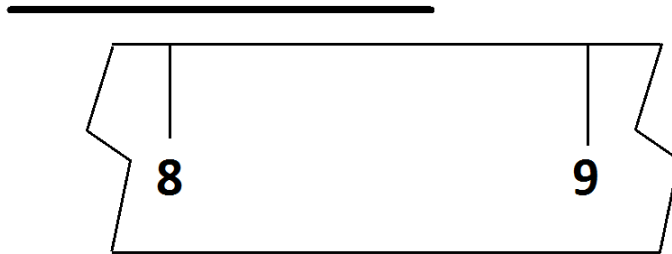
DILATION METHOD: Construct a ray XY parallel to \overline{AB} . On the parallel ray, use a compass to mark off n equally spaced points X_1, X_2, \dots, X_n so that $XX_n \neq AB$. Lines \overrightarrow{AX} and $\overrightarrow{BX_n}$ intersect at a point O . Construct the rays $\overrightarrow{OX_i}$ that meet \overline{AB} in points A_i .

Problem Set

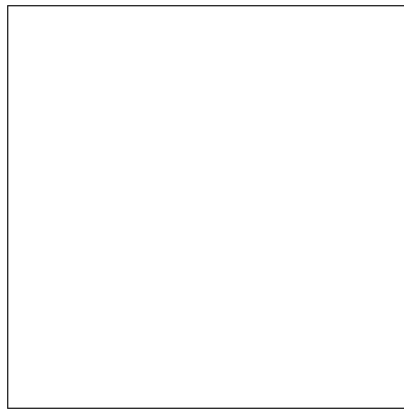
1. Pretend you are the king or queen and that the length of your foot is the official measurement for one foot. Draw a line segment on a piece of paper that is the length of your foot. (You may have to remove your shoe.) Use the method above to find the length of 1 inch in your kingdom.
2. Using a ruler, draw a segment that is 10 cm. This length is referred to as a decimeter. Use the side splitter method to divide your segment into ten equal-sized pieces. What should be the length of each of those pieces based on your construction? Check the length of the pieces using a ruler. Are the lengths of the pieces accurate?
3. Repeat Problem 2 using the dilation method. What should be the length of each of those pieces based on your construction? Check the length of the pieces using a ruler. Are the lengths of the pieces accurate?
4. A portion of a ruler that measured whole centimeters is shown below. Determine the location of $5\frac{2}{3}$ cm on the portion of the ruler shown.



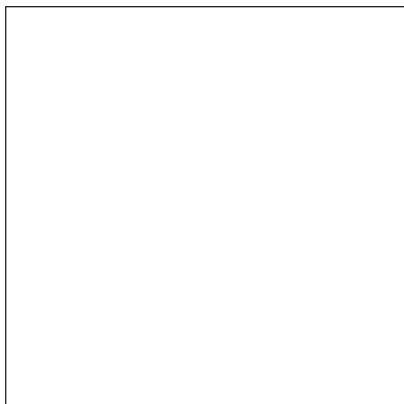
5. Merrick has a ruler that measures in inches only. He is measuring the length of a line segment that is between 8 and 9 in. Divide the one-inch section of Merrick’s ruler below into eighths to help him measure the length of the segment.



6. Use the dilation method to create an equally spaced 3×3 grid in the following square.



7. Use the side splitter method to create an equally spaced 3×3 grid in the following square.

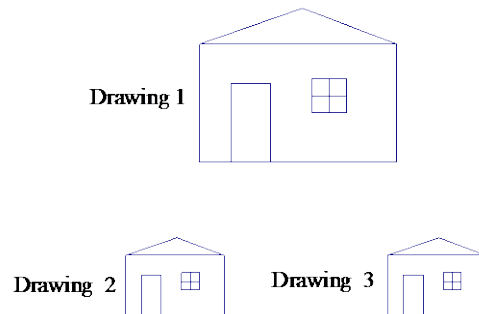


Lesson 11: Dilations from Different Centers

Classwork

Exploratory Challenge 1

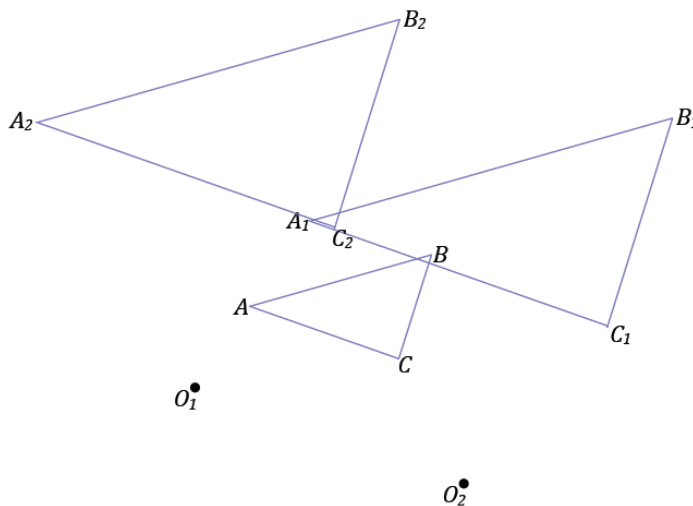
Drawing 2 and Drawing 3 are both scale drawings of Drawing 1.



- Determine the scale factor and center for each scale drawing. Take measurements as needed.
- Is there a way to map Drawing 2 onto Drawing 3 or map Drawing 3 onto Drawing 2?
- Generalize the parameters of this example and its results.

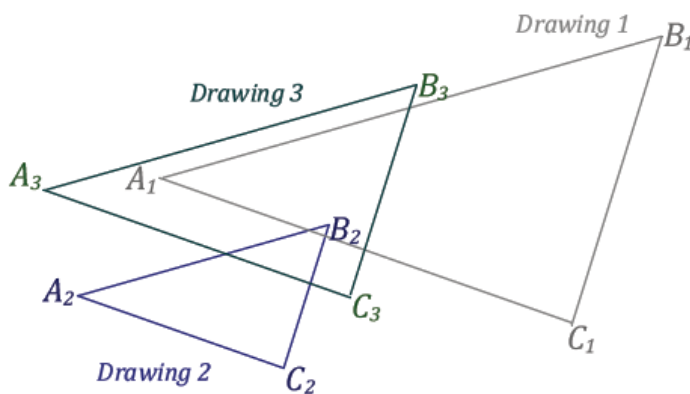
Exercise 1

Triangle ABC has been dilated with scale factor $\frac{1}{2}$ from centers O_1 and O_2 . What can you say about line segments A_1A_2 , B_1B_2 , C_1C_2 ?



Exploratory Challenge 2

If Drawing 2 is a scale drawing of Drawing 1 with scale factor r_1 , and Drawing 3 is a scale drawing of Drawing 2 with scale factor r_2 , what is the relationship between Drawing 3 and Drawing 1?



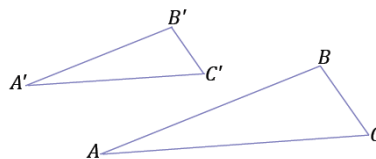
- a. Determine the scale factor and center for each scale drawing. Take measurements as needed.

- b. What is the scale factor going from Drawing 1 to Drawing 3? Take measurements as needed.
- c. Compare the centers of dilations of Drawing 1 (to Drawing 2) and of Drawing 2 (to Drawing 3). What do you notice about these centers relative to the center of the composition of dilations O_3 ?
- d. Generalize the parameters of this example and its results.

Exercise 2

Triangle ABC has been dilated with scale factor $\frac{2}{3}$ from center O_1 to get triangle $A'B'C'$, and then triangle $A'B'C'$ is dilated from center O_2 with scale factor $\frac{1}{2}$ to get triangle $A''B''C''$. Describe the dilation that maps triangle ABC to triangle $A''B''C''$. Find the center and scale factor for that dilation.

O_1 •



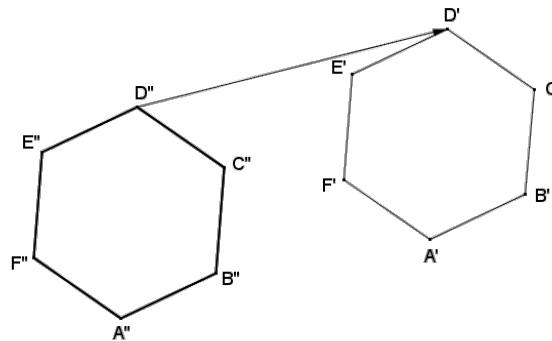
O_2 •

Lesson Summary

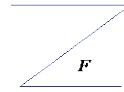
In a series of dilations, the scale factor that maps the original figure onto the final image is the product of all the scale factors in the series of dilations.

Problem Set

- In Lesson 7, the dilation theorem for line segments said that if two different length line segments in the plane were parallel to each other, then a dilation exists mapping one segment onto the other. Explain why the line segments must be different lengths for a dilation to exist.
- Regular hexagon $A'B'C'D'E'F'$ is the image of regular hexagon $ABCDEF$ under a dilation from center O_1 , and regular hexagon $A''B''C''D''E''F''$ is the image of regular hexagon $ABCDEF$ under a dilation from center O_2 . Points $A', B', C', D', E',$ and F' are also the images of points $A'', B'', C'', D'', E'',$ and F'' , respectively, under a translation along vector $\overrightarrow{D''D'}$. Find a possible regular hexagon $ABCDEF$.

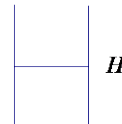


3. A dilation with center O_1 and scale factor $\frac{1}{2}$ maps figure F to figure F' . A dilation with center O_2 and scale factor $\frac{3}{2}$ maps figure F' to figure F'' . Draw figures F' and F'' , and then find the center O and scale factor r of the dilation that takes F to F'' .



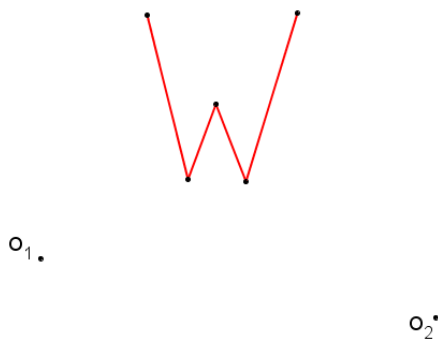
$O_1 \bullet \quad \bullet O_2$

4. If a figure T is dilated from center O_1 with a scale factor $r_1 = \frac{3}{4}$ to yield image T' , and figure T' is then dilated from center O_2 with a scale factor $r_2 = \frac{4}{3}$ to yield figure T'' . Explain why $T \cong T''$.
5. A dilation with center O_1 and scale factor $\frac{1}{2}$ maps figure H to figure H' . A dilation with center O_2 and scale factor 2 maps figure H' to figure H'' . Draw figures H' and H'' . Find a vector for a translation that maps H to H'' .



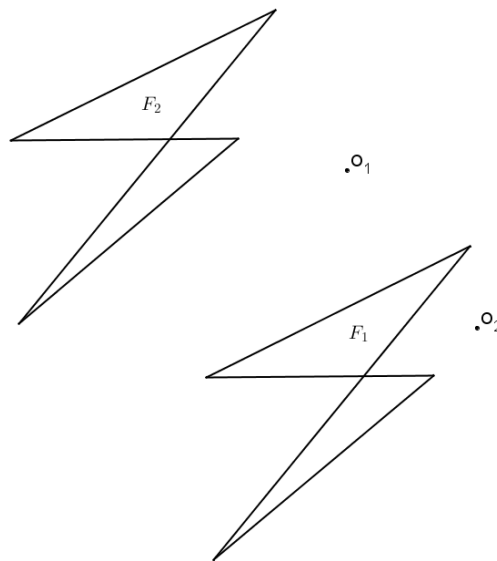
$O_1 \bullet \quad \bullet O_2$

6. Figure W is dilated from O_1 with a scale factor $r_1 = 2$ to yield W' . Figure W' is then dilated from center O_2 with a scale factor $r_2 = \frac{1}{4}$ to yield W'' .



- Construct the composition of dilations of figure W described above.
- If you were to dilate figure W'' , what scale factor would be required to yield an image that is congruent to figure W ?
- Locate the center of dilation that maps W'' to W using the scale factor that you identified in part (b).

7. Figures F_1 and F_2 in the diagram below are dilations of F from centers O_1 and O_2 , respectively.



- a. Find F .
 - b. If $F_1 \cong F_2$, what must be true of the scale factors r_1 and r_2 of each dilation?
 - c. If $F_1 \cong F_2$, what must be true of the scale factors r_1 and r_2 of each dilation?
8. Use a coordinate plane to complete each part below using $U(2,3)$, $V(6,6)$, and $W(6,-1)$.
- a. Dilate $\triangle UVW$ from the origin with a scale factor $r_1 = 2$. List the coordinate of image points U' , V' , and W' .
 - b. Dilate $\triangle UVW$ from $(0,6)$ with a scale factor of $r_2 = \frac{3}{4}$. List the coordinates of image points U'' , V'' , and W'' .
 - c. Find the scale factor, r_3 , from $\triangle U'V'W'$ to $\triangle U''V''W''$.
 - d. Find the coordinates of the center of dilation that maps $\triangle U'V'W'$ to $\triangle U''V''W''$.

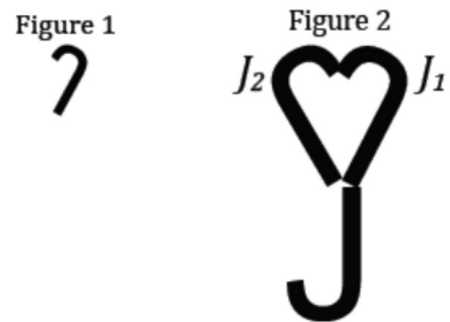
Lesson 12: What Are Similarity Transformations, and Why Do We Need Them?

Classwork

Opening Exercise

Observe Figures 1 and 2 and the images of the intermediate figures between Figures 1 and 2. Figures 1 and 2 are called *similar*.

What observations can you make about Figures 1 and 2?



Definition:

A _____ (or _____) is a composition of a finite number of dilations or basic rigid motions. The *scale factor* of a similarity transformation is the product of the scale factors of the dilations in the composition; if there are no dilations in the composition, the scale factor is defined to be 1.

Definition:

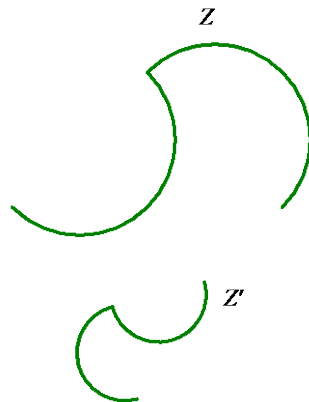
Two figures in a plane are _____ if there exists a similarity transformation taking one figure onto the other figure.

Definition	Characteristics
Examples	Non-Examples

similar

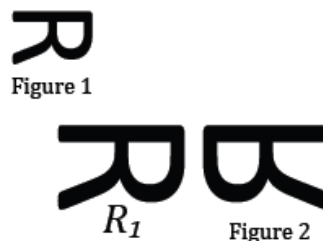
Example 1

Figure Z' is similar to Figure Z . Describe a transformation that will map Figure Z onto Figure Z' ?

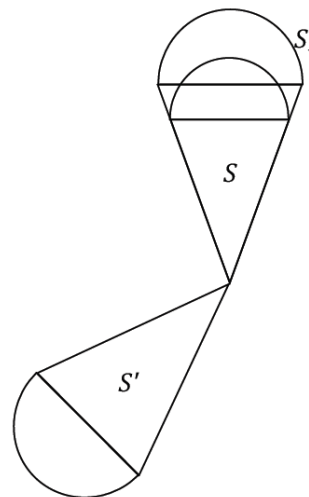


Exercises 1–3

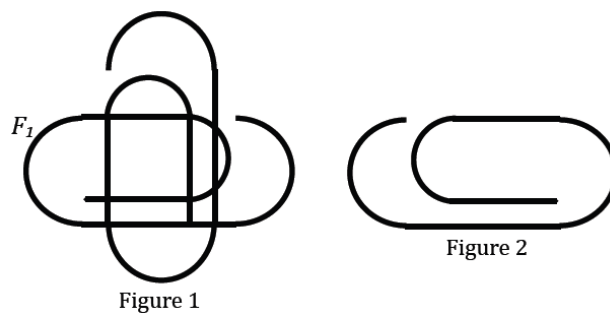
- Figure 1 is similar to Figure 2. Which transformations compose the similarity transformation that maps Figure 1 onto Figure 2?



- Figure S is similar to Figure S' . Which transformations compose the similarity transformation that maps S onto S' ?

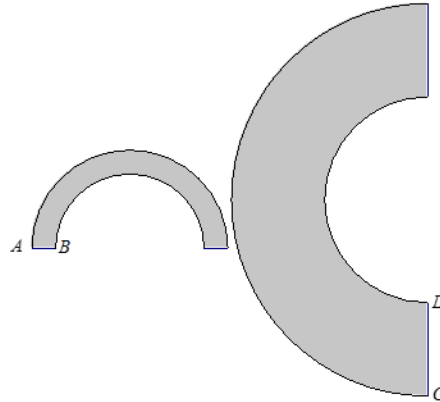


- Figure 1 is similar to Figure 2. Which transformations compose the similarity transformation that maps Figure 1 onto Figure 2?



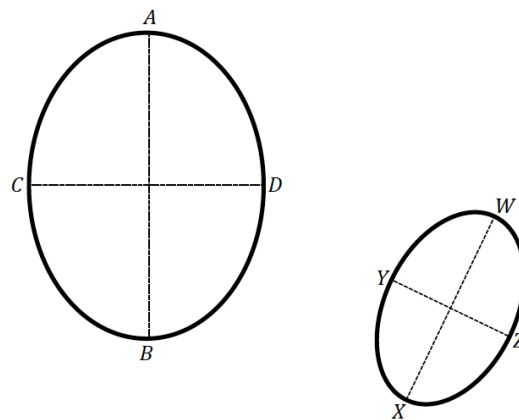
Example 2

Show that no sequence of basic rigid motions and dilations takes the small figure to the large figure. Take measurements as needed.



Exercises 4–5

- Is there a sequence of dilations and basic rigid motions that takes the large figure to the small figure? Take measurements as needed.



- What purpose do transformations serve? Compare and contrast the application of rigid motions to the application of similarity transformations.

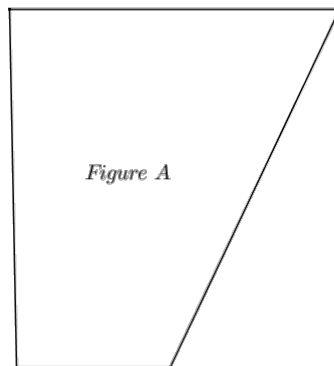
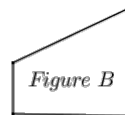
Lesson Summary

Two figures are similar if there exists a similarity transformation that maps one figure onto the other.

A similarity transformation is a composition of a finite number of dilations or rigid motions.

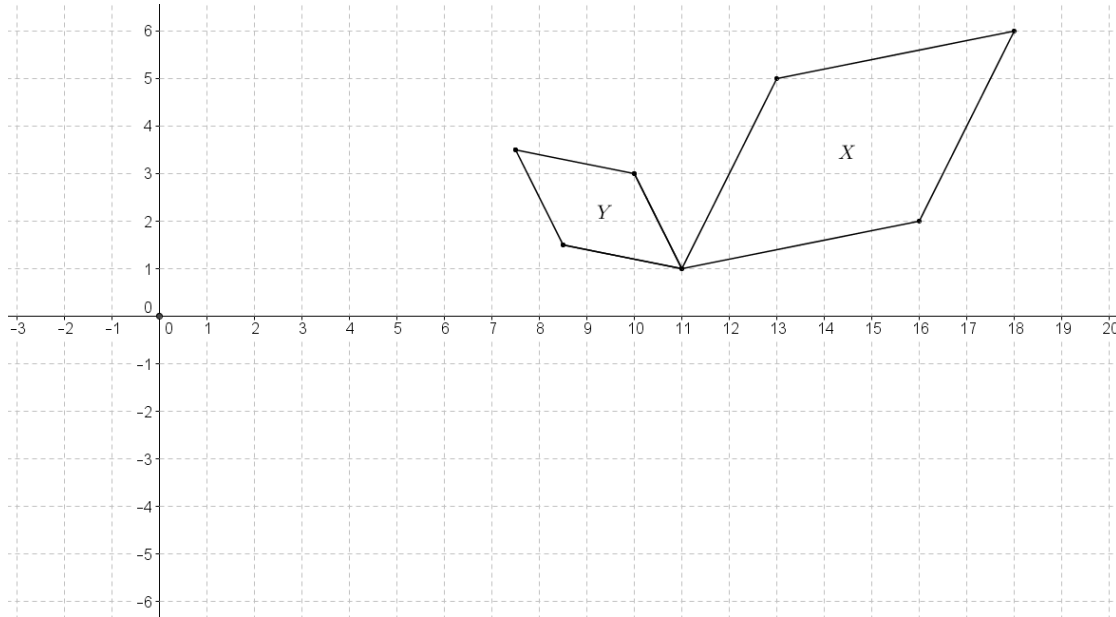
Problem Set

1. What is the relationship between scale drawings, dilations, and similar figures?
 - a. How are scale drawings and dilations alike?
 - b. How can scale drawings and dilations differ?
 - c. What is the relationship of similar figures to scale drawings and dilations?
2. Given the diagram below, identify a similarity transformation, if one exists, mapping Figure A onto Figure B. If one does not exist, explain why.

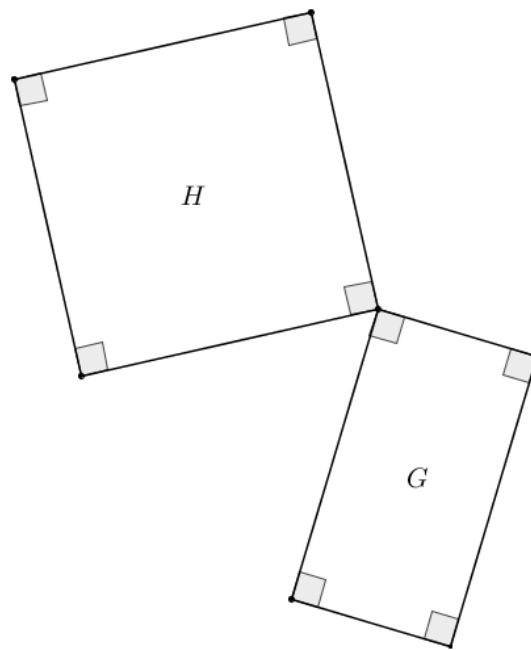


3. Teddy correctly identified a similarity transformation with at least one dilation that maps Figure I onto Figure II. Megan correctly identified a congruence transformation that maps Figure I onto Figure II. What must be true about Teddy's similarity transformation?

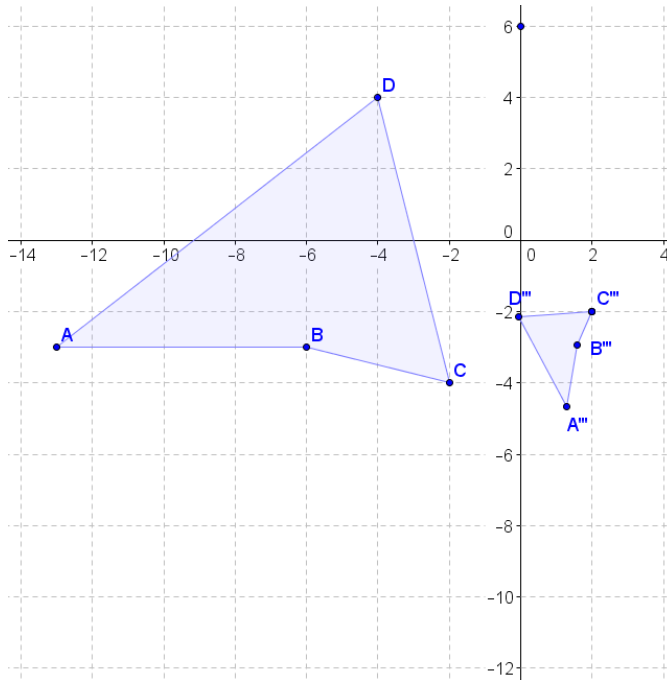
4. Given the coordinate plane shown, identify a similarity transformation, if one exists, mapping X onto Y . If one does not exist, explain why.



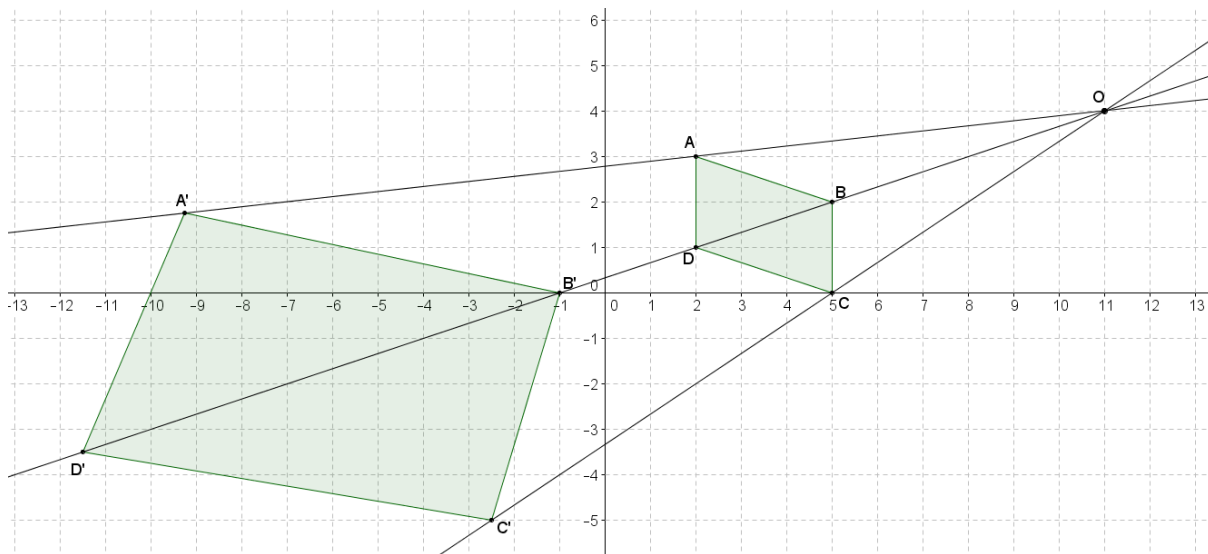
5. Given the diagram below, identify a similarity transformation, if one exists, that maps G onto H . If one does not exist, explain why. Provide any necessary measurements to justify your answer.



6. Given the coordinate plane shown, identify a similarity transformation, if one exists, that maps $ABCD$ onto $A'''B'''C'''D'''$. If one does not exist, explain why.



7. The diagram below shows a dilation of the plane...or does it? Explain your answer.

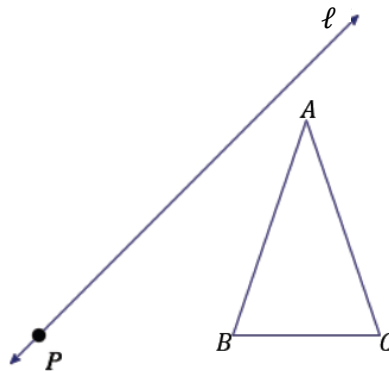


Lesson 13: Properties of Similarity Transformations

Classwork

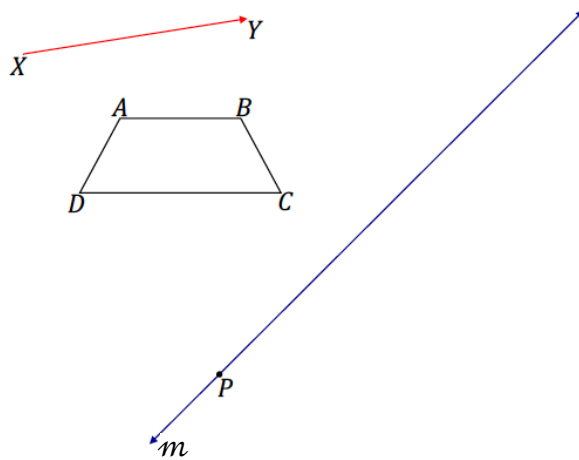
Example 1

Similarity transformation G consists of a rotation about the point P by 90° , followed by a dilation centered at P with scale factor $r = 2$, and then a reflection across line ℓ . Find the image of the triangle.



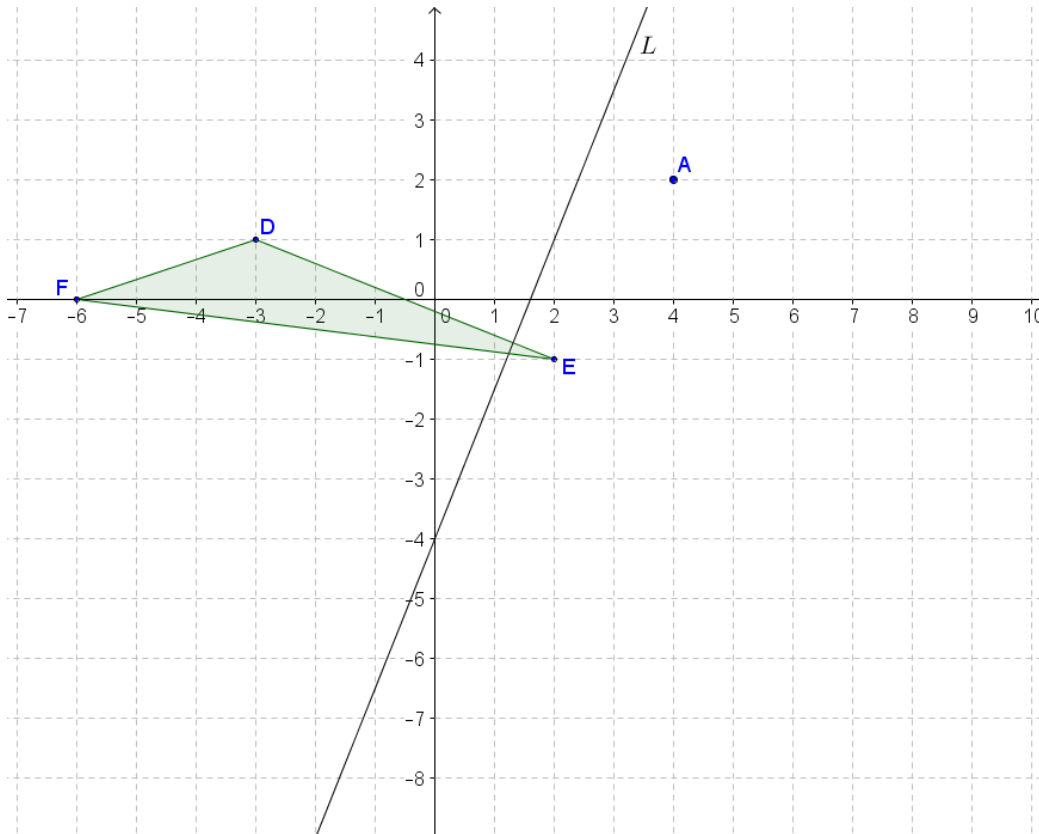
Example 2

A similarity transformation G applied to trapezoid $ABCD$ consists of a translation by vector \overrightarrow{XY} , followed by a reflection across line m , and then followed by a dilation centered at P with scale factor $r = 2$. Recall that we can describe the same sequence using the following notation: $D_{P,2}(r_m(T_{XY}(ABCD)))$. Find the image of $ABCD$.



Exercise 1

A similarity transformation for triangle DEF is described by $r_n \left(D_{A, \frac{1}{2}} \left(R_{A, 90^\circ}(DEF) \right) \right)$. Locate and label the image of triangle DEF under the similarity.



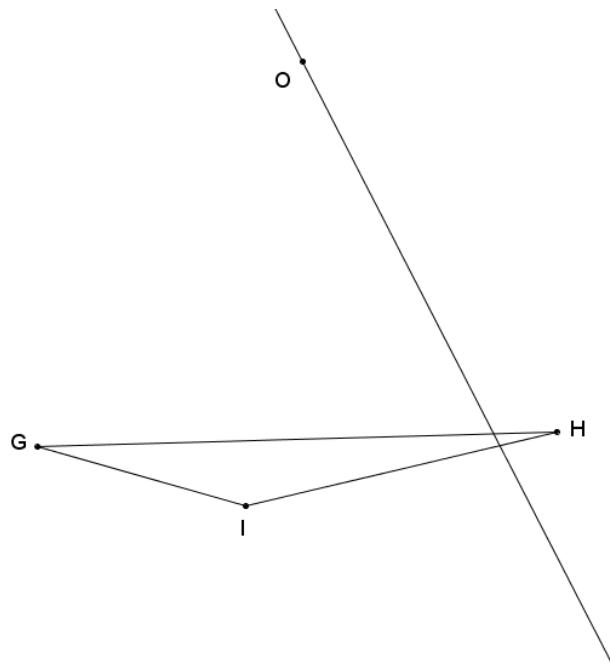
Lesson Summary

Properties of similarity transformations:

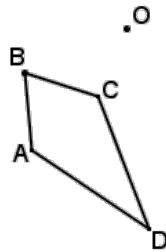
1. Distinct points are mapped to distinct points.
2. Each point P' in the plane has a pre-image.
3. There is a scale factor r for G , so that for any pair of points P and Q with images $P' = G(P)$ and $Q' = G(Q)$, then $P'Q' = rPQ$.
4. A similarity transformation sends lines to lines, rays to rays, line segments to line segments, and parallel lines to parallel lines.
5. A similarity transformation sends angles to angles of equal measure.
6. A similarity transformation maps a circle of radius R to a circle of radius rR , where r is the scaling factor of the similarity transformation.

Problem Set

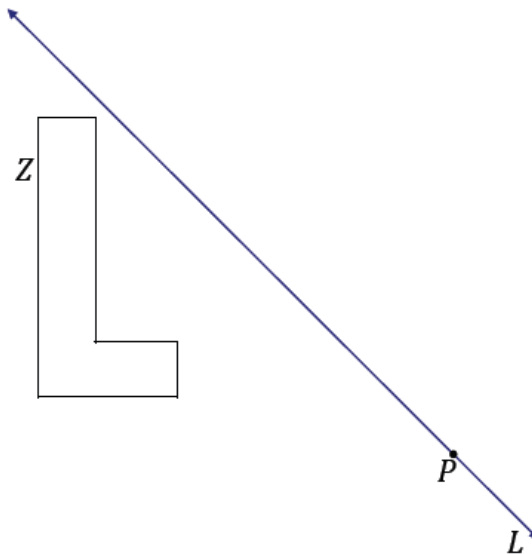
1. A similarity transformation consists of a reflection over line ℓ , followed by a dilation from O with a scale factor of $r = \frac{3}{4}$. Use construction tools to find $\triangle G''H''I''$.



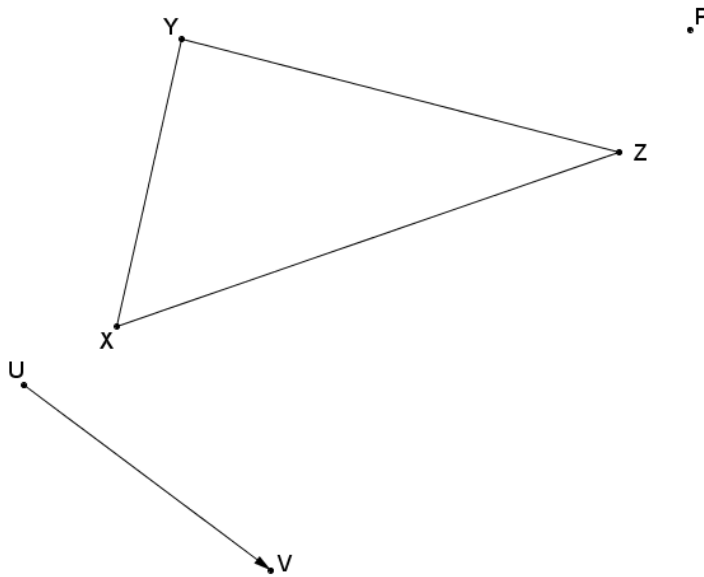
2. A similarity transformation consists of a dilation from point O with a scale factor of $r = 2\frac{1}{2}$, followed by a rotation about O of -90° . Use construction tools to find kite $A''B''C''D''$.



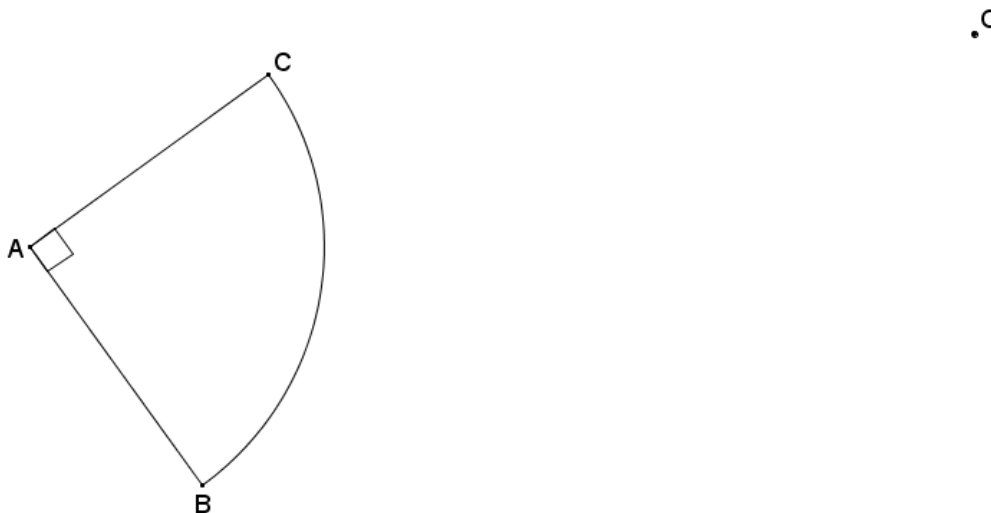
3. For the Figure Z , find the image of $r_{\ell}(R_{P,90^\circ}(D_{P,\frac{1}{2}}(Z)))$.



4. A similarity transformation consists of a translation along vector \overrightarrow{UV} , followed by a rotation of 60° about P , then a dilation from P with scale factor $r = \frac{1}{3}$. Use construction tools to find $\triangle X''Y''Z''$.



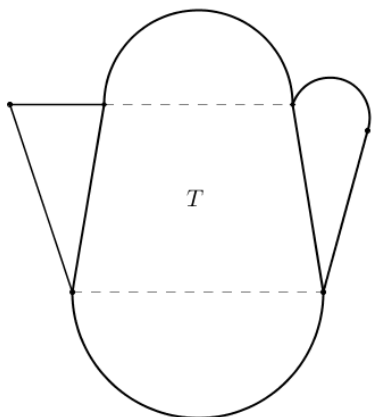
5. Given the quarter-circular figure determined by points A , B , and C , a similarity transformation consists of a -65° rotation about point B , followed by a dilation from point O with a scale factor of $r = \frac{1}{2}$. Find the image of the figure determined by points A'' , B'' , and C'' .



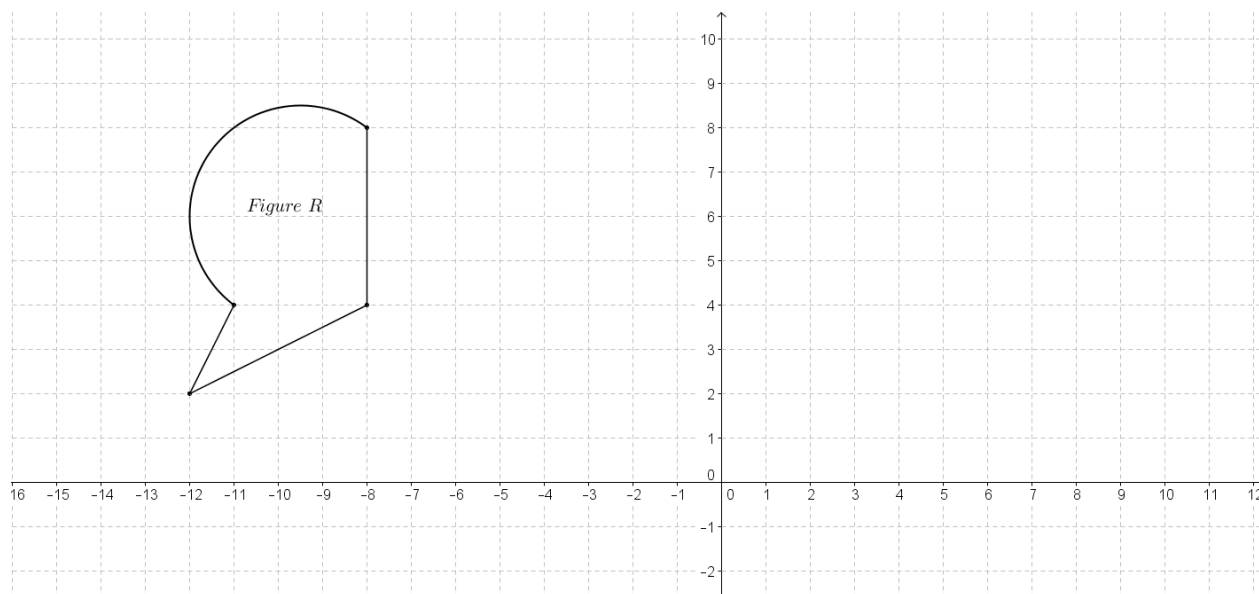
Describe a different similarity transformation that would map quarter-circle ABC to quarter-circle $A''B''C''$.

6. A similarity transformation consists of a dilation from center O with a scale factor of $\frac{1}{2}$, followed by a rotation of 60° about point O . Complete the similarity transformation on Figure T to complete the drawing of Figure T'' .

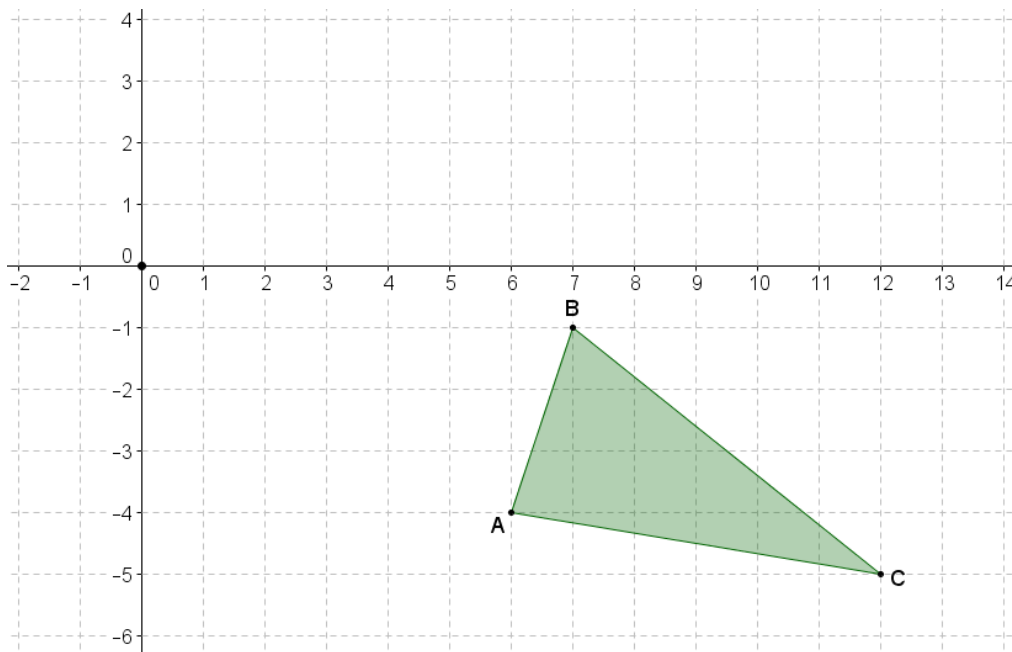
.o



7. Given Figure R on the coordinate plane shown below, a similarity transformation consists of a dilation from $(0,6)$ with a scale factor of $\frac{1}{4}$, followed by a reflection over line $x = -1$, then by a vertical translation of 5 units down. Find the image of Figure R .



8. Given $\triangle ABC$, with vertices $A(2, -7)$, $B(-2, -1)$, and $C(3, -4)$, locate and label the image of the triangle under the similarity transformation $D_{B', \frac{1}{2}}(R_{A, 120^\circ}(r_{x=2}(ABC)))$.
9. In Problem 8, describe the relationship of A''' to $\overline{AB'}$, and explain your reasoning.
10. Given $O(-8, 3)$ and quadrilateral $BCDE$, with $B(-5, 1)$, $C(-6, -1)$, $D(-4, -1)$, and $E(-4, 2)$, what are the coordinates of the vertices of the image of $BCDE$ under the similarity transformation $r_{x-axis}(D_{O, 3}(BCDE))$?
11. Given triangle ABC as shown on the diagram of the coordinate plane:
 - a. Perform a translation so that vertex A maps to the origin.
 - b. Next, dilate the image $A'B'C'$ from the origin using a scale factor of $\frac{1}{3}$.
 - c. Finally, translate the image $A''B''C''$ so that the vertex A'' maps to the original point A .
 - d. Using transformations, describe how the resulting image $A'''B'''C'''$ relates to the original figure ABC .



- 12.
- a. In the coordinate plane, name the single transformation that is the result of the composition of the two dilations shown below:
 $D_{(0,0),2}$ followed by $D_{(0,4),\frac{1}{2}}$
(Hint: Try it!)
- b. In the coordinate plane, name the single transformation that is the result of the composition of the two dilations shown below:
 $D_{(0,0),2}$ followed by $D_{(4,4),\frac{1}{2}}$
(Hint: Try it!)
- c. Using the results from parts (a) and (b), describe what happens to the origin under both similarity transformations.

Lesson 14: Similarity

Classwork

Example 1

We said that for a figure A in the plane, it must be true that $A \sim A$. Describe why this must be true.

Example 2

We said that for figures A and B in the plane so that $A \sim B$, then it must be true that $B \sim A$. Describe why this must be true.

Example 3

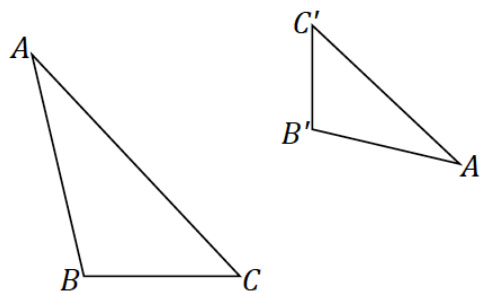
Based on the definition of *similar*, how would you show that any two circles are similar?

Example 4

Suppose $\triangle ABC \leftrightarrow \triangle DEF$ and under this correspondence, corresponding angles are equal and corresponding sides are proportional. Does this guarantee that $\triangle ABC$ and $\triangle DEF$ are similar?

Example 5

- a. In the diagram below, $\triangle ABC \sim \triangle A'B'C'$. Describe a similarity transformation that maps $\triangle ABC$ to $\triangle A'B'C'$.



- b. Joel says the sequence must require a dilation and three rigid motions, but Sharon is sure there is a similarity composed of just a dilation and just two rigid motions. Who is right?

Lesson Summary

Similarity is reflexive because a figure is similar to itself.

Similarity is symmetric because once a similarity transformation is determined to take a figure to another, there are inverse transformations that will take the figure back to the original.

Problem Set

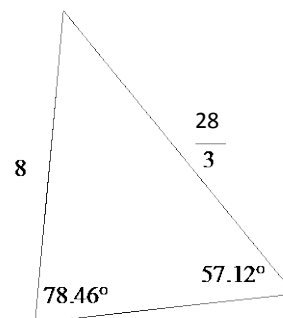
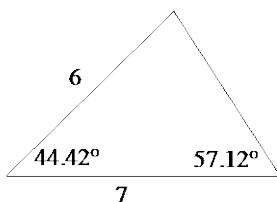
1. If you are given any two congruent triangles, describe a sequence of basic rigid motions that will take one to the other.
2. If you are given two similar triangles that are not congruent triangles, describe a sequence of dilations and basic rigid motions that will take one to the other.
3. Given two line segments, \overline{AB} and \overline{CD} , of different lengths, answer the following questions.
 - a. It is always possible to find a similarity transformation that maps \overline{AB} to \overline{CD} sending A to C and B to D . Describe one such similarity transformation.
 - b. If you are given that \overline{AB} and \overline{CD} are not parallel, are not congruent, do not share any points, and do not lie in the same line, what is the least number of transformations needed in a sequence to map \overline{AB} to \overline{CD} ? Which transformations make this work?
 - c. If you performed a similarity transformation that instead takes A to D and B to C , either describe what mistake was made in the similarity transformation, or describe what additional transformation is needed to fix the error so that A maps to C and B maps to D .
4. We claim that similarity is transitive, i.e., that if A , B , and C are figures in the plane such that $A \sim B$ and $B \sim C$, then $A \sim C$. Describe why this must be true.
5. Given two line segments, \overline{AB} and \overline{CD} , of different lengths, we have seen that it is always possible to find a similarity transformation that maps \overline{AB} to \overline{CD} , sending A to C and B to D with one rotation and one dilation. Can you do this with one reflection and one dilation?
6. Given two triangles, $\triangle ABC \sim \triangle DEF$, is it always possible to rotate $\triangle ABC$ so that the sides of $\triangle ABC$ are parallel to the corresponding sides in $\triangle DEF$; i.e., $\overline{AB} \parallel \overline{DE}$, etc.?

5. Do you think that what you observed will be true when you construct a pair of triangles with two pairs of equal angles? Explain.

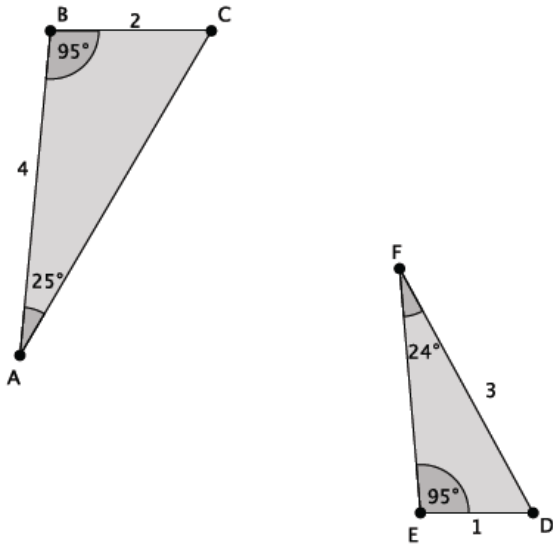
Exercises 6–10

6. Draw another two triangles of different sizes with two pairs of equal angles. Then measure the lengths of the corresponding sides to verify that the ratio of their lengths is proportional. Use a ruler, compass, or protractor, as necessary.

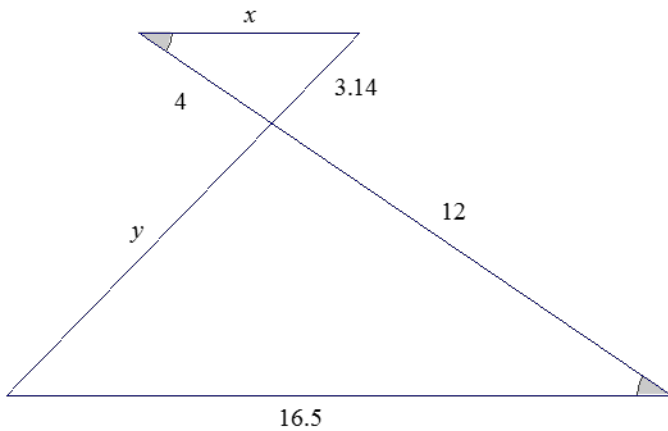
7. Are the triangles shown below similar? Explain. If the triangles are similar, identify any missing angle and side length measures.



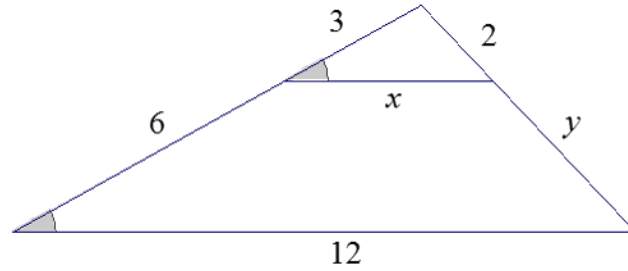
8. Are the triangles shown below similar? Explain. If the triangles are similar identify any missing angle and side length measures.



9. The triangles shown below are similar. Use what you know about similar triangles to find the missing side lengths x and y .

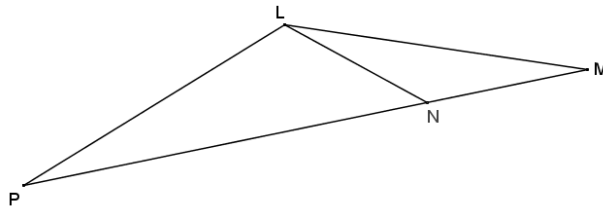


10. The triangles shown below are similar. Write an explanation to a student, Claudia, of how to find the lengths of x and y .



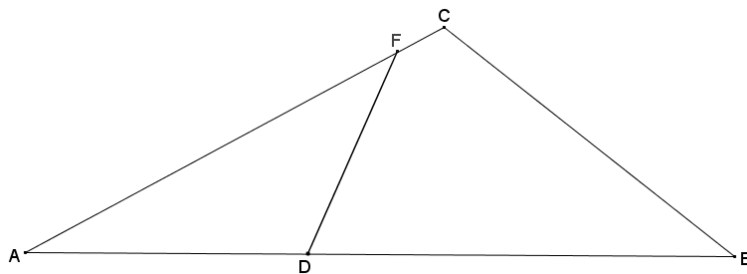
Problem Set

1. In the figure to the right, $\triangle LMN \sim \triangle MPL$.

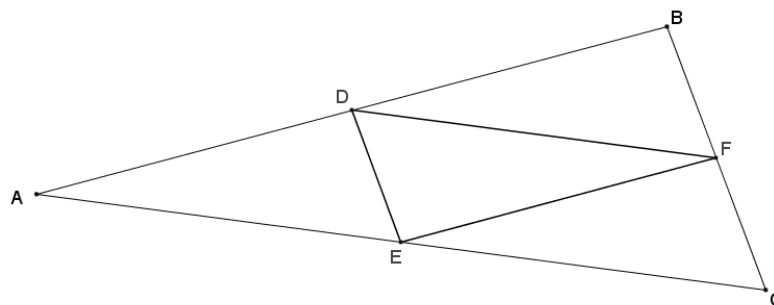


- a. Classify $\triangle LMP$ based on what you know about similar triangles and justify your reasoning.
 - b. If $m\angle P = 20^\circ$, find the remaining angles in the diagram.
2. In the diagram below, $\triangle ABC \sim \triangle AFD$. Determine whether the following statements must be true from the given information, and explain why.

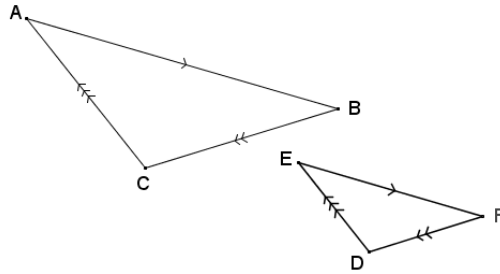
- a. $\triangle CAB \sim \triangle DAF$
- b. $\triangle ADF \sim \triangle CAB$
- c. $\triangle BCA \sim \triangle ADF$
- d. $\triangle ADF \sim \triangle ACB$



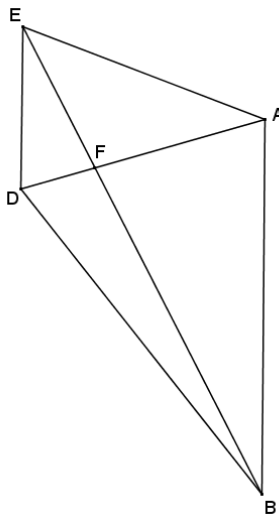
3. In the diagram below, D is the midpoint of \overline{AB} , F is the midpoint of \overline{BC} , and E is the midpoint of \overline{AC} . Prove that $\triangle ABC \sim \triangle FED$.



4. Use the diagram below to answer each part.



- a. If $\overline{AC} \parallel \overline{ED}$, $\overline{AB} \parallel \overline{EF}$, and $\overline{CB} \parallel \overline{DF}$, prove that the triangles are similar.
 - b. The triangles are not congruent. Find the dilation that takes one to the other.
5. Given trapezoid $ABDE$, and $\overline{AB} \parallel \overline{ED}$, prove that $\triangle AFB \sim \triangle DEF$.

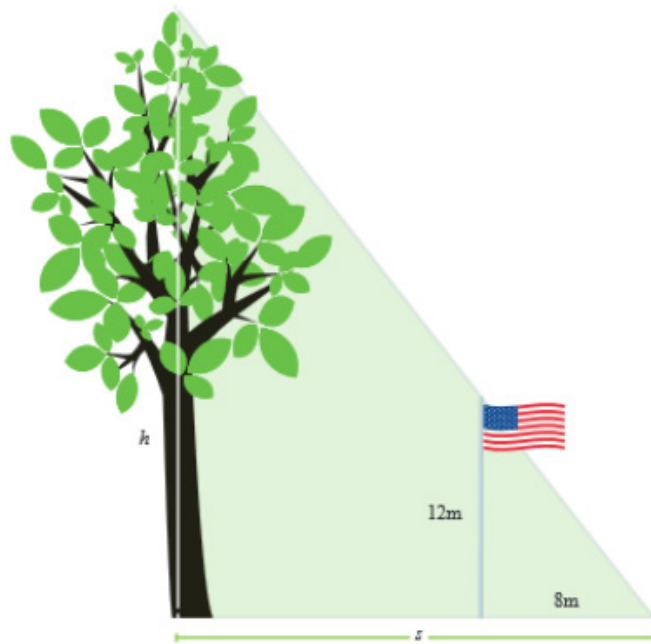


Lesson 16: Between-Figure and Within-Figure Ratios

Classwork

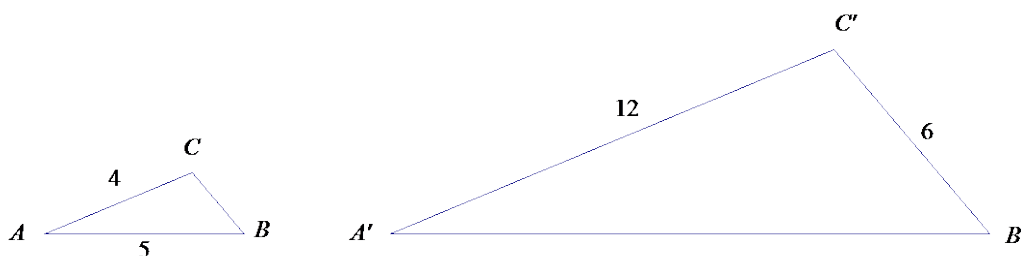
Opening Exercise

At a certain time of day, a 12 meter flagpole casts an 8 m shadow. Write an equation that would allow you to find the height, h , of the tree that uses the length, s , of the tree's shadow.

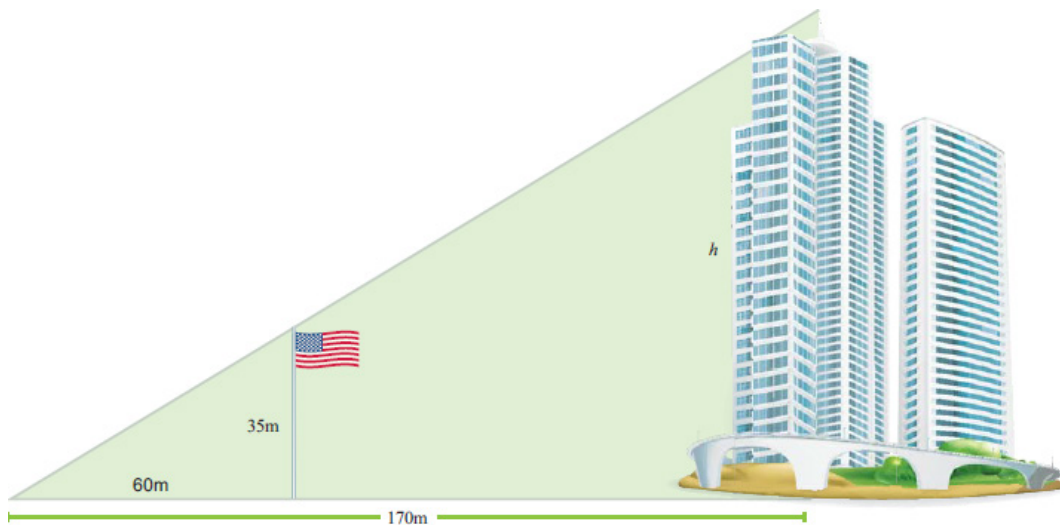


Example 1

Given $\triangle ABC \sim \triangle A'B'C'$, find the missing side lengths.



Example 2

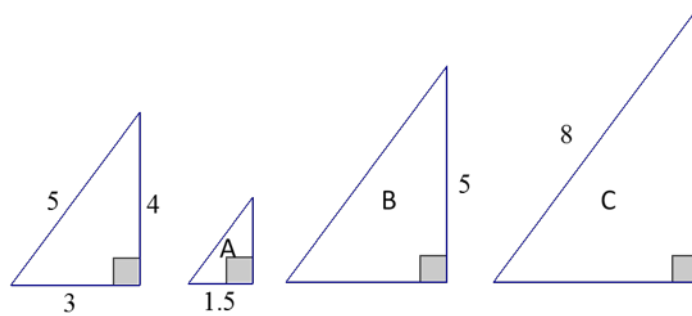


In the diagram above a large flagpole stands outside of an office building. Marquis realizes that when he looks up from the ground, 60 m away from the flagpole, that the top of the flagpole and the top of the building line up. If the flagpole is 35 m tall, and Marquis is 170 m from the building, how tall is the building?

- Are the triangles in the diagram similar? Explain.
- Determine the height of the building using what you know about scale factors.
- Determine the height of the building using ratios between similar figures.
- Determine the height of the building using ratios within similar figures.

Example 3

The following right triangles are similar. We will determine the unknown side lengths by using ratios within the first triangle. For each of the triangles below we define the base as the horizontal length of the triangle and the height as the vertical length.



- Write and find the value of the ratio that compares the height to the hypotenuse of the leftmost triangle.
- Write and find the value of the ratio that compares the base to the hypotenuse of the leftmost triangle.
- Write and find the value of the ratio that compares the height to the base of the leftmost triangle.
- Use the triangle with lengths 3–4–5 and triangle *A* to answer the following questions.
 - Which ratio can be used to determine the height of triangle *A*?
 - Which ratio can be used to determine the hypotenuse of triangle *A*?
 - Find the unknown lengths of triangle *A*.

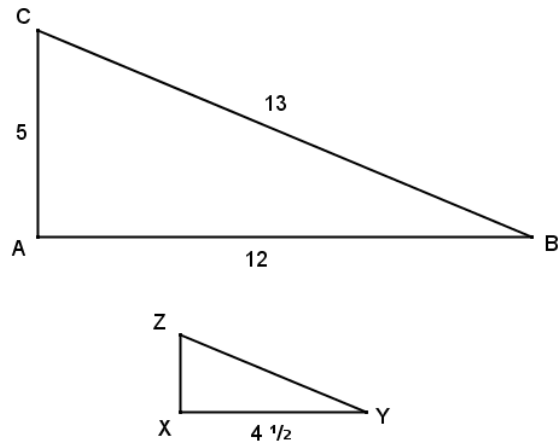
- e. Use the triangle with lengths 3–4–5 and triangle B to answer the following questions.
- Which ratio can be used to determine the base of triangle B ?
 - Which ratio can be used to determine the hypotenuse of triangle B ?
 - Find the unknown lengths of Triangle B .
- f. Use the triangle with lengths 3–4–5 and triangle C to answer the following questions.
- Which ratio can be used to determine the height of triangle C ?
 - Which ratio can be used to determine the base of triangle C ?
 - Find the unknown lengths of triangle C .
- g. Explain the relationship of the ratio of the corresponding sides within a figure to the ratio of corresponding sides within a similar figure.
- h. How does the relationship you noted in part (g) allow you to determine the length of an unknown side of a triangle?

Problem Set

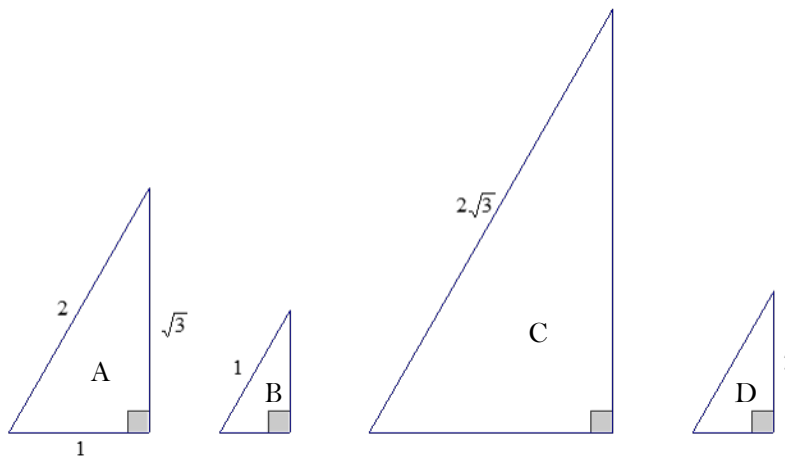
1. $\triangle DEF \sim \triangle ABC$. All side length measurements are in centimeters. Use between ratios and/or within ratios to determine the unknown side lengths.

2. Given $\triangle ABC \sim \triangle XYZ$, answer the following questions.

- Write and find the value of the ratio that compares the height \overline{AC} to the hypotenuse of $\triangle ABC$.
- Write and find the value of the ratio that compares the base \overline{AB} to the hypotenuse of $\triangle ABC$.
- Write and find the value of the ratio that compares the height \overline{AC} to the base \overline{AB} of $\triangle ABC$.
- Use within-figure ratios to find the corresponding height of $\triangle XYZ$.
- Use within-figure ratios to find the hypotenuse of $\triangle XYZ$.

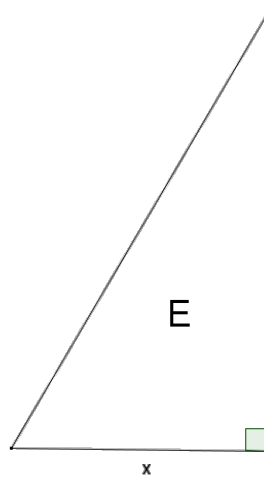


3. Right triangles A , B , C , and D are similar. Determine the unknown side lengths of each triangle by using ratios of side lengths within triangle A .

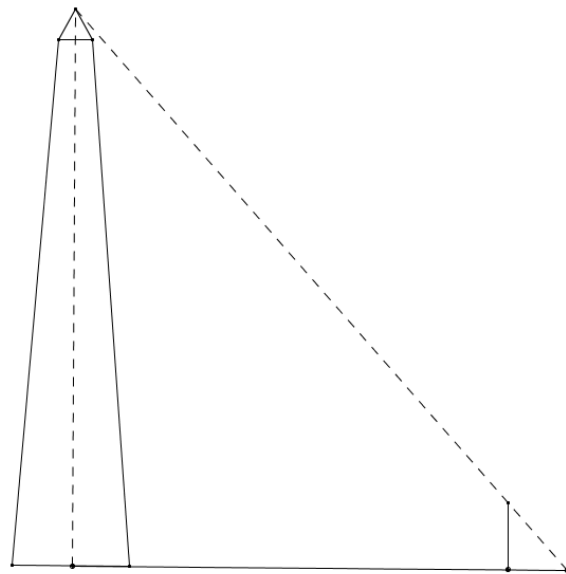


- Write and find the value of the ratio that compares the height to the hypotenuse of triangle A .
- Write and find the value of the ratio that compares the base to the hypotenuse of triangle A .
- Write and find the value of the ratio that compares the height to the base of triangle A .
- Which ratio can be used to determine the height of triangle B ? Find the height of triangle B .
- Which ratio can be used to determine the base of triangle B ? Find the base of triangle B .
- Find the unknown lengths of triangle C .
- Find the unknown lengths of triangle D .

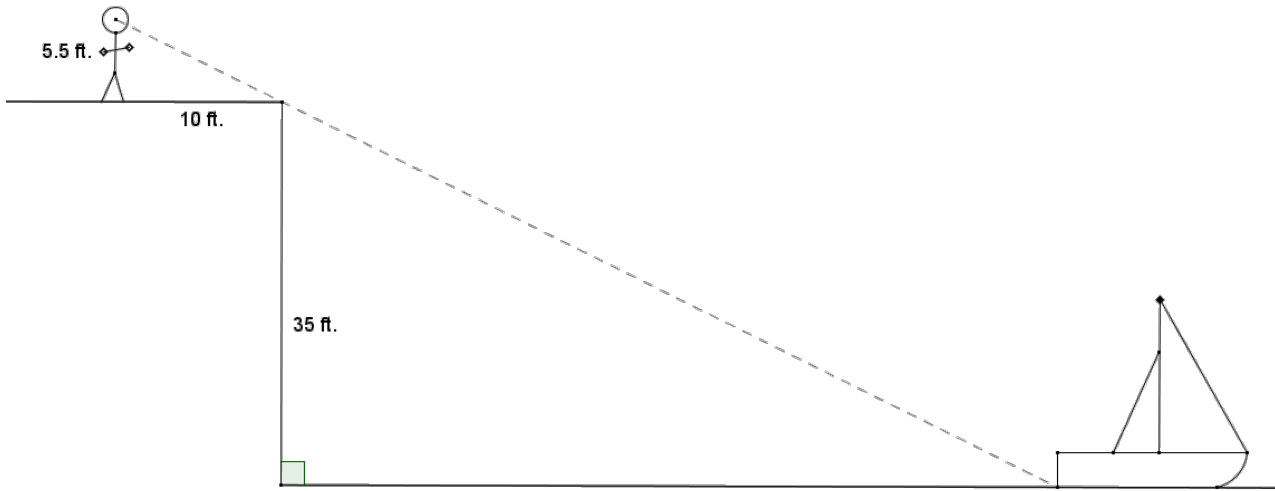
- h. Triangle E is also similar to triangles A , B , C , and D . Find the lengths of the missing sides in terms of x .



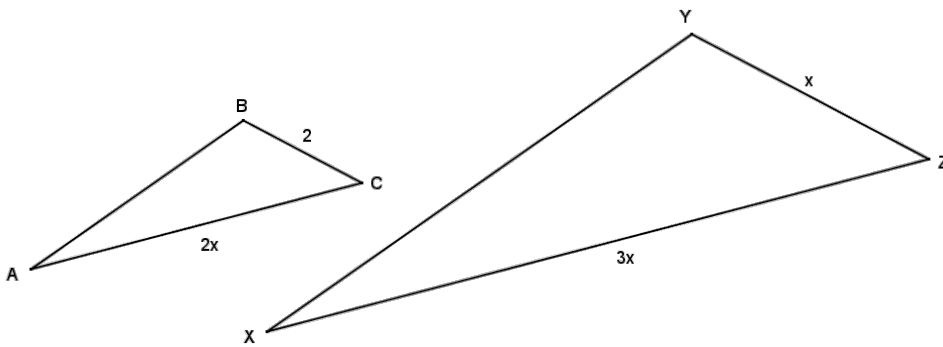
4. Brian is photographing the Washington Monument and wonders how tall the monument is. Brian places his 5 ft. camera tripod approximately 100 yd. from the base of the monument. Lying on the ground, he visually aligns the top of his tripod with the top of the monument and marks his location on the ground approximately 2 ft. 9 in. from the center of his tripod. Use Brian’s measurements to approximate the height of the Washington Monument.



5. Catarina’s boat has come untied and floated away on the lake. She is standing atop a cliff that is 35 feet above the water in a lake. If she stands 10 feet from the edge of the cliff, she can visually align the top of the cliff with the water at the back of her boat. Her eye level is $5\frac{1}{2}$ feet above the ground. Approximately how far out from the cliff is Catarina’s boat?



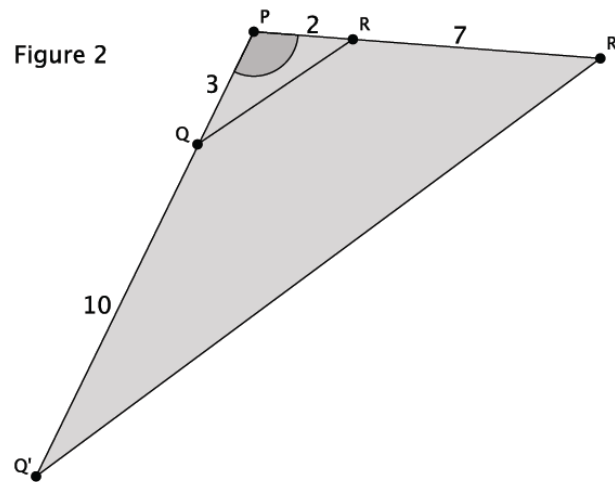
6. Given the diagram below and $\triangle ABC \sim \triangle XYZ$, find the unknown lengths x , $2x$, and $3x$.



c. Compare the corresponding side lengths of $\triangle ABC$ and $\triangle AB'C'$. What do you notice?

d. Based on your work in parts (a)–(c), draw a conclusion about the relationship between $\triangle ABC$ and $\triangle AB'C'$. Explain your reasoning.

2. Examine the figure, and answer the questions to determine whether or not the triangles shown are similar.



a. What information is given about the triangles in Figure 2?

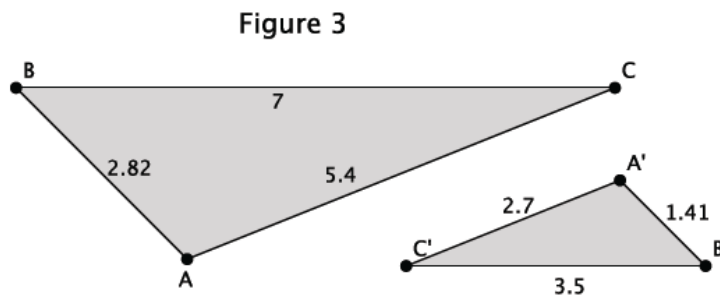
- b. How can the information provided be used to determine whether $\triangle PQR$ is similar to $\triangle PQ'R'$?

- c. Compare the corresponding side lengths of $\triangle PQR$ and $\triangle PQ'R'$. What do you notice?

- d. Based on your work in parts (a)–(c), draw a conclusion about the relationship between $\triangle PQR$ and $\triangle PQ'R'$. Explain your reasoning.

Exploratory Challenge 2/Exercises 3–4

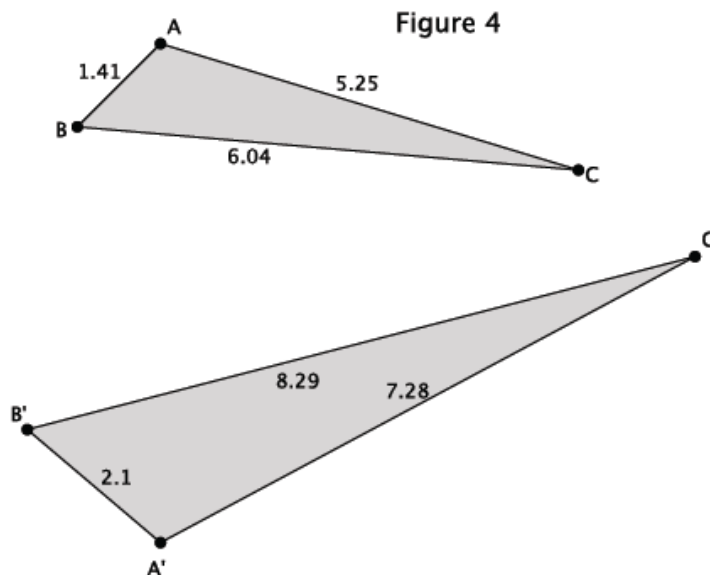
- 3. Examine the figure and answer the questions to determine whether or not the triangles shown are similar.



- a. What information is given about the triangles in Figure 3?

- b. How can the information provided be used to determine whether $\triangle ABC$ is similar to $\triangle AB'C'$?
- c. Compare the corresponding side lengths of $\triangle ABC$ and $\triangle AB'C'$. What do you notice?
- d. Based on your work in parts (a)–(c), make a conjecture about the relationship between $\triangle ABC$ and $\triangle AB'C'$. Explain your reasoning.

4. Examine the figure and answer the questions to determine whether or not the triangles shown are similar.



- a. What information is given about the triangles in Figure 4?

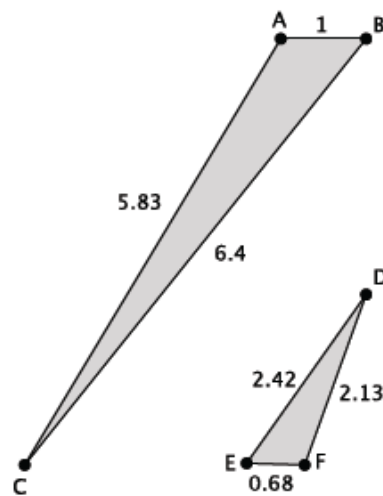
- b. How can the information provided be used to determine whether $\triangle ABC$ is similar to $\triangle AB'C'$?

- c. Compare the corresponding side lengths of $\triangle ABC$ and $\triangle AB'C'$. What do you notice?

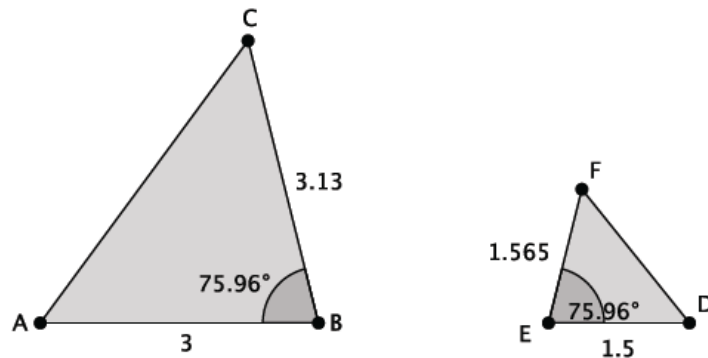
- d. Based on your work in parts (a)–(c), make a conjecture about the relationship between $\triangle ABC$ and $\triangle AB'C'$. Explain your reasoning.

Exercises 5–10

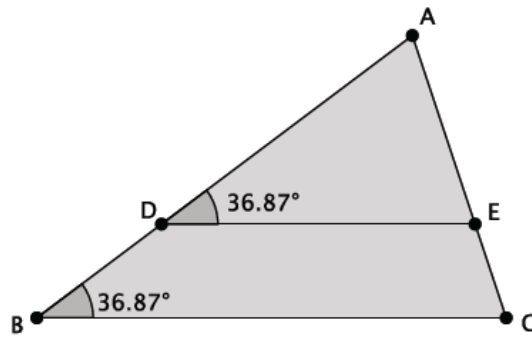
5. Are the triangles shown below similar? Explain. If the triangles are similar, write the similarity statement.



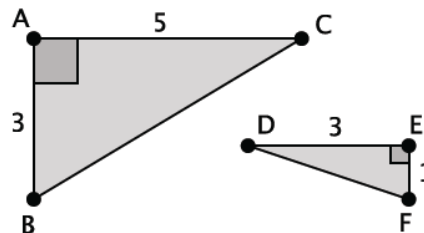
6. Are the triangles shown below similar? Explain. If the triangles are similar, write the similarity statement.



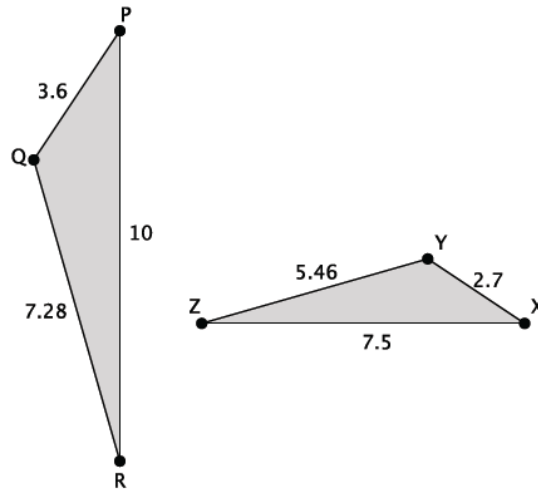
7. Are the triangles shown below similar? Explain. If the triangles are similar, write the similarity statement.



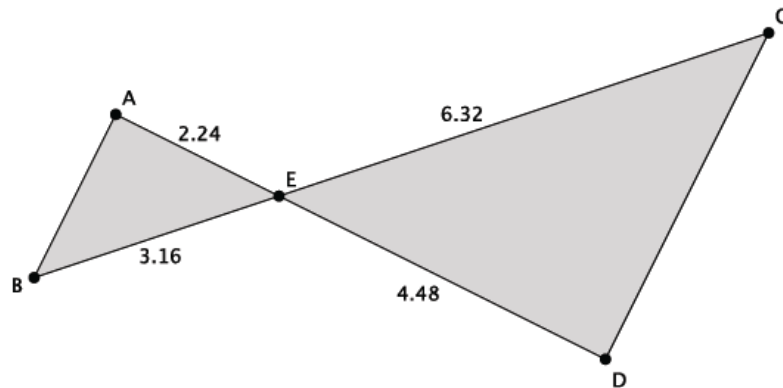
8. Are the triangles shown below similar? Explain. If the triangles are similar, write the similarity statement.



9. Are the triangles shown below similar? Explain. If the triangles are similar, write the similarity statement.



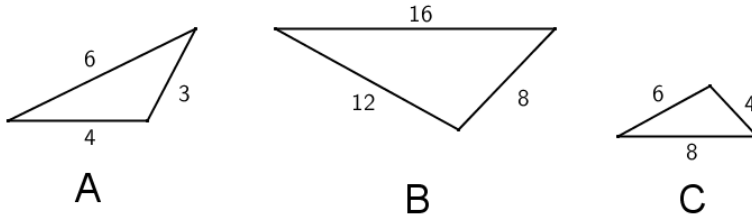
10. Are the triangles shown below similar? Explain. If the triangles are similar, write the similarity statement.



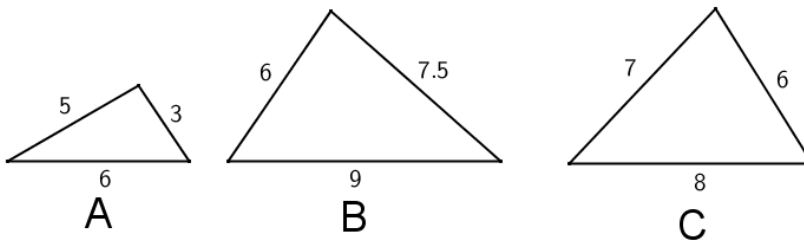
Problem Set

1. For each part (a) through (d) below, state which of the three triangles, if any, are similar and why.

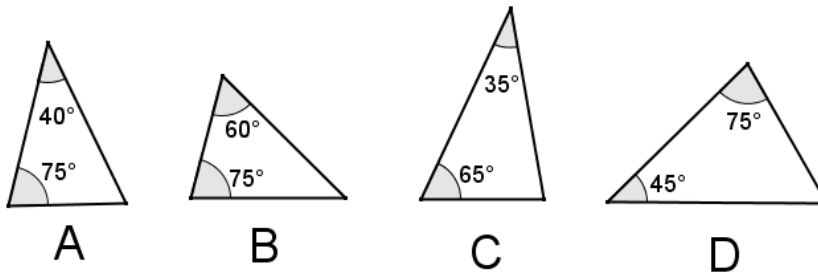
a.



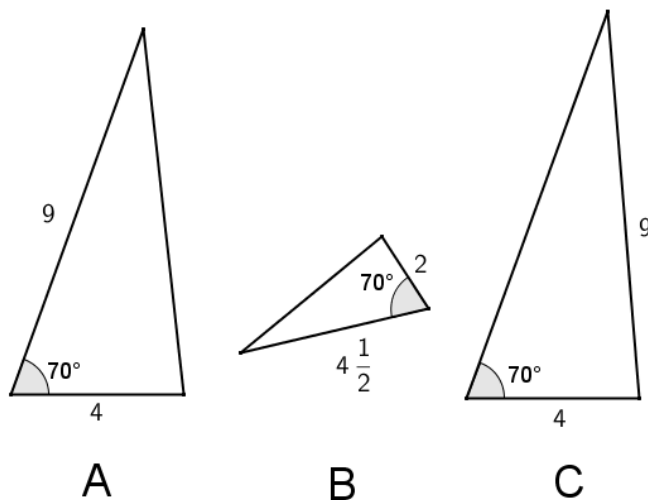
b.



c.

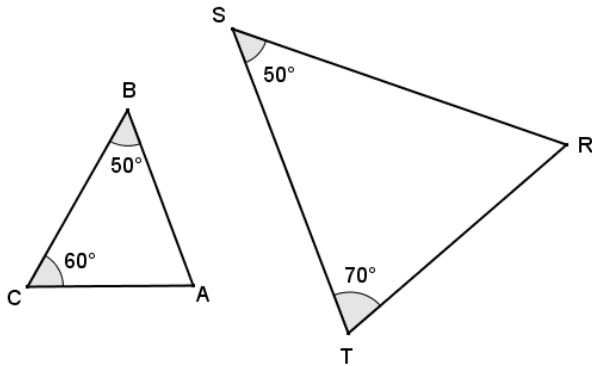


d.

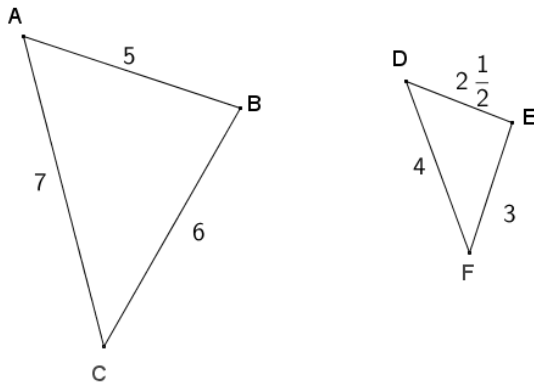


2. For each given pair of triangles, determine if the triangles are similar or not, and provide your reasoning. If the triangles are similar, write a similarity statement relating the triangles.

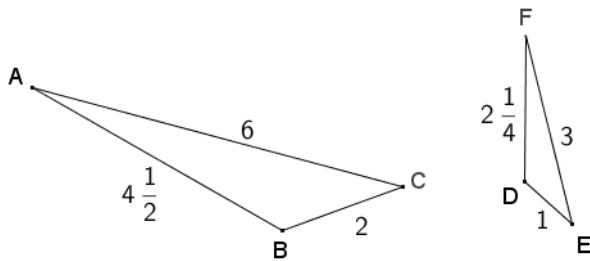
a.



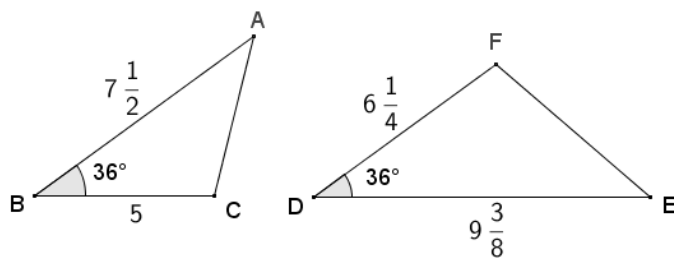
b.



c.

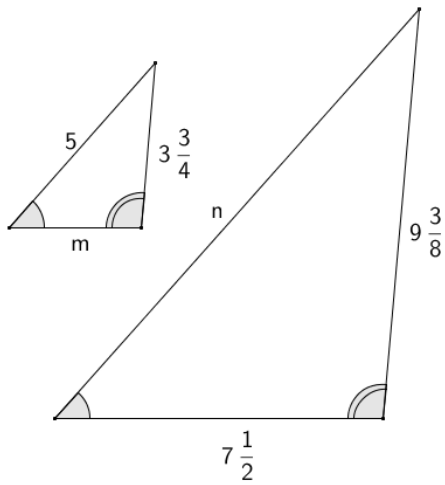


d.

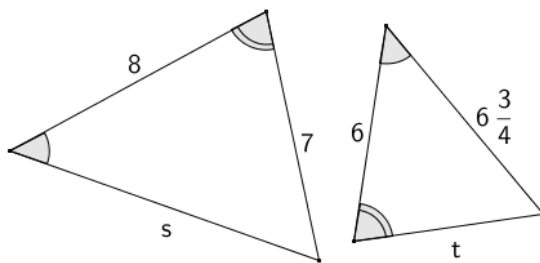


3. For each pair of similar triangles below, determine the unknown lengths of the sides labeled with letters.

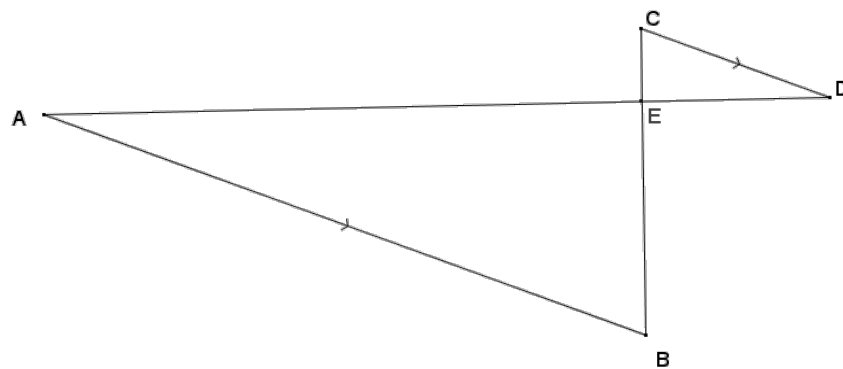
a.



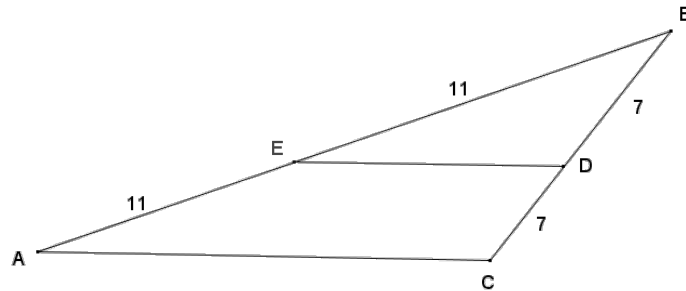
b.



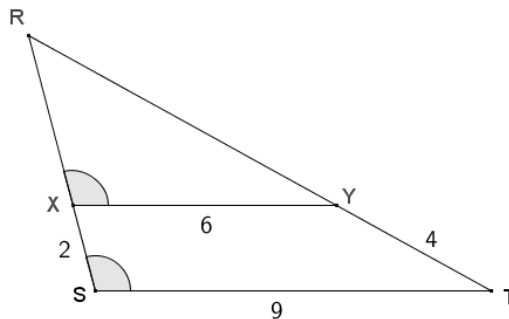
4. Given that \overline{AD} and \overline{BC} intersect at E , and $\overline{AB} \parallel \overline{CD}$, show that $\triangle ABE \sim \triangle DCE$.



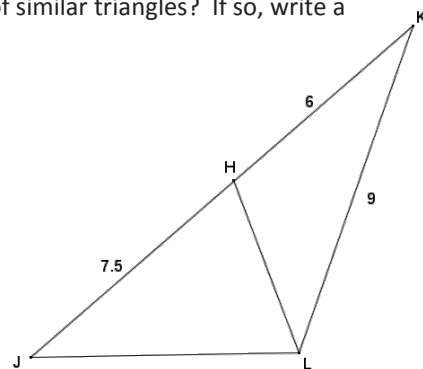
5. Given $BE = 11$, $EA = 11$, $BD = 7$, and $DC = 7$, show that $\triangle BED \sim \triangle BAC$.



6. Given the diagram below, X is on \overline{RS} and Y is on \overline{RT} , $XS = 2$, $XY = 6$, $ST = 9$, and $YT = 4$.



- Show that $\triangle RXY \sim \triangle RST$.
 - Find RX and RY .
7. One triangle has a 120° angle, and a second triangle has a 65° angle. Is it possible that the two triangles are similar? Explain why or why not.
8. A right triangle has a leg that is 12 cm long, and another right triangle has a leg that is 6 cm long. Are the two triangles similar or not? If so, explain why. If not, what other information would be needed to show they are similar?
9. Given the diagram below, $JH = 7.5$, $HK = 6$, and $KL = 9$, is there a pair of similar triangles? If so, write a similarity statement and explain why. If not, explain your reasoning.



Lesson 18: Similarity and the Angle Bisector Theorem

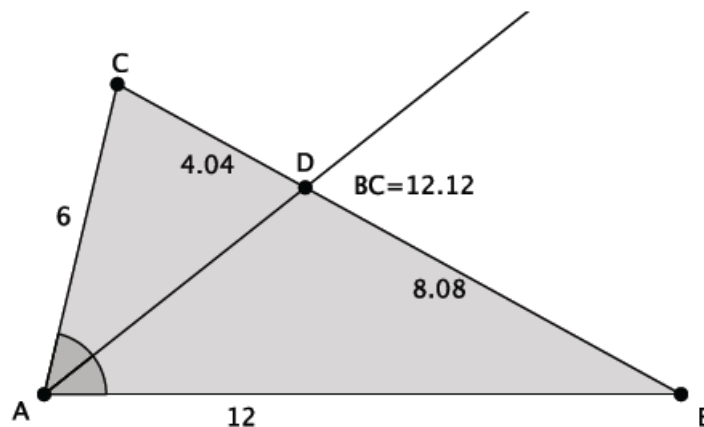
Classwork

Opening Exercise

- What is an angle bisector?
- Describe the angle relationships formed when parallel lines are cut by a transversal.
- What are the properties of an isosceles triangle?

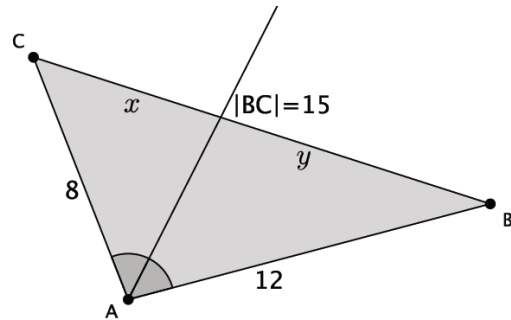
Discussion

In the diagram below, the angle bisector of $\angle A$ in $\triangle ABC$ meets side BC at point D . Does the angle bisector create any observable relationships with respect to the side lengths of the triangle?

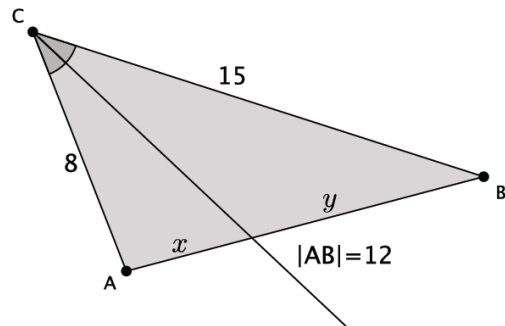


Exercises 1–4

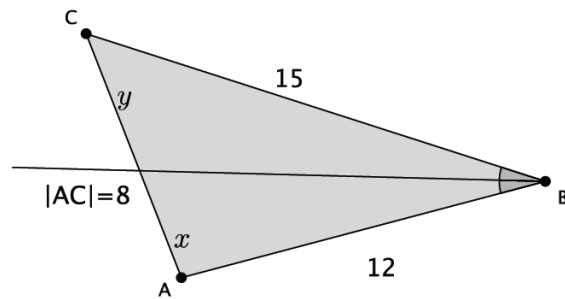
- The sides of a triangle are 8, 12, and 15. An angle bisector meets the side of length 15. Find the lengths x and y . Explain how you arrived at your answers.



- The sides of a triangle are 8, 12, and 15. An angle bisector meets the side of length 12. Find the lengths x and y .



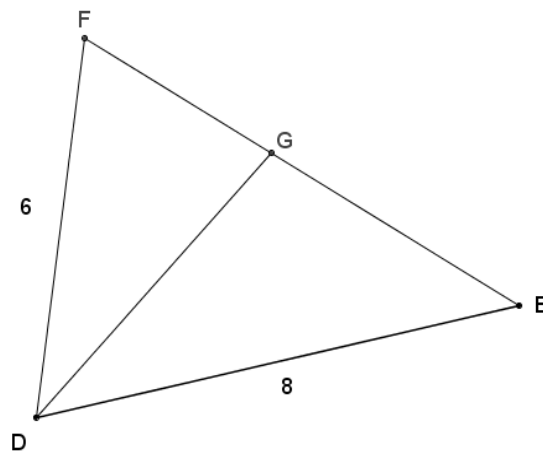
3. The sides of a triangle are 8, 12, and 15. An angle bisector meets the side of length 8. Find the lengths x and y .



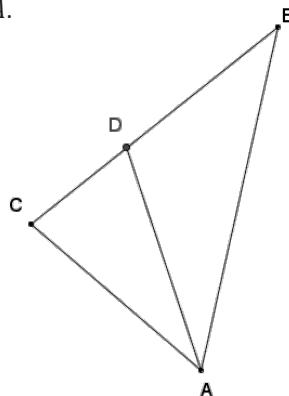
4. The angle bisector of an angle splits the opposite side of a triangle into lengths 5 and 6. The perimeter of the triangle is 33. Find the lengths of the other two sides.

Problem Set

- The sides of a triangle have lengths of 5, 8, and $6\frac{1}{2}$. An angle bisector meets the side of length $6\frac{1}{2}$. Find the lengths x and y .
- The sides of a triangle are $10\frac{1}{2}$, $16\frac{1}{2}$, and 9. An angle bisector meets the side of length 9. Find the lengths x and y .
- In the diagram of triangle DEF below, \overline{DG} is an angle bisector, $DE = 8$, $DF = 6$, and $EF = 8\frac{1}{6}$. Find FG and EG .

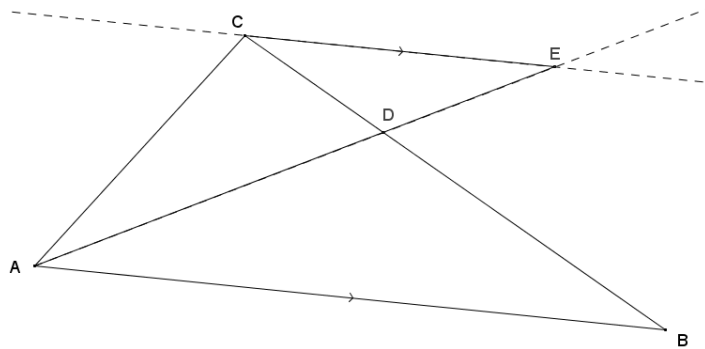


- $\angle BAD \cong \angle DAC$, show that $BD:BA = CD:CA$.

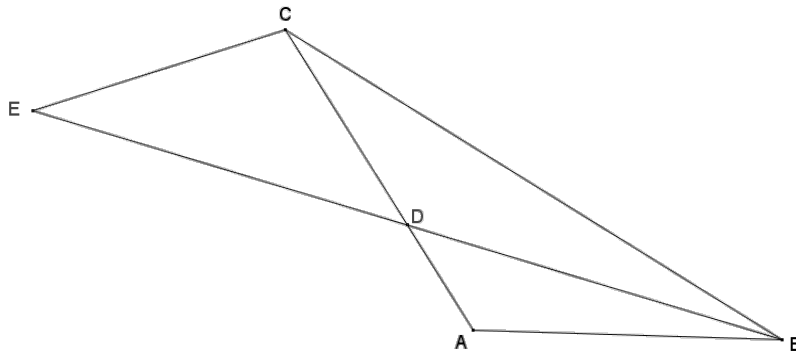


- The perimeter of triangle LMN is 32 cm. \overline{NX} is the angle bisector of angle N , $LX = 3$ cm, and $XM = 5$ cm. Find LN and MN .
- Given $CD = 3$, $DB = 4$, $BF = 4$, $FE = 5$, $AB = 6$, and $\angle CAD \cong \angle DAB \cong \angle BAF \cong \angle FAE$, find the perimeter of quadrilateral $AEBC$.

7. If \overline{AE} meets \overline{BC} at D such that $CD:BD = CA:BA$, show that $\angle CAD \cong \angle BAD$. Explain how this proof relates to the angle bisector theorem.



8. In the diagram below, $\overline{ED} \cong \overline{DB}$, \overline{BE} bisects $\angle ABC$, $AD = 4$, and $DC = 8$. Prove that $\triangle ADB \sim \triangle CEB$.

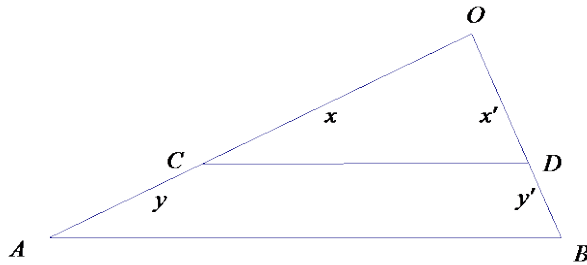


Lesson 19: Families of Parallel Lines and the Circumference of the Earth

Classwork

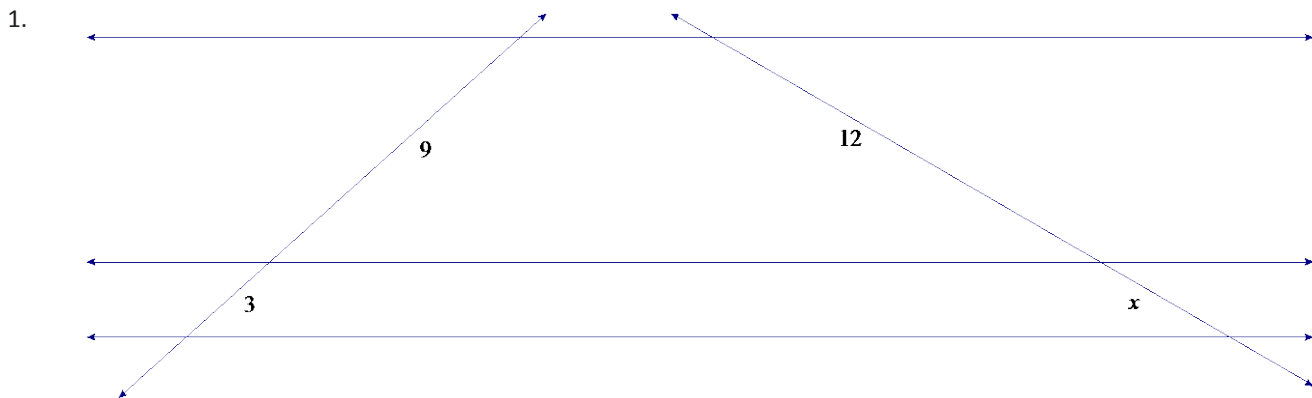
Opening Exercise

Show $x : y = x' : y'$ is equivalent to $x : x' = y : y'$.

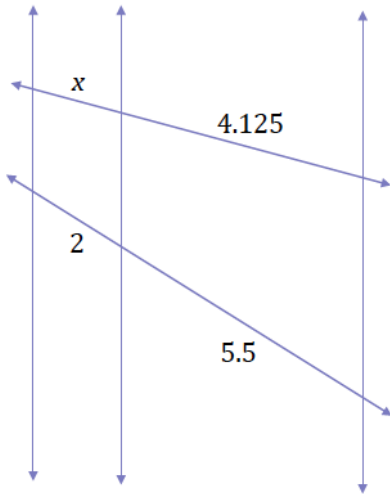


Exercises 1–2

Lines that appear to be parallel are in fact parallel.

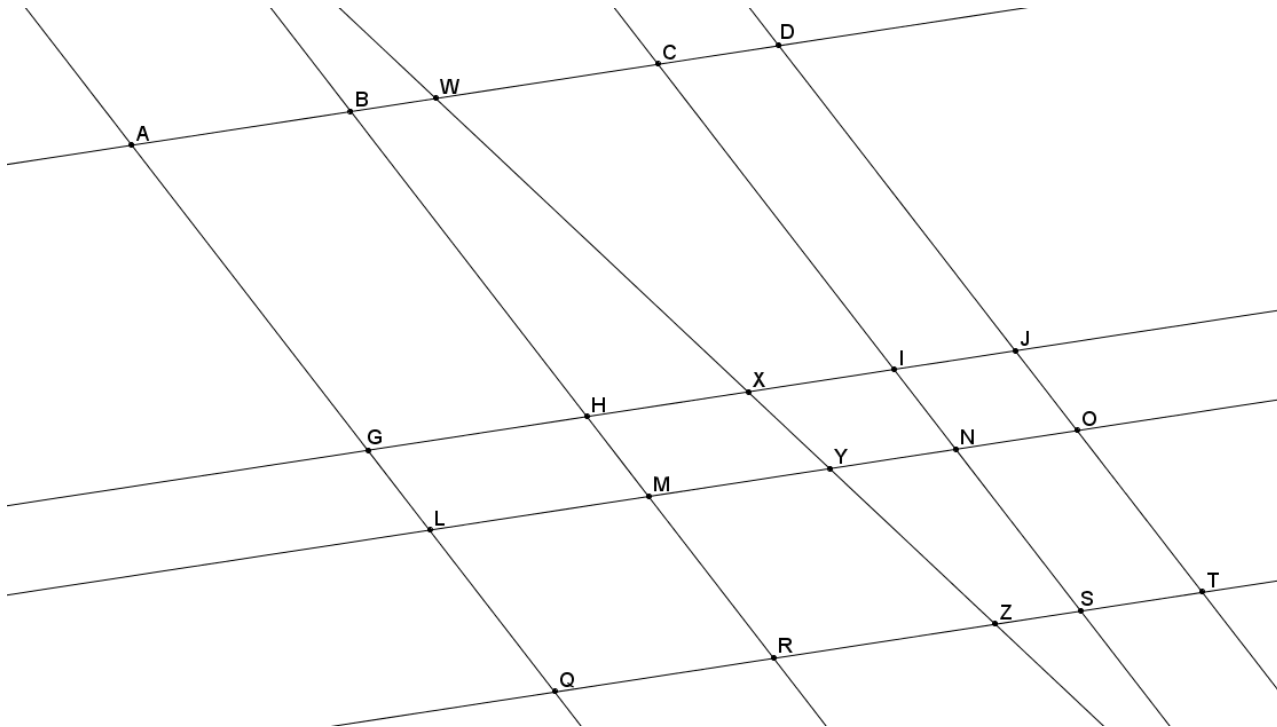


2.



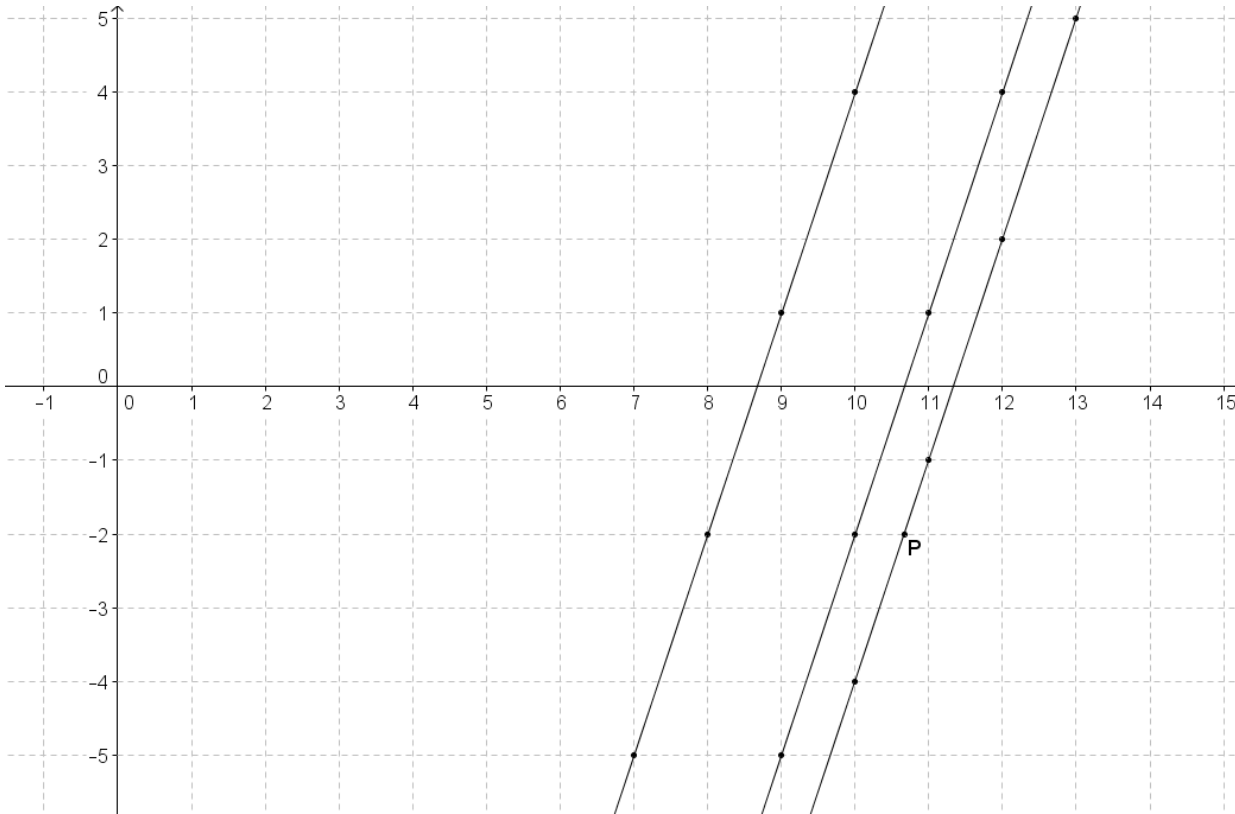
Problem Set

1. Given the diagram shown, $\overline{AD} \parallel \overline{GJ} \parallel \overline{LO} \parallel \overline{QT}$, and $\overline{AQ} \parallel \overline{BR} \parallel \overline{CS} \parallel \overline{DT}$. Use the additional information given in each part below to answer the questions:

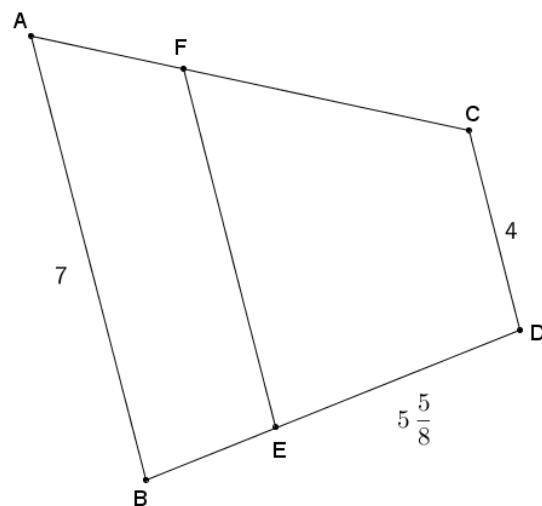


- a. If $GL = 4$, what is HM ?
- b. If $GL = 4$, $LQ = 9$, and $XY = 5$, what is YZ ?
- c. Using information from part (b), if $CI = 18$, what is WX ?

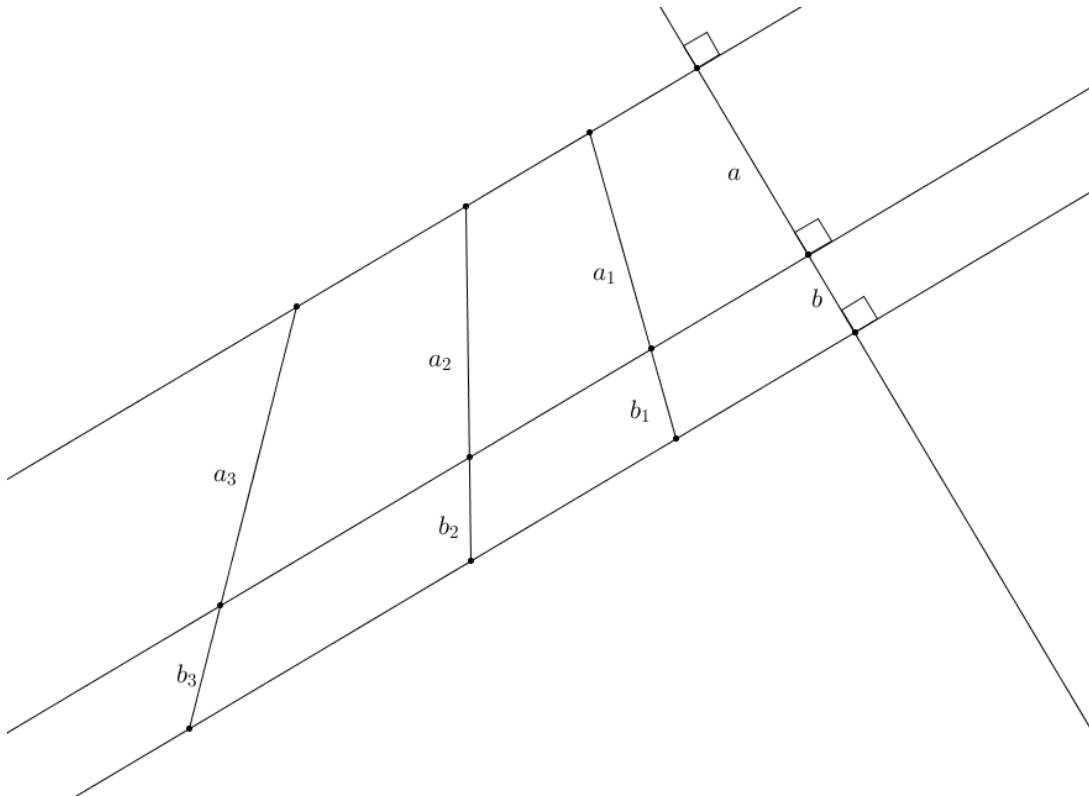
2. Use your knowledge about families of parallel lines to find the coordinates of point P on the coordinate plane below.



3. $ACDB$ and $FCDE$ are both trapezoids with bases \overline{AB} , \overline{FE} , and \overline{CD} . The perimeter of trapezoid $ACDB$ is $24\frac{1}{2}$. If the ratio of $AF:FC$ is $1:3$, and $ED = 5\frac{5}{8}$, find AF , FC , and BE .



4. Given the diagram and the ratio of $a : b$ is $3 : 2$, answer each question below.



- Write an equation for a_n in terms of b_n .
 - Write an equation for b_n in terms of a_n .
 - Use one of your equations to find b_1 in terms of a if $a_1 = 1.2(a)$.
 - What is the relationship between b_1 and b ?
 - What constant, c , relates b_1 and b ? Is this surprising? Why or why not?
 - Using the formula $a_n = c \cdot a_{n-1}$, find a_3 in terms of a .
 - Using the formula $b_n = c \cdot b_{n-1}$, find b_3 in terms of b .
 - Use your answers from parts (f) and (g) to calculate the value of the ratio of $a_3 : b_3$?
5. Julius wants to try to estimate the circumference of the earth based on measurements made near his home. He cannot find a location near his home where the sun is straight overhead. Will he be able to calculate the circumference of the earth? If so, explain and draw a diagram to support your claim.

Lesson 20: How Far Away Is the Moon?

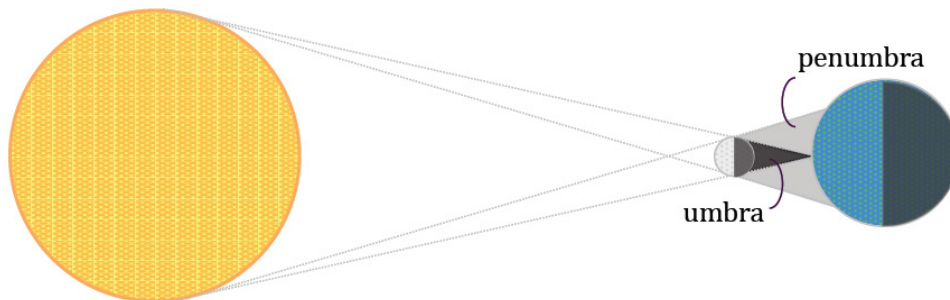
Classwork

Opening Exercise

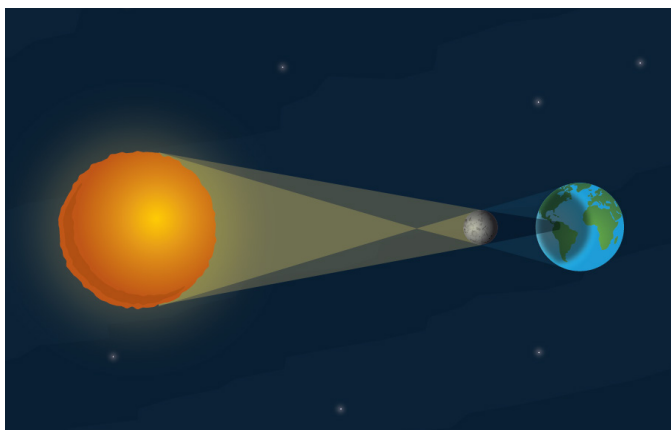
What is a solar eclipse? What is a lunar eclipse?

Discussion

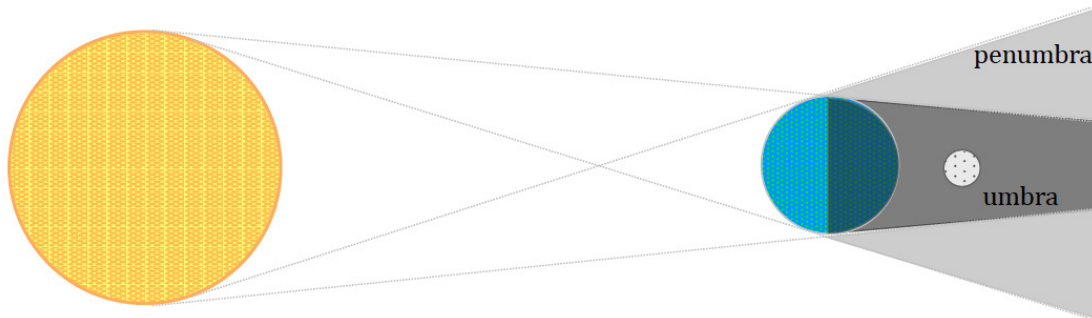
Solar Eclipse



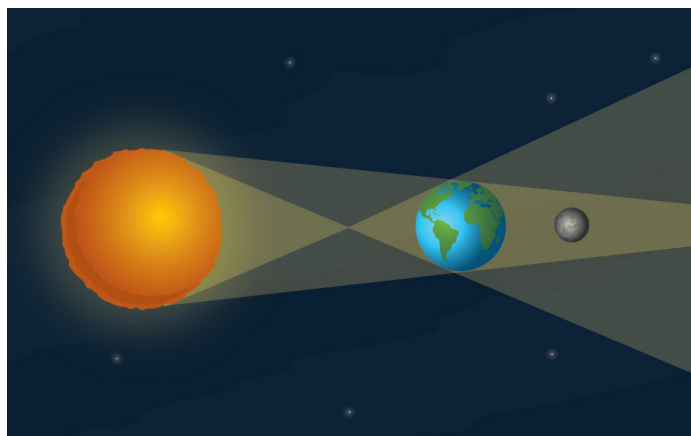
3D view:



Lunar Eclipse



3D view:



Example

- a. If the circumference of the earth is about 25,000 miles, what is the Earth's diameter in miles?

b. Using (a), what is the moon's diameter in miles?

c. How far away is the moon in miles?

Problem Set

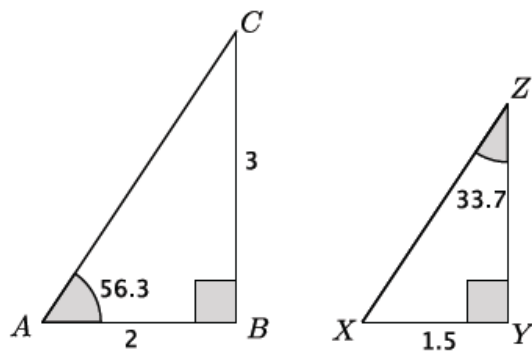
1. If the sun and the moon do not have the same diameter, explain how the sun's light can be covered by the moon during a solar eclipse.
2. What would a lunar eclipse look like when viewed from the moon?
3. Suppose you live on a planet with a moon, where during a solar eclipse, the moon appears to be half the diameter of the sun.
 - a. Draw a diagram of how the moon would look against the sun during a solar eclipse.
 - b. A 1 inch diameter marble held 100 inches away on the planet barely blocks the sun. How many moon diameters away is the moon from the planet? Draw and label a diagram to support your answer.
 - c. If the diameter of the moon is approximately $\frac{3}{5}$ of the diameter of the planet, and the circumference of the planet is 185,000 miles, approximately how far is the moon from the planet?

Lesson 21: Special Relationships Within Right Triangles—Dividing into Two Similar Sub-Triangles

Classwork

Opening Exercise

Use the diagram below to complete parts (a)–(c).



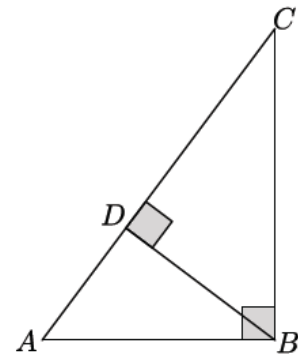
- Are the triangles shown above similar? Explain.
- Determine the unknown lengths of the triangles.

- c. Explain how you found the lengths in part (a).

Example 1

Recall that an altitude of a triangle is a perpendicular line segment from a vertex to the line determined by the opposite side. In triangle $\triangle ABC$ below, \overline{BD} is the altitude from vertex B to the line containing \overline{AC} .

How many triangles do you see in the figure?



Identify the three triangles by name.

Note that there are many ways to name the three triangles. Ensure that the names students give show corresponding angles.

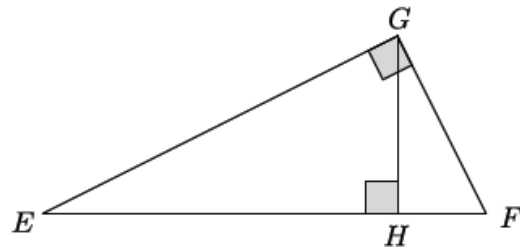
We want to consider the altitude of a right triangle from the right angle to the hypotenuse. The altitude of a right triangle splits the triangle into two right triangles, each of which shares a common acute angle with the original triangle. In $\triangle ABC$, the altitude \overline{BD} divides the right triangle into two sub-triangles, $\triangle BDC$ and $\triangle ADB$.

Is $\triangle ABC \sim \triangle BDC$? Is $\triangle ABC \sim \triangle ADB$? Explain.

Is $\triangle ABC \sim \triangle DBC$? Explain.

Since $\triangle ABC \sim \triangle BDC$ and $\triangle ABC \sim \triangle ADB$, can we conclude that $\triangle BDC \sim \triangle ADB$? Explain.

Identify the altitude drawn in triangle $\triangle EFG$.



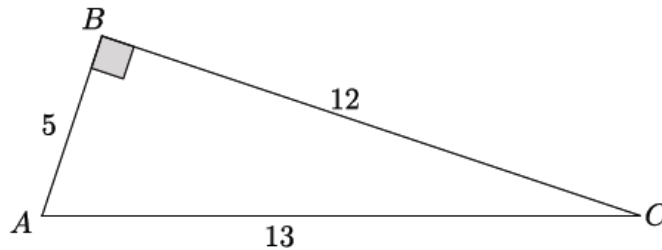
As before, the altitude divides the triangle into three triangles. Identify them by name so that the corresponding angles match up.

Does the altitude divide $\triangle EFG$ into three similar sub-triangles as the altitude did with $\triangle ABC$?

The fact that the altitude drawn from the right angle of a right triangle divides the triangle into two similar sub-triangles which are also similar to the original triangle allows us to determine the unknown lengths of right triangles.

Example 2

Consider the right triangle $\triangle ABC$ below.



Draw the altitude \overline{BD} from vertex B to the line containing \overline{AC} . Label the segment \overline{AD} as x , the segment \overline{DC} as y , and the segment \overline{BD} as z .

Find the values of x , y , and z .

Now we will look at a different strategy for determining the lengths of x , y , and z . The strategy requires that we complete a table of ratios that compares different parts of each triangle.

Make a table of ratios for each triangle that relates the sides listed in the column headers.

	shorter leg: hypotenuse	longer leg: hypotenuse	shorter leg: longer leg
$\triangle ABC$			
$\triangle ADB$			
$\triangle CDB$			

Our work in Example 1 showed us that $\triangle ABC \sim \triangle ADB \sim \triangle CDB$. Since the triangles are similar, the ratios of their corresponding sides will be equal. For example, we can find the length of x by equating the values of *shorter leg: hypotenuse* ratios of triangles $\triangle ABC$ and $\triangle ADB$.

$$\begin{aligned}\frac{x}{5} &= \frac{5}{13} \\ 13x &= 25 \\ x &= \frac{25}{13} = 1\frac{12}{13}\end{aligned}$$

Why can we use these ratios to determine the length of x ?

Which ratios can we use to determine the length of y ?

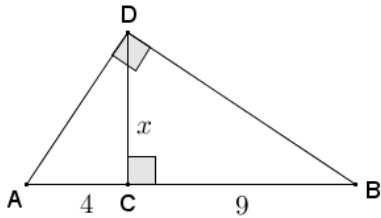
Use ratios to determine the length of z .

Since corresponding ratios within similar triangles are equal, we can solve for any unknown side length by equating the values of the corresponding ratios. In the coming lessons, we will learn about more useful ratios for determining unknown side lengths of right triangles.

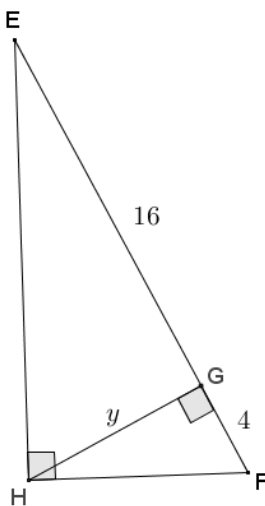
Problem Set

1. Use similar triangles to find the length of the altitudes labeled with variables in each triangle below.

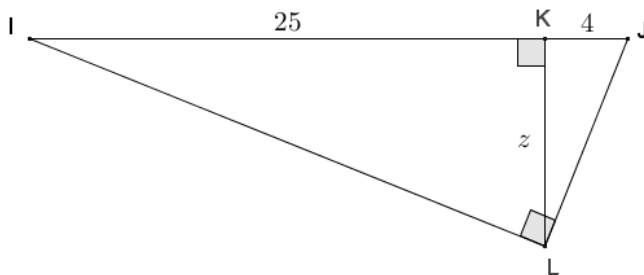
a.



b.

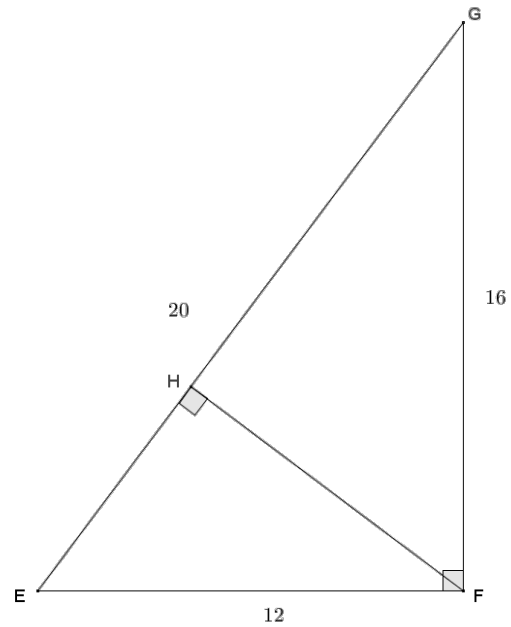


c.

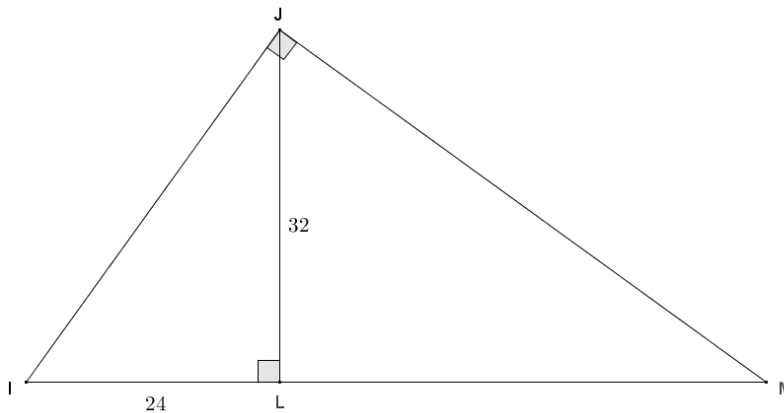


d. Describe the pattern that you see in your calculations for parts (a) through (c).

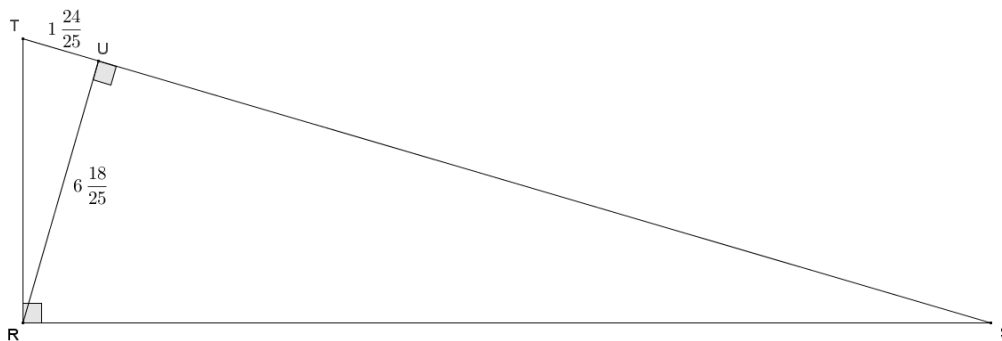
2. Given right triangle EFG with altitude \overline{FH} drawn to the hypotenuse, find the lengths of EH , FH , and GH .



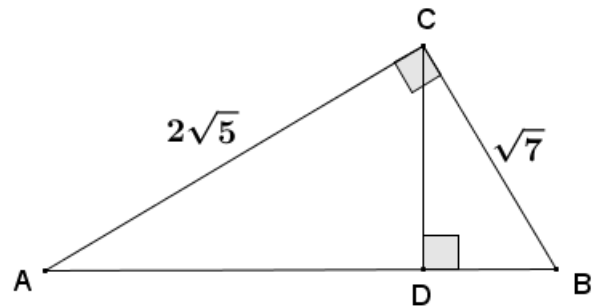
3. Given triangle IMJ with altitude \overline{JL} , $JL = 32$, and $IL = 24$, find IJ , JM , LM , and IM .



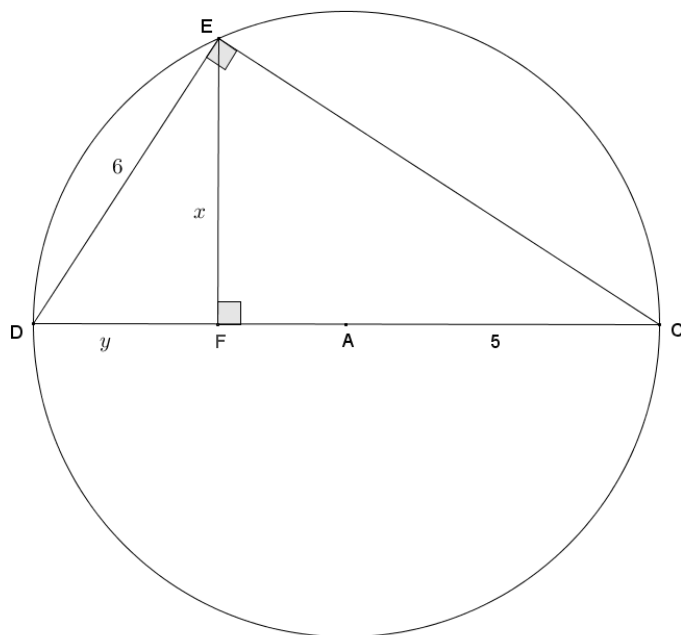
4. Given right triangle RST with altitude \overline{RU} to its hypotenuse, $TU = 1\frac{24}{25}$, and $RU = 6\frac{18}{25}$, find the lengths of the sides of $\triangle RST$.



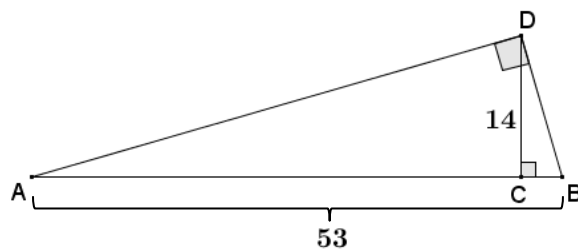
5. Given right triangle ABC with altitude \overline{CD} , find AD , BD , AB , and DC .



6. Right triangle DEC is inscribed in a circle with radius $AC = 5$. \overline{DC} is a diameter of the circle, EF is an altitude of $\triangle DEC$, and $DE = 6$. Find the lengths x and y .



7. In right triangle ABD , $AB = 53$, and altitude $DC = 14$. Find the lengths of BC and AC .



Lesson 22: Multiplying and Dividing Expressions with Radicals

Classwork

Exercises 1–5

Simplify as much as possible.

1. $\sqrt{17^2} =$

2. $\sqrt{5^{10}} =$

3. $\sqrt{4x^4} =$

4. Complete parts (a) through (c).

a. Compare the value of $\sqrt{36}$ to the value of $\sqrt{9} \times \sqrt{4}$.

- b. Make a conjecture about the validity of the following statement. For nonnegative real numbers a and b , $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$. Explain.
- c. Does your conjecture hold true for $a = -4$ and $b = -9$?
5. Complete parts (a) through (c).
- a. Compare the value of $\sqrt{\frac{100}{25}}$ to the value of $\frac{\sqrt{100}}{\sqrt{25}}$.
- b. Make a conjecture about the validity of the following statement. For nonnegative real numbers a and b , when $b \neq 0$, $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$. Explain.
- c. Does your conjecture hold true for $a = -100$ and $b = -25$?

Exercises 6–17

Simplify each expression as much as possible and rationalize denominators when applicable.

6. $\sqrt{72} =$

7. $\sqrt{\frac{17}{25}} =$

8. $\sqrt{32x} =$

9. $\sqrt{\frac{1}{3}} =$

10. $\sqrt{54x^2} =$

11. $\frac{\sqrt{36}}{\sqrt{18}} =$

12. $\sqrt{\frac{4}{x^4}} =$

13. $\frac{4x}{\sqrt{64x^2}} =$

14. $\frac{5}{\sqrt{x^7}} =$

15. $\sqrt{\frac{x^5}{2}} =$

16. $\frac{\sqrt{18x}}{3\sqrt{x^5}} =$

17. The captain of a ship recorded the ship's coordinates, then sailed north and then west, and then recorded the new coordinates. The coordinates were used to calculate the distance they traveled, $\sqrt{578}$ km. When the captain asked how far they traveled, the navigator said, "About 24 km." Is the navigator correct? Under what conditions might he need to be more precise in his answer?

Problem Set

Express each number in its simplest radical form.

1. $\sqrt{6} \cdot \sqrt{60} =$

2. $\sqrt{108} =$

3. Pablo found the length of the hypotenuse of a right triangle to be $\sqrt{45}$. Can the length be simplified? Explain.

4. $\sqrt{12x^4} =$

5. Sarahi found the distance between two points on a coordinate plane to be $\sqrt{74}$. Can this answer be simplified? Explain.

6. $\sqrt{16x^3} =$

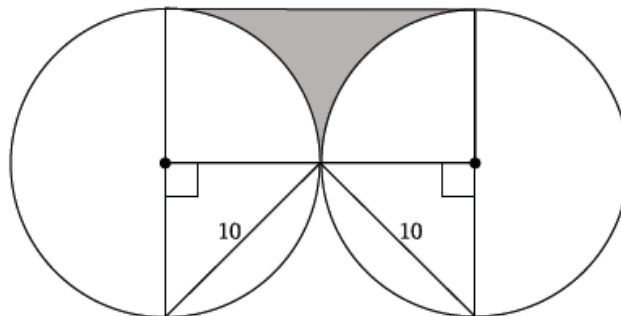
7. $\frac{\sqrt{27}}{\sqrt{3}} =$

8. Nazem and Joffrey are arguing about who got the right answer. Nazem says the answer is $\frac{1}{\sqrt{3}}$, and Joffrey says the answer is $\frac{\sqrt{3}}{3}$. Show and explain that their answers are equivalent.

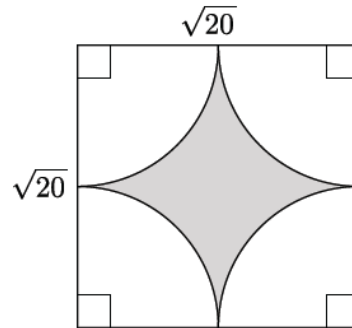
9. $\sqrt{\frac{5}{8}} =$

10. Determine the area of a square with side length $2\sqrt{7}$ in.

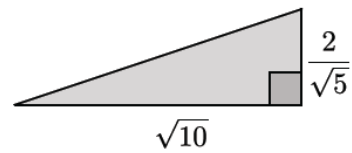
11. Determine the exact area of the shaded region shown below.



12. Determine the exact area of the shaded region shown to the right.



13. Calculate the area of the triangle to the right.



14. $\frac{\sqrt{2x^3} \cdot \sqrt{8x}}{\sqrt{x^3}} =$

15. Prove Rule 2 for square roots: $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$ ($a \geq 0, b > 0$).

Let p be the nonnegative number so that $p^2 = a$, and let q be the nonnegative number so that $q^2 = b$.

Lesson 23: Adding and Subtracting Expressions with Radicals

Classwork

Exercises 1–5

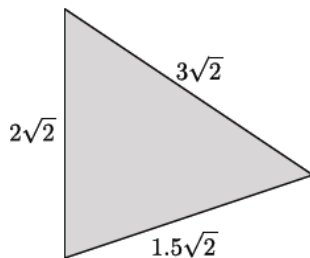
Simplify each expression as much as possible.

1. $\sqrt{32} =$

2. $\sqrt{45} =$

3. $\sqrt{300} =$

4. The triangle shown below has a perimeter of $6.5\sqrt{2}$ units. Make a conjecture about how this answer was reached.



5. The sides of a triangle are $4\sqrt{3}$, $\sqrt{12}$, and $\sqrt{75}$. Make a conjecture about how to determine the perimeter of this triangle.

Exercise 6

6. Circle the expressions that can be simplified using the distributive property. Be prepared to explain your choices.

$8.3\sqrt{2} + 7.9\sqrt{2}$
$\sqrt{13} - \sqrt{6}$
$-15\sqrt{5} + \sqrt{45}$
$11\sqrt{7} - 6\sqrt{7} + 3\sqrt{2}$
$19\sqrt{2} + 2\sqrt{8}$
$4 + \sqrt{11}$
$\sqrt{7} + 2\sqrt{10}$
$\sqrt{12} - \sqrt{75}$
$\sqrt{32} + \sqrt{2}$
$6\sqrt{13} + \sqrt{26}$

Example 1

Explain how the expression $8.3\sqrt{2} + 7.9\sqrt{2}$ can be simplified using the distributive property.

Explain how the expression $11\sqrt{7} - 6\sqrt{7} + 3\sqrt{2}$ can be simplified using the distributive property.

Example 2

Explain how the expression $19\sqrt{2} + 2\sqrt{8}$ can be simplified using the distributive property.

Example 3

Can the expression $\sqrt{7} + 2\sqrt{10}$ be simplified using the distributive property?

To determine if an expression can be simplified, you must first simplify each of the terms within the expression. Then, apply the distributive property, or other properties as needed, to simplify the expression.

Problem Set

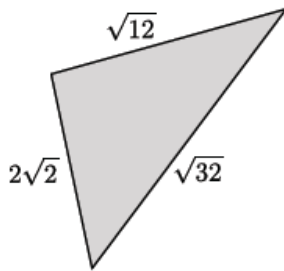
Express each answer in simplified radical form.

1. $18\sqrt{5} - 12\sqrt{5} =$

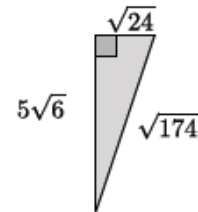
2. $\sqrt{24} + 4\sqrt{54} =$

3. $2\sqrt{7} + 4\sqrt{63} =$

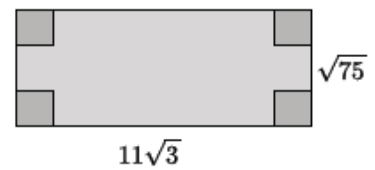
4. What is the perimeter of the triangle shown below?



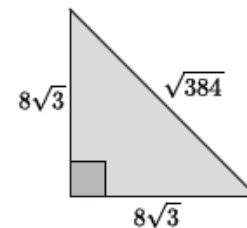
5. Determine the area and perimeter of the triangle shown. Simplify as much as possible.



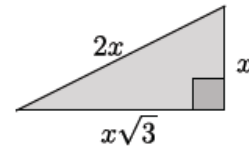
6. Determine the area and perimeter of the rectangle shown. Simplify as much as possible.



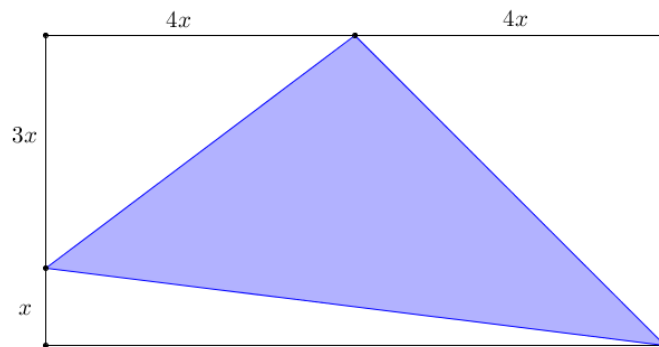
7. Determine the area and perimeter of the triangle shown. Simplify as much as possible.



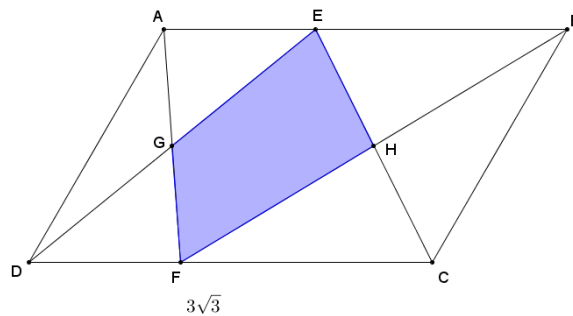
8. Determine the area and perimeter of the triangle shown. Simplify as much as possible.



9. The area of the rectangle shown in the diagram below is 160 square units. Determine the area and perimeter of the shaded triangle. Write your answers in simplest radical form, and then approximate to the nearest tenth.

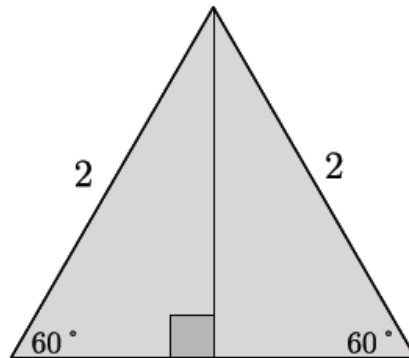


10. Parallelogram $ABCD$ has an area of $9\sqrt{3}$. $DC = 3\sqrt{3}$, and G and H are midpoints of \overline{DE} and \overline{CE} , respectively. Find the area of the shaded region. Write your answer in simplest radical form.



Exploratory Challenge/Exercises 4–5

4. An equilateral triangle has sides of length 2 and angle measures of 60° , as shown below. The altitude from one vertex to the opposite side divides the triangle into two right triangles.

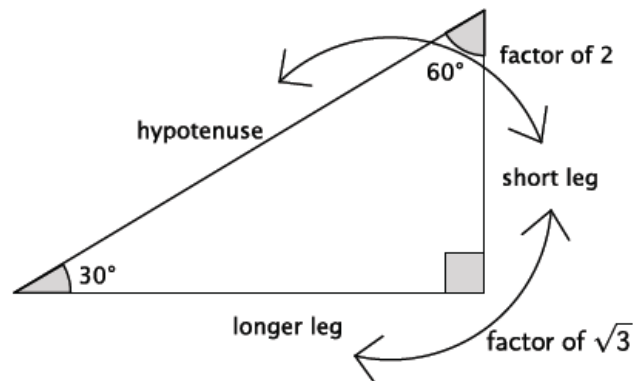


- Are those triangles congruent? Explain.
- What is the length of the shorter leg of each of the right triangles? Explain.
- Use the Pythagorean theorem to determine the length of the altitude.
- Write the ratio that represents *shorter leg*: *hypotenuse*.

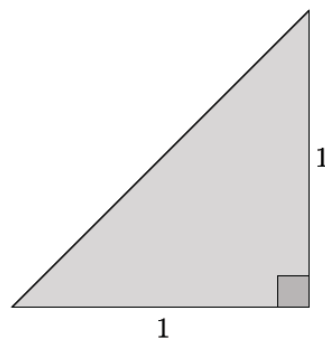
- e. Write the ratio that represents *longer leg: hypotenuse*.

- f. Write the ratio that represents *shorter leg: longer leg*.

- g. By the AA criterion, any triangles with measures 30–60–90 will be similar to this triangle. If a 30–60–90 triangle has a hypotenuse of length 16, what are the lengths of the legs?



5. An isosceles right triangle has leg lengths of 1, as shown.



- a. What are the measures of the other two angles? Explain.

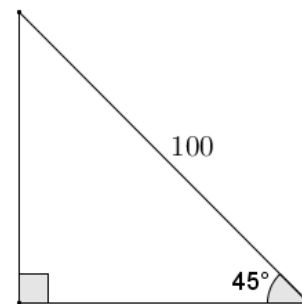
- b. Use the Pythagorean theorem to determine the length of the hypotenuse of the right triangle.
- c. Is it necessary to write all three ratios: *shorter leg: hypotenuse*, *longer leg: hypotenuse*, and *shorter leg: longer leg*? Explain.
- d. Write the ratio that represents *leg: hypotenuse*.
- e. By the AA criterion, any triangles with measures 45–45–90 will be similar to this triangle. If a 45–45–90 triangle has a hypotenuse of length 20, what are the lengths of the legs?

Problem Set

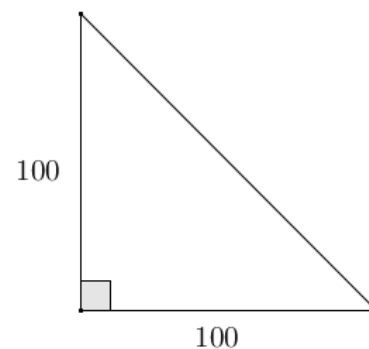
1. In each row of the table below are the lengths of the legs and hypotenuses of different right triangles. Find the missing side lengths in each row, in simplest radical form.

Leg ₁	Leg ₂	Hypotenuse
15		25
15	36	
3		7
100	200	

2. Claude sailed his boat due south for 38 miles, then due west for 25 miles. Approximately how far is Claude from where he began?
3. Find the lengths of the legs in the triangle given the hypotenuse with length 100.



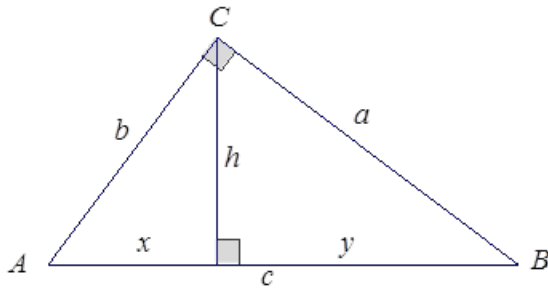
4. Find the length of the hypotenuse in the right triangle given that the legs have lengths of 100.



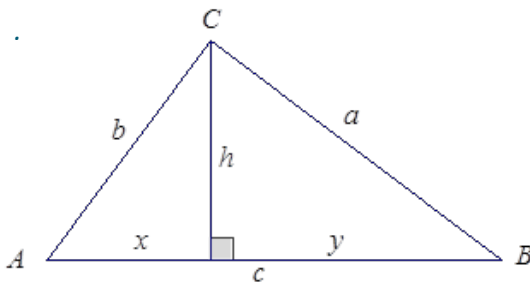
5. Each row in the table below shows the side lengths of a different 30–60–90 right triangle. Complete the table with the missing side lengths in simplest radical form. Use the relationships of the values in the first three rows to complete the last row. How could the expressions in the last row be used?

Shorter Leg	Longer Leg	Hypotenuse
25		50
15		
	3	$2\sqrt{3}$
x		

6. In right triangle ABC with $\angle C$ a right angle, an altitude of length h is dropped to side AB that splits the side AB into segments of length x and y . Use the Pythagorean theorem to show $h^2 = xy$.



7. In triangle ABC , the altitude from $\angle C$ splits side AB into two segments of lengths x and y . If h denotes the length of the altitude and $h^2 = xy$, use the Pythagorean theorem and its converse to show that triangle ABC is a right triangle with $\angle C$ a right angle.



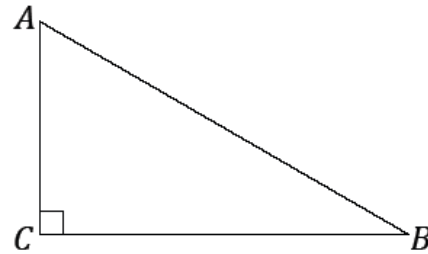
Lesson 25: Incredibly Useful Ratios

Classwork

Exercises 1–3

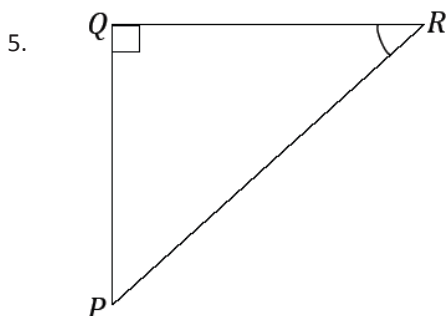
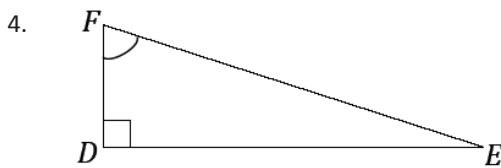
Use the right triangle $\triangle ABC$ to answer Exercises 1–3.

1. Name the side of the triangle opposite $\angle A$.
2. Name the side of the triangle opposite $\angle B$.
3. Name the side of the triangle opposite $\angle C$.

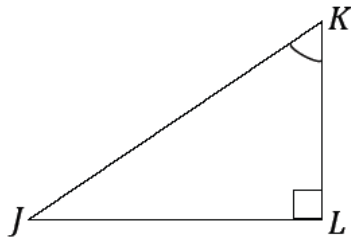


Exercises 4–6

For each exercise, label the appropriate sides as adjacent, opposite, and hypotenuse, with respect to the marked acute angle.



6.



Exploratory Challenge

Note: Angle measures are approximations.

For each triangle in your set, determine missing angle measurements and side lengths. Side lengths should be measured to one decimal place. Make sure that each of the $\frac{adj}{hyp}$ and $\frac{opp}{hyp}$ ratios are set up and missing values are calculated to one decimal place.

Group 1					
	Triangle	Angle Measures	Length Measures	$\frac{opp}{hyp}$	$\frac{adj}{hyp}$
1.	$\triangle ABC$			$\frac{12}{13} \approx 0.92$	$\frac{5}{13} \approx 0.38$
2.	$\triangle DEF$	$m\angle D \approx 53^\circ$	$DE = 3$ cm $EF = 4$ cm $DF = 5$ cm		
3.	$\triangle GHI$	$m\angle I \approx 41^\circ$	$GH = 5.3$ cm	$\frac{5.3}{8} \approx 0.66$	$= 0.75$
4.	$\triangle JKL$		$KL = 6.93$ cm $JL = 8$ cm	$\frac{6.93}{8} =$	$\frac{5}{8} \approx 0.87$
5.	$\triangle MNO$			$\frac{4}{8.5} \approx 0.47$	$\frac{7.5}{8.5} \approx 0.88$

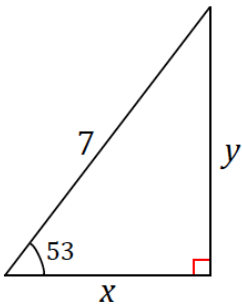
Group 2					
	Triangle	Angle Measures	Length Measures	$\frac{opp}{hyp}$	$\frac{adj}{hyp}$
1.	$\triangle A'B'C'$			$\frac{6}{6.5} \approx 0.92$	$\frac{2.5}{6.5} \approx 0.38$
2.	$\triangle D'E'F'$	$m\angle D' \approx 53^\circ$	$D'E' = 6$ cm $E'F' = 8$ cm $D'F' = 10$ cm		
3.	$\triangle G'H'I'$	$m\angle I' \approx 41^\circ$	$G'H' = 7.9$ cm	$\frac{7.9}{\quad} \approx 0.66$	$= 0.75$
4.	$\triangle J'K'L'$		$K'L' = 10.4$ cm $J'L' = 12$ cm	$\frac{\quad}{12} =$	$\frac{\quad}{12} \approx 0.87$
5.	$\triangle M'N'O'$			$\frac{8}{17} \approx 0.47$	$\frac{15}{17} \approx 0.88$

With a partner, discuss what you can conclude about each pair of triangles between the two sets?

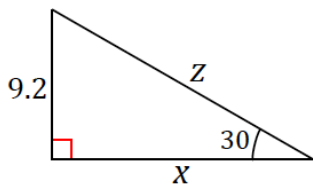
Exercises 7–10

For each question, approximate the unknown lengths to one decimal place. Refer back to your completed chart from the Exploratory Challenge; each indicated acute angle is the same approximated acute angle measure as in the chart. Set up and label the appropriate length ratios, using the terms *opp*, *adj*, and *hyp* in the set up of each ratio.

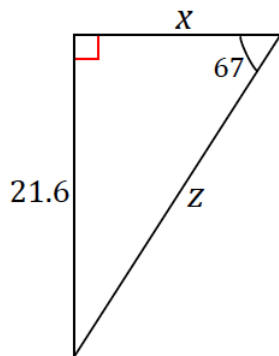
7.



8.

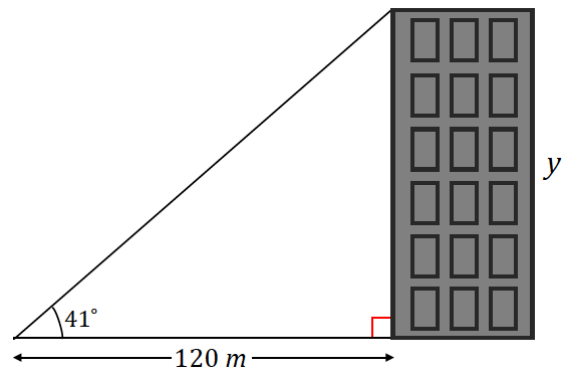


9.



10. From a point 120 m away from a building, Serena measures the angle between the ground and the top of a building and finds it measures 41° .

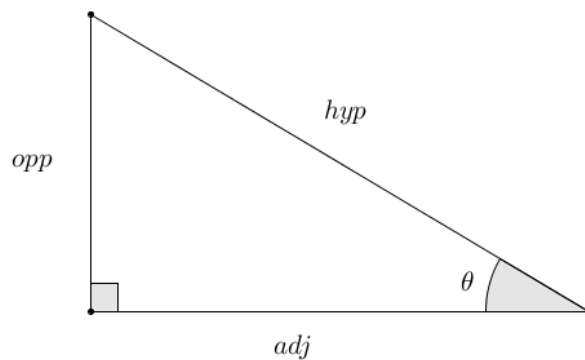
What is the height of the building? Round to the nearest meter.



Problem Set

The table below contains the values of the ratios $\frac{opp}{hyp}$ and $\frac{adj}{hyp}$ for a variety of right triangles based on a given acute angle, θ , from each triangle. Use the table and the diagram of the right triangle below to complete each problem.

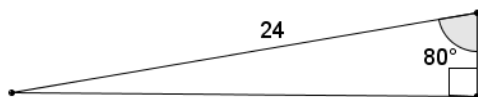
θ (degrees)	0	10	20	30	40	45	50	60	70	80	90
$\frac{opp}{hyp}$	0	0.1736	0.3420	$\frac{1}{2} = 0.5$	0.6428	0.7071	0.7660	0.8660	0.9397	0.9848	1
$\frac{adj}{hyp}$	1	0.9848	0.9397	0.8660	0.7660	0.7071	0.6428	$\frac{1}{2} = 0.5$	0.3420	0.1736	0



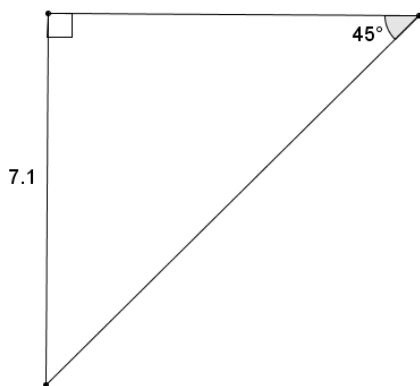
NOT DRAWN TO SCALE

For each problem, approximate the unknown lengths to one decimal place. Write the appropriate length ratios, using the terms *opp*, *adj*, and *hyp* in the set up of each ratio.

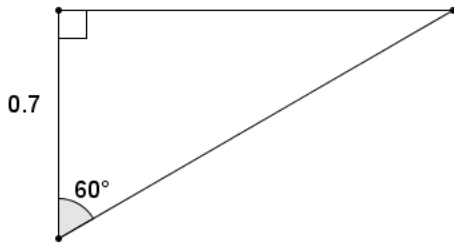
- Find the approximate length of the leg opposite the 80° angle.



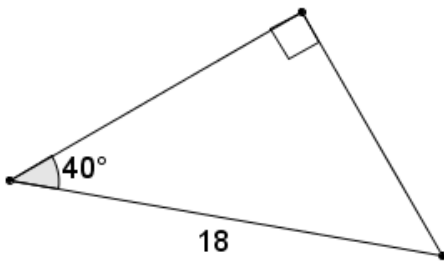
- Find the approximate length of the hypotenuse.



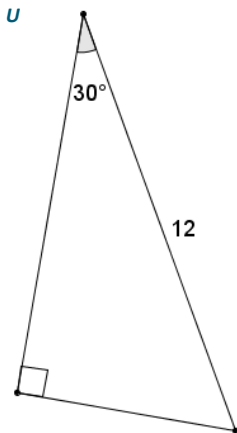
3. Find the approximate length of the hypotenuse.



4. Find the approximate length of the leg adjacent to the 40° angle.

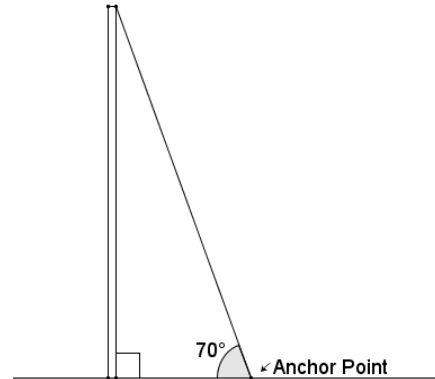


5. Find the length of both legs of the right triangle below. Indicate which leg is adjacent and which is opposite the given angle of 30°.



6. Three city streets form a right triangle. Main Street and State Street are perpendicular. Laura Street and State Street intersect at a 50° angle. The distance along Laura Street to Main Street is 0.8 mile. If Laura Street is closed between Main Street and State Street for a festival, approximately how far (to the nearest tenth) will someone have to travel to get around the festival if they take only Main Street and State Street?

7. A cable anchors a utility pole to the ground as shown in the picture. The cable forms an angle of 70° with the ground. The distance from the base of the utility pole to the anchor point on the ground is 3.8 meters. Approximately how long is the support cable?



8. Indy says that the ratio of $\frac{opp}{adj}$ for an angle of 0° has a value of 0 because the opposite side of the triangle has a length of 0. What does she mean?

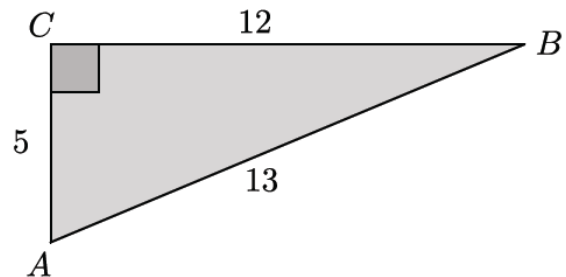
Lesson 26: Definition of Sine, Cosine, and Tangent

Classwork

Exercises 1–3

1. Identify the $\frac{opp}{hyp}$ ratios for angles $\angle A$ and $\angle B$.

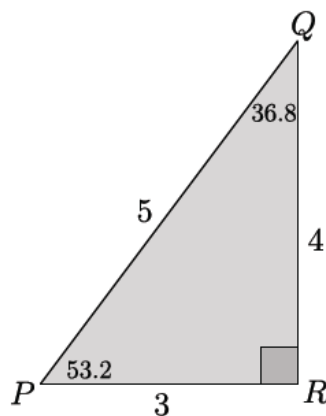
2. Identify the $\frac{adj}{hyp}$ ratios for angles $\angle A$ and $\angle B$.



3. Describe the relationship between the ratios for angles $\angle A$ and $\angle B$.

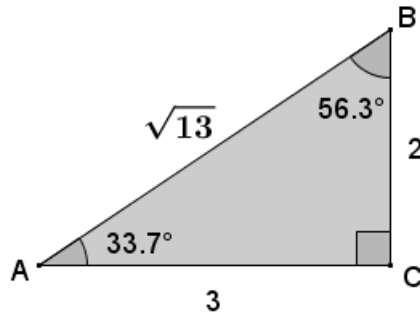
Exercises 4–9

4. In $\triangle PQR$, $m\angle P = 53.2^\circ$ and $m\angle Q = 36.8^\circ$. Complete the following table.



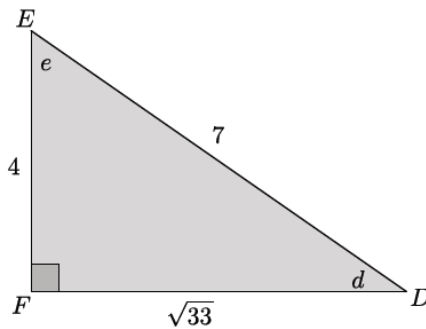
Measure of Angle	Sine $\left(\frac{opp}{hyp}\right)$	Cosine $\left(\frac{adj}{hyp}\right)$	Tangent $\left(\frac{opp}{adj}\right)$
53.2			
36.8			

5. In the triangle below, $m\angle A = 33.7^\circ$ and $m\angle B = 56.3^\circ$. Complete the following table.



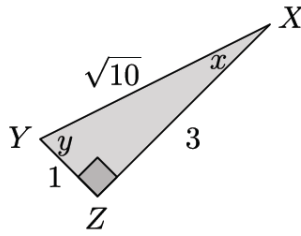
Measure of Angle	Sine	Cosine	Tangent
33.7			
56.3			

6. In the triangle below, let e be the measure of $\angle E$ and d be the measure of $\angle D$. Complete the following table.



Measure of Angle	Sine	Cosine	Tangent
d			
e			

7. In the triangle below, let x be the measure of $\angle X$ and y be the measure of $\angle Y$. Complete the following table.



Measure of Angle	Sine	Cosine	Tangent
x			
y			

8. Tamer did not finish completing the table below for a diagram similar to the previous problems that the teacher had on the board where p was the measure of $\angle P$ and q was the measure of $\angle Q$. Use any patterns you notice from Exercises 1–4 to complete the table for Tamer.

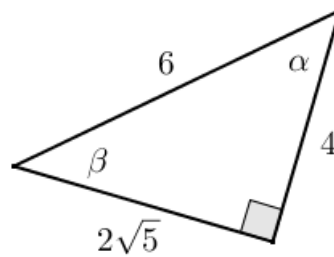
Measure of Angle	Sine	Cosine	Tangent
p	$\sin p = \frac{11}{\sqrt{157}}$	$\cos p = \frac{6}{\sqrt{157}}$	$\tan p = \frac{11}{6}$
q			

9. Explain how you were able to determine the sine, cosine, and tangent of $\angle Q$ in Exercise 7.

Problem Set

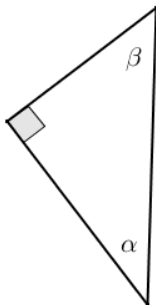
1. Given the triangle in the diagram, complete the following table.

Angle Measure	sin	cos	tan
α			
β			



2. Given the table of values below (not in simplest radical form), label the sides and angles in the right triangle.

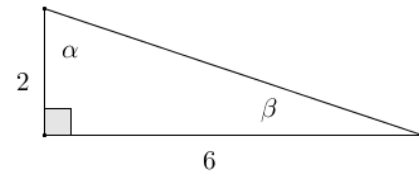
Angle Measure	sin	cos	tan
α	$\frac{4}{2\sqrt{10}}$	$\frac{2\sqrt{6}}{2\sqrt{10}}$	$\frac{4}{2\sqrt{6}}$
β	$\frac{2\sqrt{6}}{2\sqrt{10}}$	$\frac{4}{2\sqrt{10}}$	$\frac{2\sqrt{6}}{4}$



3. Given $\sin \alpha$ and $\sin \beta$, complete the missing values in the table. You may draw a diagram to help you.

Angle Measure	sin	cos	tan
α	$\frac{\sqrt{2}}{3\sqrt{3}}$	$\frac{5}{3\sqrt{3}}$	
β			

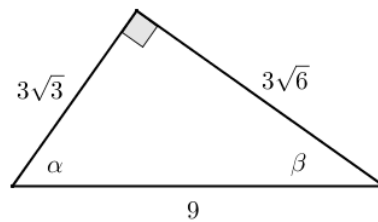
4. Given the triangle shown to the right, fill in the missing values in the table.



Angle Measure	sin	cos	tan
α			
β			

5. Jules thinks that if α and β are two different acute angle measures, then $\sin \alpha \neq \sin \beta$. Do you agree or disagree? Explain.

6. Given the triangle in the diagram, complete the following table.



Angle Measure	sin	cos	tan
α			
β			

Rewrite the values from the table in simplest terms.

Angle Measure	sin	cos	tan
α			
β			

Draw and label the sides and angles of a right triangle using the values of the ratios sin and cos. How is the new triangle related to the original triangle?

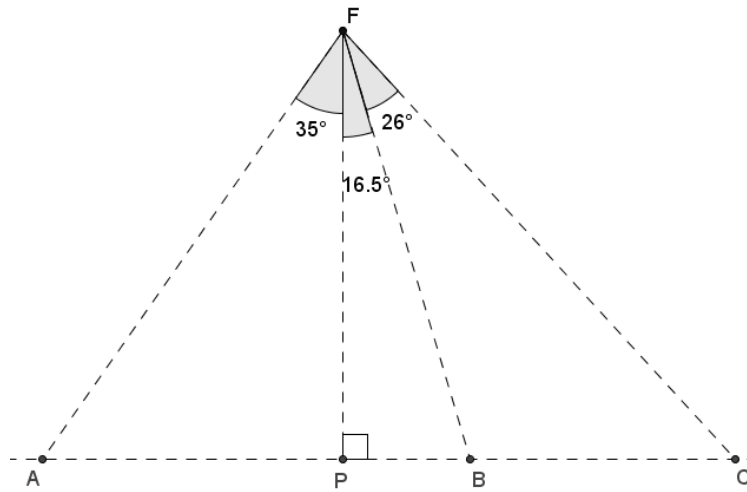
7. Given $\tan \alpha$ and $\cos \beta$, in simplest terms, find the missing side lengths of the right triangle if one leg of the triangle has a length of 4. Draw and label the sides and angles of the right triangle.

Angle	sin θ	cos θ	tan θ
α			
β			

8. Eric wants to hang a rope bridge over a small ravine so that it is easier to cross. To hang the bridge, he needs to know how much rope is needed to span the distance between two trees that are directly across from each other on either side of the ravine. Help Eric devise a plan using sine, cosine, and tangent to determine the approximate distance from tree A to tree B without having to cross the ravine.



9. A fisherman is at point F on the open sea and has three favorite fishing locations. The locations are indicated by points A , B , and C . The fisherman plans to sail from F to A , then to B , then to C , then back to F . If the fisherman is 14 miles from \overline{AC} , find the total distance that he will sail.



Lesson 27: Sine and Cosine of Complementary and Special Angles

Classwork

Example 1

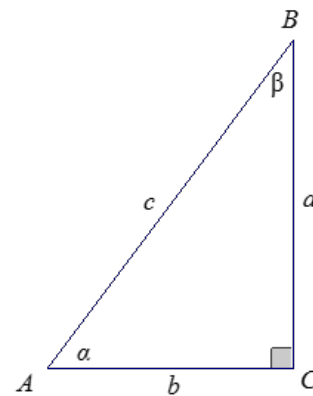
If α and β are the measurements of complementary angles, then we are going to show that $\sin \alpha = \cos \beta$.

In right triangle ABC , the measurement of acute angle $\angle A$ is denoted by α , and the measurement of acute angle $\angle B$ is denoted by β .

Determine the following values in the table:

$\sin \alpha$	$\sin \beta$	$\cos \alpha$	$\cos \beta$

What can you conclude from the results?

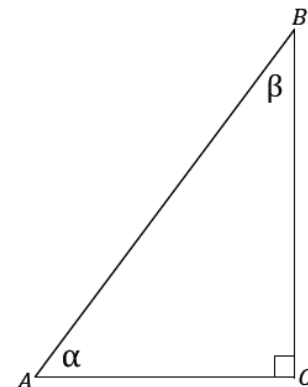


Exercises 1–3

1. Consider the right triangle ABC so that $\angle C$ is a right angle, and the degree measures of $\angle A$ and $\angle B$ are α and β , respectively.

a. Find $\alpha + \beta$.

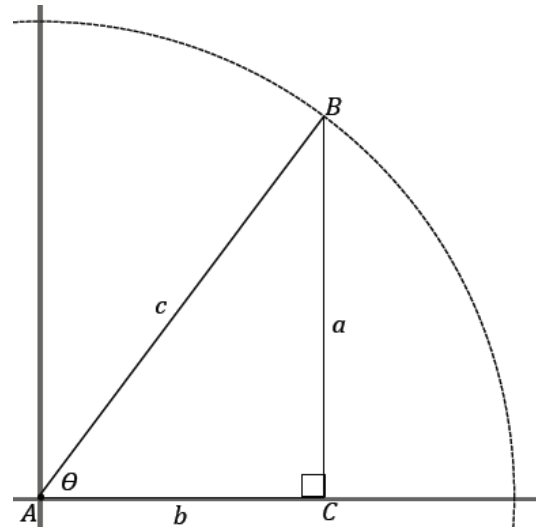
b. Use trigonometric ratios to describe $\frac{BC}{AB}$ two different ways.



- c. Use trigonometric ratios to describe $\frac{AC}{AB}$ two different ways.
- d. What can you conclude about $\sin \alpha$ and $\cos \beta$?
- e. What can you conclude about $\cos \alpha$ and $\sin \beta$?
2. Find values for θ that make each statement true.
- a. $\sin \theta = \cos (25)$
- b. $\sin 80 = \cos \theta$
- c. $\sin \theta = \cos (\theta + 10)$
- d. $\sin (\theta - 45) = \cos (\theta)$
3. For what angle measurement must sine and cosine have the same value? Explain how you know.

Example 2

What is happening to a and b as θ changes? What happens to $\sin \theta$ and $\cos \theta$?



Example 3

There are certain special angles where it is possible to give the exact value of sine and cosine. These are the angles that measure 0° , 30° , 45° , 60° , and 90° ; these angle measures are frequently seen.

You should memorize the sine and cosine of these angles with quick recall just as you did your arithmetic facts.

- a. Learn the following sine and cosine values of the key angle measurements.

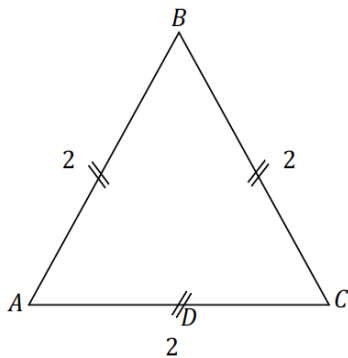
θ	0°	30°	45°	60°	90°
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

We focus on an easy way to remember the entries in the table. What do you notice about the table values?

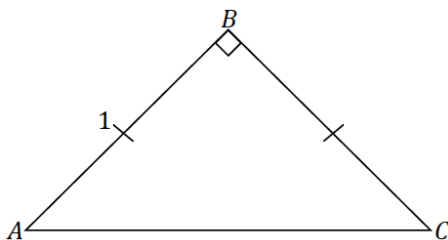
This is easily explained because the pairs $(0,90)$, $(30,60)$, and $(45,45)$ are the measures of complementary angles. So, for instance, $\sin 30 = \cos 60$.

The sequence $0, \frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, 1$ may be easier to remember as the sequence $\frac{\sqrt{0}}{2}, \frac{\sqrt{1}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{4}}{2}$.

- b. $\triangle ABC$ is equilateral, with side length 2; D is the midpoint of side AC . Label all side lengths and angle measurements for $\triangle ABD$. Use your figure to determine the sine and cosine of 30 and 60.



- c. Draw an isosceles right triangle with legs of length 1. What are the measures of the acute angles of the triangle? What is the length of the hypotenuse? Use your triangle to determine sine and cosine of the acute angles.

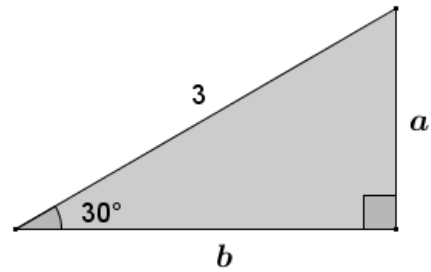


Parts (b) and (c) demonstrate how the sine and cosine values of the mentioned special angles can be found. These triangles are common to trigonometry; we refer to the triangle in part (b) as a 30–60–90 triangle and the triangle in part (c) as a 45–45–90 triangle.

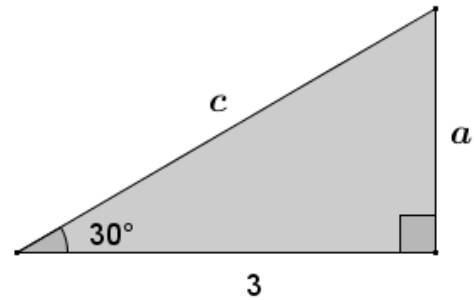
30–60–90 Triangle, side length ratio 1: 2: $\sqrt{3}$	45–45–90 Triangle, side length ratio 1: 1: $\sqrt{2}$
2: 4: $2\sqrt{3}$	2: 2: $2\sqrt{2}$
3: 6: $3\sqrt{3}$	3: 3: $3\sqrt{2}$
4: 8: $4\sqrt{3}$	4: 4: $4\sqrt{2}$
x : $2x$: $x\sqrt{3}$	x : x : $x\sqrt{2}$

Exercises 4–5

4. Find the missing side lengths in the triangle.

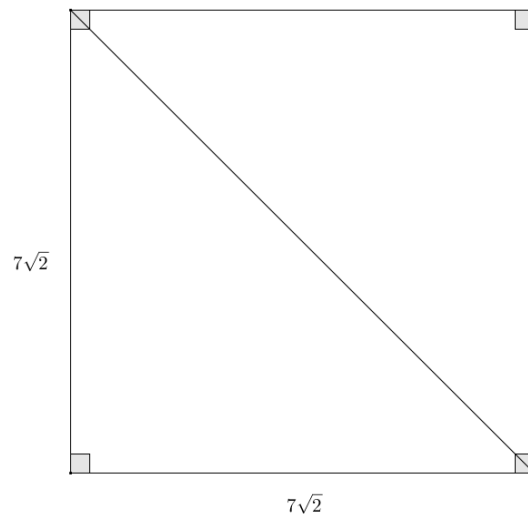


5. Find the missing side lengths in the triangle.



Problem Set

- Find the value of θ that makes each statement true.
 - $\sin \theta = \cos(\theta + 38)$
 - $\cos \theta = \sin(\theta - 30)$
 - $\sin \theta = \cos(3\theta + 20)$
 - $\sin\left(\frac{\theta}{3} + 10\right) = \cos \theta$
- Make a prediction about how the sum $\sin 30 + \cos 60$ will relate to the sum $\sin 60 + \cos 30$.
 - Use the sine and cosine values of special angles to find the sum: $\sin 30 + \cos 60$.
 - Find the sum $\sin 60 + \cos 30$.
 - Was your prediction a valid prediction? Explain why or why not.
- Langdon thinks that the sum $\sin 30 + \sin 30$ is equal to $\sin 60$. Do you agree with Langdon? Explain what this means about the sum of the sines of angles.
- A square has side lengths of $7\sqrt{2}$. Use sine or cosine to find the length of the diagonal of the square. Confirm your answer using the Pythagorean theorem.



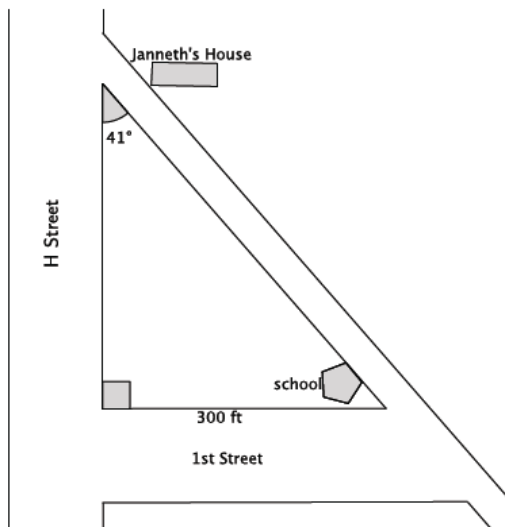
- Given an equilateral triangle with sides of length 9, find the length of the altitude. Confirm your answer using the Pythagorean theorem.

Lesson 28: Solving Problems Using Sine and Cosine

Classwork

Exercise 1–4

1.
 - a. The bus drops you off at the corner of H Street and 1st Street, approximately 300 ft. from school. You plan to walk to your friend Janneth's house after school to work on a project. Approximately how many feet will you have to walk from school to Janneth's house? Round your answer to the nearest foot. (Hint: Use the ratios you developed in Lesson 25.)



- b. In real life, it is unlikely that you would calculate the distance between school and Janneth's house in this manner. Describe a similar situation in which you might actually want to determine the distance between two points using a trigonometric ratio.

2. Use a calculator to find the sine and cosine of θ . Give your answer rounded to the ten-thousandth place.

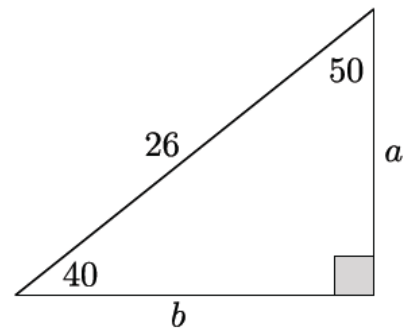
θ	0	10	20	30	40	50	60	70	80	90
$\sin \theta$										
$\cos \theta$										

3. What do you notice about the numbers in the row $\sin \theta$ compared with the numbers in the row $\cos \theta$?

4. Provide an explanation for what you noticed in Exercise 2.

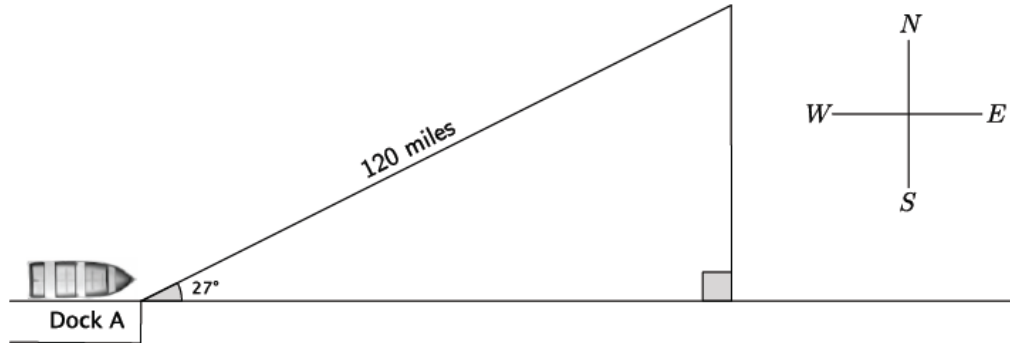
Example 1

Find the values of a and b .



Exercise 5

5. A shipmate set a boat to sail exactly 27° NE from the dock. After traveling 120 miles, the shipmate realized he had misunderstood the instructions from the captain; he was supposed to set sail going directly east!

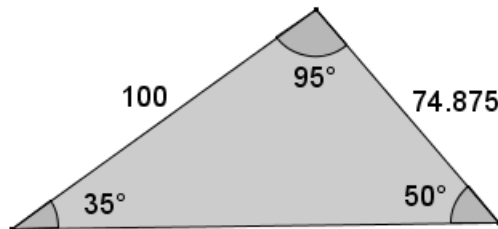


- a. How many miles will the shipmate have to travel directly south before he is directly east of the dock? Round your answer to the nearest mile.
- b. How many extra miles does the shipmate travel by going the wrong direction compared to going directly east? Round your answer to the nearest mile.

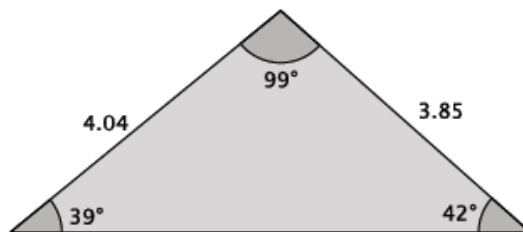
Example 2

Johanna borrowed some tools from a friend so that she could precisely, but not exactly, measure the corner space in her backyard to plant some vegetables. She wants to build a fence to prevent her dog from digging up the seeds that she plants. Johanna returned the tools to her friend before making the most important measurement: the one that would give the length of the fence!

Johanna decided that she could just use the Pythagorean theorem to find the length of the fence she'd need. Is the Pythagorean theorem applicable in this situation? Explain.

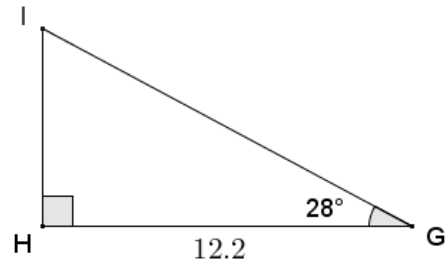
**Exercise 6**

6. The measurements of the triangle shown below are rounded to the nearest hundredth. Calculate the missing side length to the nearest hundredth.

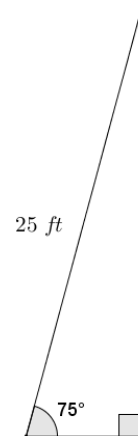


Problem Set

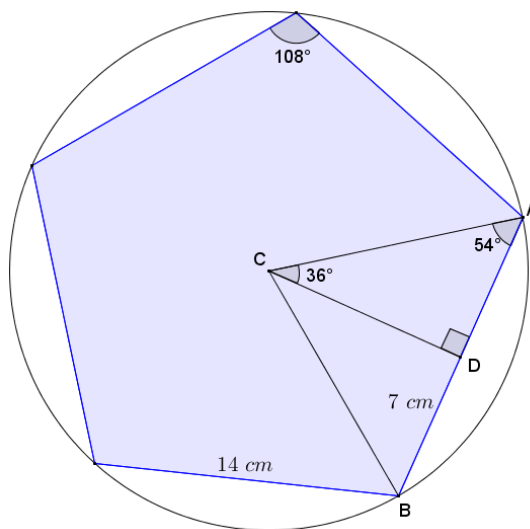
- Given right triangle GHI , with right angle at H , $GH = 12.2$, and $m\angle G = 28^\circ$, find the measures of the remaining sides and angle to the nearest tenth.



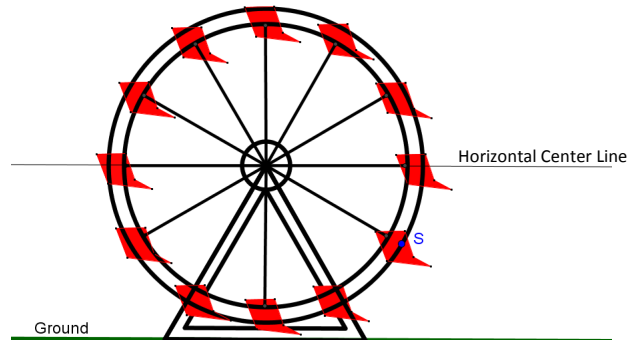
- The Occupational Safety and Health Administration (OSHA) provides standards for safety at the workplace. A ladder is leaned against a vertical wall according to OSHA standards and forms an angle of approximately 75° with the floor.
 - If the ladder is 25 ft. long, what is the distance from the base of the ladder to the base of the wall?
 - How high on the wall does the ladder make contact?
 - Describe how to safely set a ladder according to OSHA standards without using a protractor.



- A regular pentagon with side lengths of 14 cm is inscribed in a circle. What is the radius of the circle?



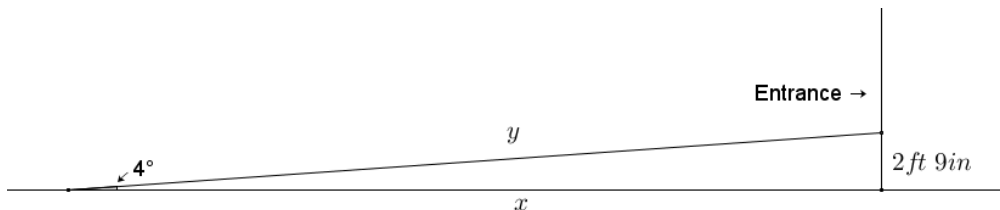
4. The circular frame of a Ferris wheel is suspended so that it sits 4 ft. above the ground and has a radius of 30 ft. A segment joins center C to point S on the circle. If \overline{CS} makes an angle of 48° with the horizon, what is the distance of point S to the ground?



5. Tim is a contractor who is designing a wheelchair ramp for handicapped access to a business. According to the Americans with Disabilities Act (ADA), the maximum slope allowed for a public wheelchair ramp forms an angle of approximately 4.76° to level ground. The length of a ramp's surface cannot exceed 30 ft. without including a flat 5 ft. \times 5 ft. platform (minimum dimensions) on which a person can rest, and such a platform must be included at the bottom and top of any ramp.

Tim designs a ramp that forms an angle of 4° to the level ground to reach the entrance of the building. The entrance of the building is 2 ft. 9 in. above the ground. Let x and y as shown in Tim's initial design below be the indicated distances in feet.

- a. Assuming that the ground in front of the building's entrance is flat, use Tim's measurements and the ADA requirements to complete and/or revise his wheelchair ramp design.



(For more information, see section 405 of the 2010 ADA Standards for Accessible Design at the following link: <http://www.ada.gov/regs2010/2010ADASTandards/2010ADASTandards.htm#pgfid-1006877>.)

- b. What is the total distance from the start of the ramp to the entrance of the building in your design?
6. Tim is designing a roof truss in the shape of an isosceles triangle. The design shows the base angles of the truss to have measures of 18.5° . If the horizontal base of the roof truss is 36 ft. across, what is the height of the truss?

Lesson 29: Applying Tangents

Classwork

Opening Exercise

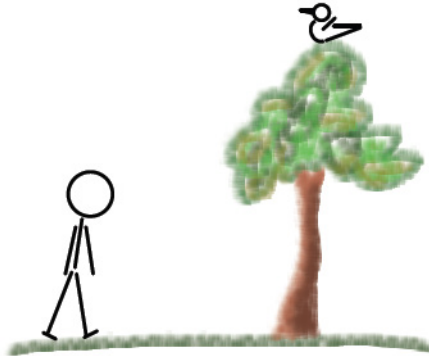
- a. Use a calculator to find the tangent of θ . Give your answer correct to four decimal places.

θ	0	10	20	30	40	50	60	70	80	90
$\sin \theta$	0	0.1736	0.3420	0.5	0.6428	0.7660	0.8660	0.9397	0.9848	1
$\cos \theta$	1	0.9848	0.9397	0.8660	0.7660	0.6428	0.5	0.3420	0.1736	0
$\frac{\sin \theta}{\cos \theta}$										
$\tan \theta$										

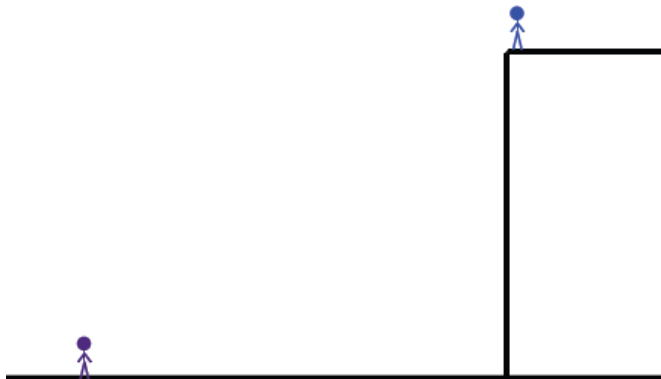
- b. The table from Lesson 29 is provided here for you. In the row labeled $\frac{\sin \theta}{\cos \theta}$, divide the sine values by the cosine values. What do you notice?

Example 1

Scott, whose eye level is 1.5 m above the ground, stands 30 m from a tree. The angle of elevation of a bird at the top of the tree is 36° . How far above ground is the bird?

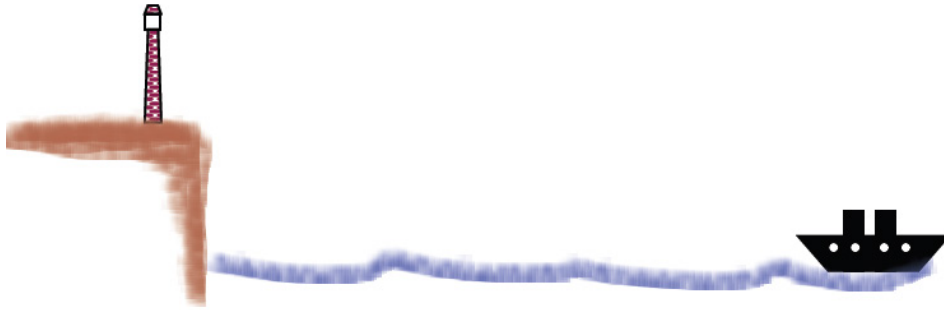
**Example 2**

From an angle of depression of 40° , John watches his friend approach his building while standing on the rooftop. The rooftop is 16 m from the ground, and John's eye level is at about 1.8 m from the rooftop. What is the distance between John's friend and the building?



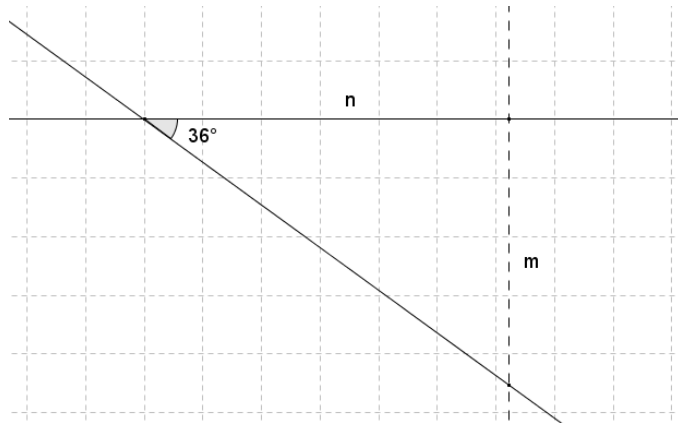
Exercise 1

Standing on the gallery of a lighthouse (the deck at the top of a lighthouse), a person spots a ship at an angle of depression of 20° . The lighthouse is 28 m tall and sits on a cliff 45 m tall as measured from sea level. What is the horizontal distance between the lighthouse and the ship? Sketch a diagram to support your answer.



Exercise 2

A line on the coordinate plane makes an angle of depression of 36° . Find the slope of the line, correct to four decimal places.

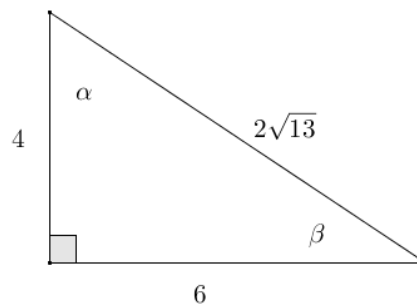


Problem Set

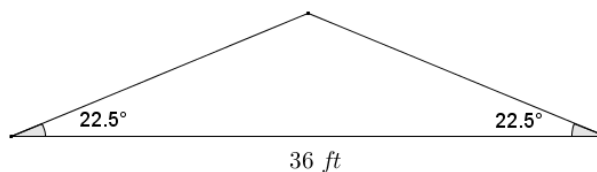
1. A line in the coordinate plane has an angle of elevation of 53° . Find the slope of the line correct to four decimal places.
2. A line in the coordinate plane has an angle of depression of 25° . Find the slope of the line correct to four decimal places.
3. In Problems 1 and 2, why do the lengths of the legs of the right triangles formed not affect the slope of the line?
4. Given the angles of depression below, determine the slope of the line with the indicated angle correct to four decimal places.
 - a. 35° angle of depression
 - b. 49° angle of depression
 - c. 80° angle of depression
 - d. 87° angle of depression
 - e. 89° angle of depression
 - f. 89.9° angle of depression
 - g. What appears to be happening to the slopes (and tangent values) as the angles of depression get closer to 90° ?
 - h. Find the slopes of angles of depression that are even closer to 90° than 89.9° . Can the value of the tangent of 90° be defined? Why or why not?

5. For the indicated angle, express the quotient in terms of sine, cosine, or tangent. Then write the quotient in simplest terms.

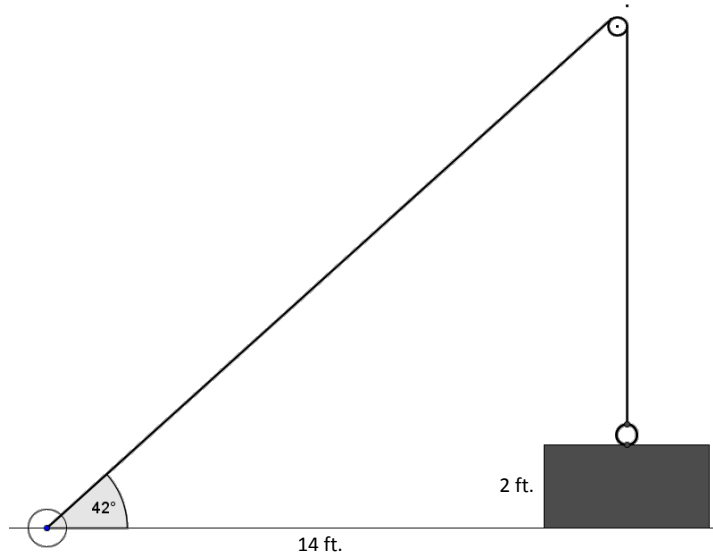
- a. $\frac{4}{2\sqrt{13}}$; α
- b. $\frac{6}{4}$; α
- c. $\frac{4}{2\sqrt{13}}$; β
- d. $\frac{4}{6}$; β



6. The pitch of a roof on a home is expressed as a ratio of vertical rise: horizontal run where the run has a length of 12 units. If a given roof design includes an angle of elevation of 22.5° , and the roof spans 36 ft. as shown in the diagram, determine the pitch of the roof. Then determine the distance along one of the two sloped surfaces of the roof.



7. An anchor cable supports a vertical utility pole forming a 51° angle with the ground. The cable is attached to the top of the pole. If the distance from the base of the pole to the base of the cable is 5 meters, how tall is the pole?
8. A winch is a tool that rotates a cylinder, around which a cable is wound. When the winch rotates in one direction, it draws the cable in. Joey is using a winch and a pulley (as shown in the diagram) to raise a heavy box off the floor and onto a cart. The box is 2 ft. tall, and the winch is 14 ft. horizontally from where cable drops down vertically from the pulley. The angle of elevation to the pulley is 42° . What is the approximate length of cable required to connect the winch and the box?



Lesson 30: Trigonometry and the Pythagorean Theorem

Classwork

Exercises 1–2

1. In a right triangle, with acute angle of measure θ , $\sin \theta = \frac{1}{2}$. What is the value of $\cos \theta$? Draw a diagram as part of your response.

2. In a right triangle, with acute angle of measure θ , $\sin \theta = \frac{7}{9}$. What is the value of $\tan \theta$? Draw a diagram as part of your response.

Example 1

- a. What common right triangle was probably modeled in the construction of the triangle in Figure 2? Use $\sin 53^\circ \approx 0.8$.

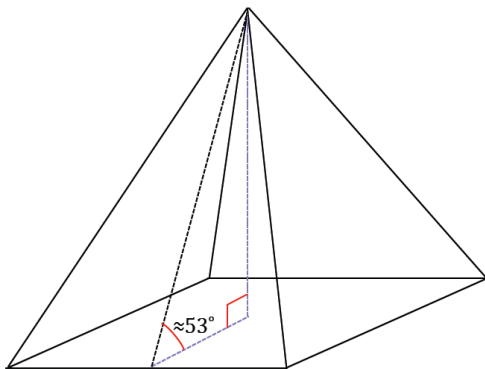
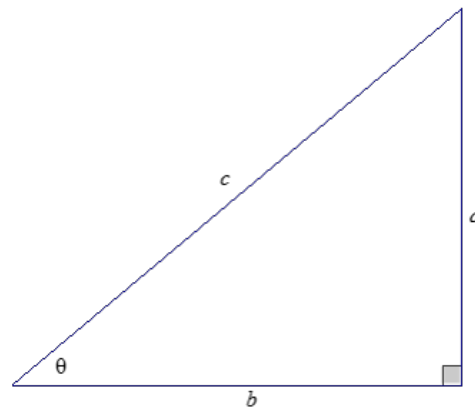


Figure 1

- b. The actual angle between the base and lateral faces of the pyramid is actually closer to 52° . Considering the age of the pyramid, what could account for the difference between the angle measure in part (a) and the actual measure?
- c. Why do you think the architects chose to use a 3–4–5 as a model for the triangle?

Example 2

Show why $\tan \theta = \frac{\sin \theta}{\cos \theta}$.

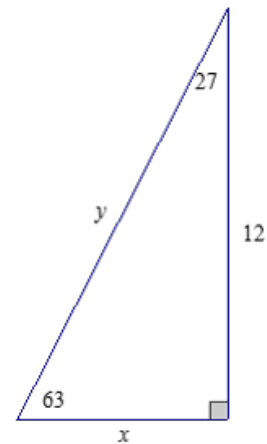


Exercises 3–4

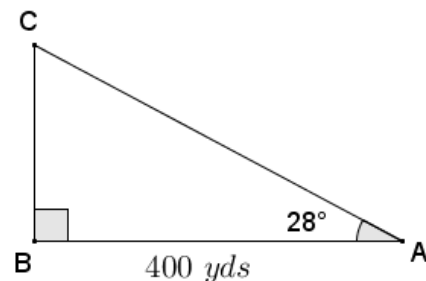
3. In a right triangle, with acute angle of measure θ , $\sin \theta = \frac{1}{2}$, use the Pythagorean identity to determine the value of $\cos \theta$.
4. Given a right triangle, with acute angle of measure θ , $\sin \theta = \frac{7}{9}$, use the Pythagorean identity to determine the value of $\tan \theta$.

Problem Set

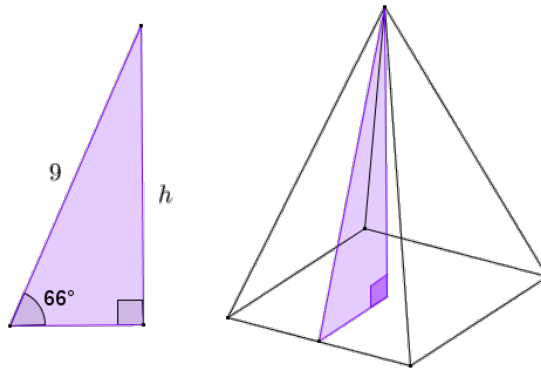
1. If $\cos \theta = \frac{4}{5}$, find $\sin \theta$ and $\tan \theta$.
2. If $\sin \theta = \frac{44}{125}$, find $\cos \theta$ and $\tan \theta$.
3. If $\tan \theta = 5$, find $\sin \theta$ and $\cos \theta$.
4. If $\sin \theta = \frac{\sqrt{5}}{5}$, find $\cos \theta$ and $\tan \theta$.
5. Find the missing side lengths of the following triangle using sine, cosine, and/or tangent. Round your answer to four decimal places.



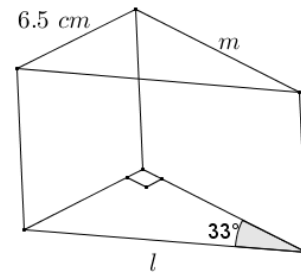
6. A surveying crew has two points A and B marked along a roadside at a distance of 400 yd. A third point C is marked at the back corner of a property along a perpendicular to the road at B . A straight path joining C to A forms a 28° angle with the road. Find the distance from the road to point C at the back of the property and the distance from A to C using sine, cosine, and/or tangent. Round your answer to three decimal places.



7. The right triangle shown is taken from a slice of a right rectangular pyramid with a square base.
- Find the height of the pyramid (to the nearest tenth).
 - Find the lengths of the sides of the base of the pyramid (to the nearest tenth).
 - Find the lateral surface area of the right rectangular pyramid.



8. A machinist is fabricating a wedge in the shape of a right triangular prism. One acute angle of the right triangular base is 33° , and the opposite side is 6.5 cm . Find the length of the edges labeled l and m using sine, cosine, and/or tangent. Round your answer to the nearest thousandth of a centimeter.



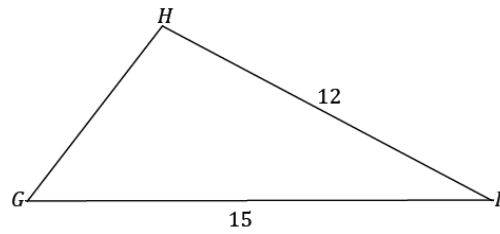
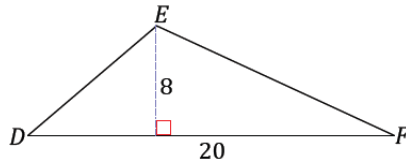
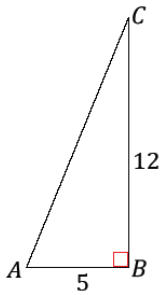
9. Let $\sin \theta = \frac{l}{m}$, where $l, m > 0$. Express $\tan \theta$ and $\cos \theta$ in terms of l and m .

Lesson 31: Using Trigonometry to Determine Area

Classwork

Opening Exercise

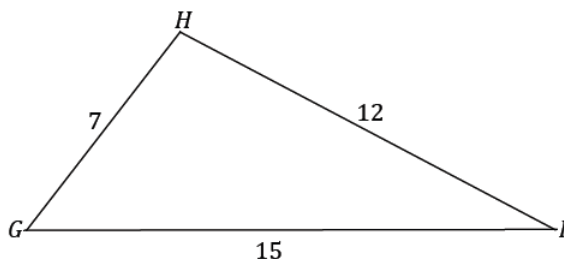
Three triangles are presented below. Determine the areas for each triangle, if possible. If it is not possible to find the area with the provided information, describe what is needed in order to determine the area.



Is there a way to find the missing information?

Example 1

What if the third side length of the triangle were provided? Is it possible to determine the area of the triangle now? Find the area of $\triangle GHI$.

**Example 2**

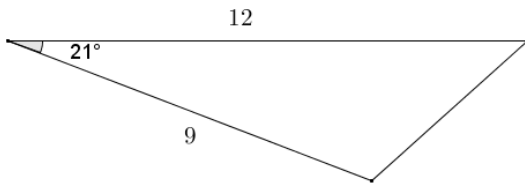
A farmer is planning how to divide his land for planting next year's crops. A triangular plot of land is left with two known side lengths measuring 500 m and 1,700 m.

What could the farmer do next in order to find the area of the plot?

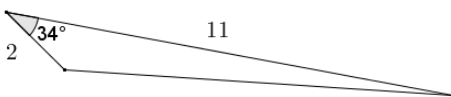
Problem Set

Find the area of each triangle. Round each answer to the nearest tenth.

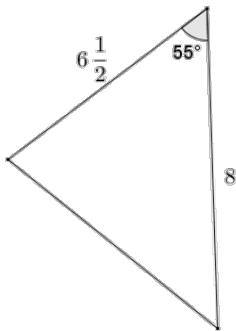
1.



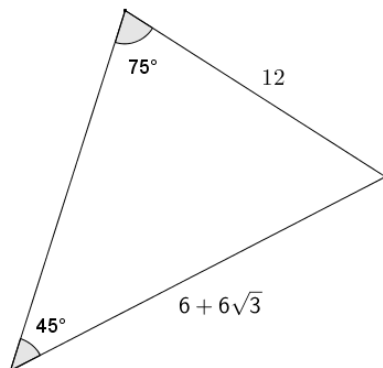
2.



3.

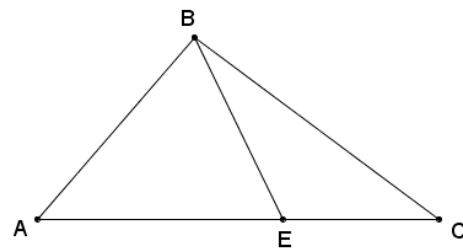


4.

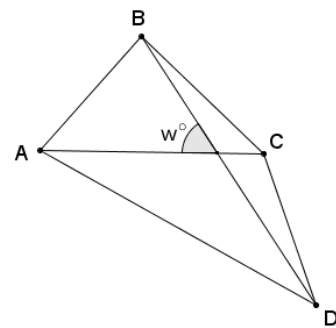


5. In $\triangle DEF$, $EF = 15$, $DF = 20$, and $\angle F = 63^\circ$. Determine the area of the triangle. Round to the nearest tenth.
6. A landscape designer is designing a flower garden for a triangular area that is bounded on two sides by the client's house and driveway. The length of the edges of the garden along the house and driveway are 18 ft. and 8 ft. respectively, and the edges come together at an angle of 80° . Draw a diagram, and then find the area of the garden to the nearest square foot.

- A right rectangular pyramid has a square base with sides of length 5. Each lateral face of the pyramid is an isosceles triangle. The angle on each lateral face between the base of the triangle and the adjacent edge is 75° . Find the surface area of the pyramid to the nearest tenth.
- The Pentagon Building in Washington D.C. is built in the shape of a regular pentagon. Each side of the pentagon measures 921 ft. in length. The building has a pentagonal courtyard with the same center. Each wall of the center courtyard has a length of 356 ft. What is the approximate area of the roof of the Pentagon Building?
- A regular hexagon is inscribed in a circle with a radius of 7. Find the perimeter and area of the hexagon.
- In the figure below, $\angle AEB$ is acute. Show that $\text{Area}(\triangle ABC) = \frac{1}{2} AC \cdot BE \cdot \sin \angle AEB$.



- Let $ABCD$ be a quadrilateral. Let w be the measure of the acute angle formed by diagonals \overline{AC} and \overline{BD} . Show that $\text{Area}(ABCD) = \frac{1}{2} AC \cdot BD \cdot \sin w$.
(Hint: Apply the result from Problem 10 to $\triangle ABC$ and $\triangle ACD$.)

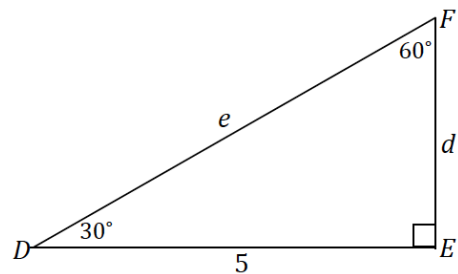


Lesson 32: Using Trigonometry to Find Side Lengths of an Acute Triangle

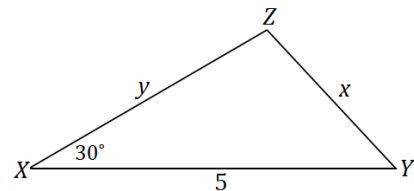
Classwork

Opening Exercise

- a. Find the lengths of d and e .

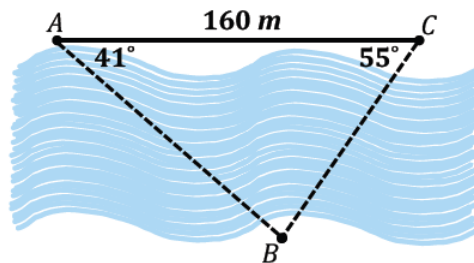


- b. Find the lengths of x and y . How is this different from part (a)?



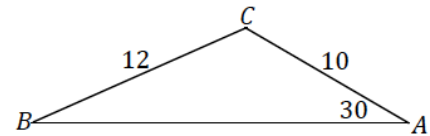
Example 1

A surveyor needs to determine the distance between two points A and B that lie on opposite banks of a river. A point C is chosen 160 meters from point A , on the same side of the river as A . The measures of angles $\angle BAC$ and $\angle ACB$ are 41° and 55° , respectively. Approximate the distance from A to B to the nearest meter.

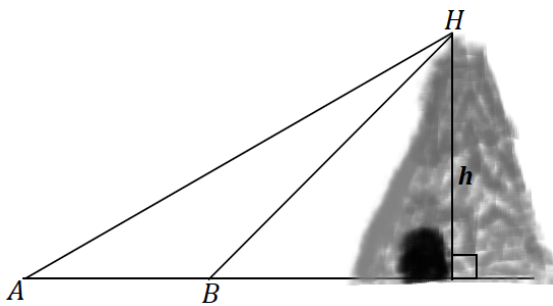


Exercises 1–2

1. In $\triangle ABC$, $m\angle A = 30^\circ$, $a = 12$, and $b = 10$. Find $\sin\angle B$. Include a diagram in your answer.

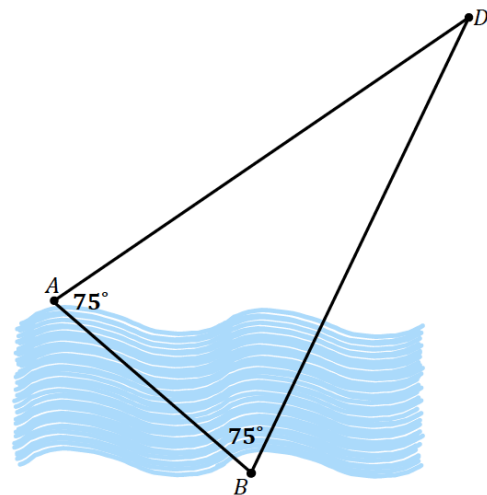


2. A car is moving towards a tunnel carved out of the base of a hill. As the accompanying diagram shows, the top of the hill, H , is sighted from two locations, A and B . The distance between A and B is 250 ft. What is the height, h , of the hill to the nearest foot?



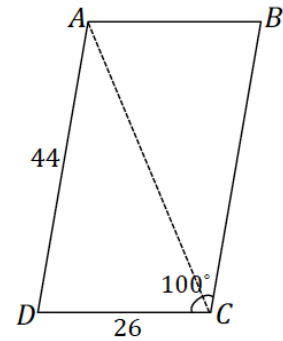
Example 2

Our friend the surveyor from Example 1 is doing some further work. He has already found the distance between points A and B (from Example 1). Now he wants to locate a point D that is equidistant from both A and B and on the same side of the river as A . He has his assistant mark the point D so that the angles $\angle ABD$ and $\angle BAD$ both measure 75° . What is the distance between D and A to the nearest meter?



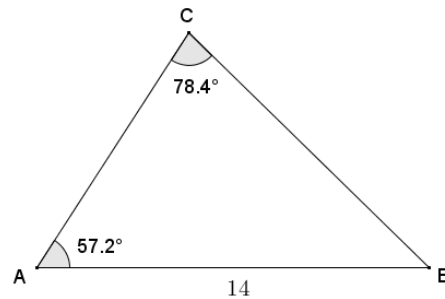
Exercise 3

3. Parallelogram $ABCD$ has sides of lengths 44 mm and 26 mm, and one of the angles has a measure of 100° . Approximate the length of diagonal AC to the nearest mm.



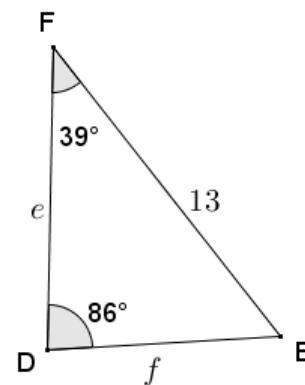
Problem Set

- Given $\triangle ABC$, $AB = 14$, $\angle A = 57.2^\circ$, and $\angle C = 78.4^\circ$, calculate the measure of angle B to the nearest tenth of a degree, and use the Law of Sines to find the lengths of AC and BC to the nearest tenth.

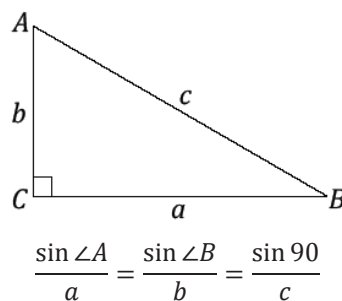


Calculate the area of $\triangle ABC$ to the nearest square unit.

- Given $\triangle DEF$, $\angle F = 39^\circ$, and $EF = 13$, calculate the measure of $\angle E$, and use the Law of Sines to find the lengths of \overline{DF} and \overline{DE} to the nearest hundredth.



- Does the law of sines apply to a right triangle? Based on $\triangle ABC$, the following ratios were set up according to the law of sines.



Fill in the partially completed work below.

$$\frac{\sin \angle A}{a} = \frac{\sin 90}{c}$$

$$\frac{\sin \angle A}{a} = \frac{\sin 90}{c}$$

$$\sin \angle A = \frac{\sin 90}{c}$$

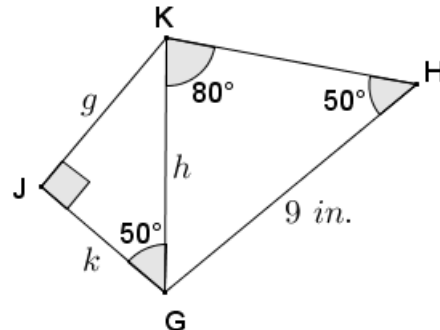
$$\frac{\sin \angle B}{b} = \frac{\sin 90}{c}$$

$$\frac{\sin \angle B}{b} = \frac{\sin 90}{c}$$

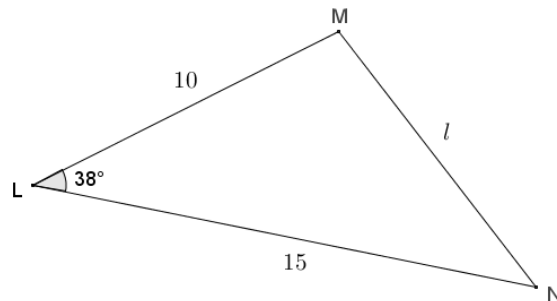
$$\sin \angle B = \frac{\sin 90}{c}$$

What conclusions can we draw?

4. Given quadrilateral $GHKJ$, $\angle H = 50^\circ$, $\angle HKG = 80^\circ$, $\angle KGJ = 50^\circ$, $\angle J$ is a right angle and $GH = 9$ in., use the law of sines to find the length of GK , and then find the lengths of \overline{GJ} and \overline{JK} to the nearest tenth of an inch.



5. Given triangle LMN , $LM = 10$, $LN = 15$, and $\angle L = 38^\circ$, use the law of cosines to find the length of \overline{MN} to the nearest tenth.



6. Given triangle ABC , $AC = 6$, $AB = 8$, and $\angle A = 78^\circ$. Draw a diagram of triangle ABC , and use the law of cosines to find the length of \overline{BC} .

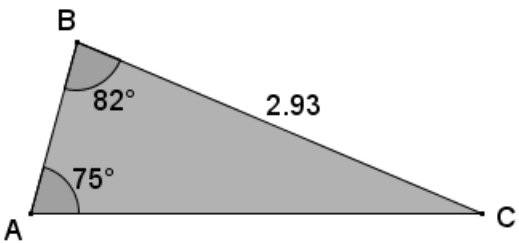
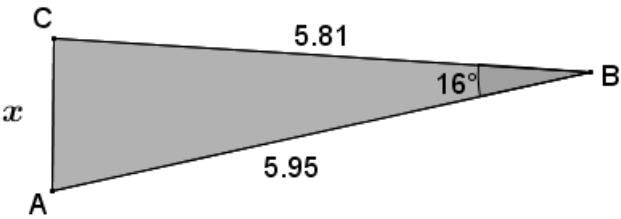
Calculate the area of triangle ABC .

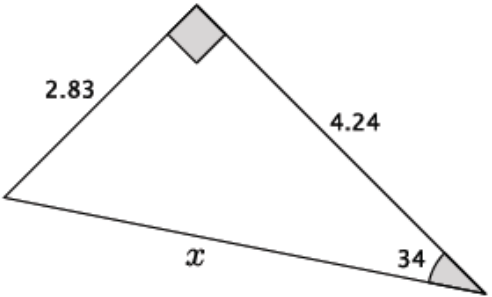
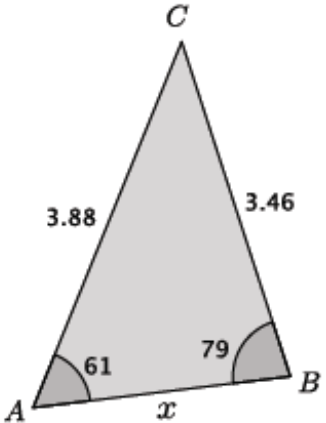
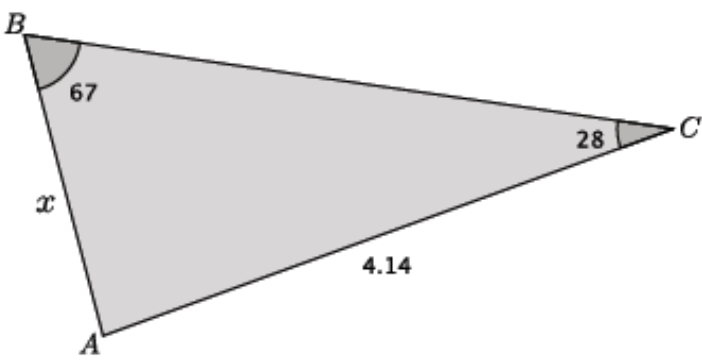
Lesson 33: Applying Laws of Sines and Cosines

Classwork

Opening Exercise

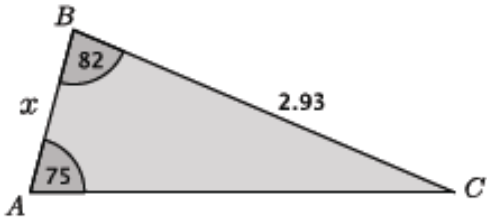
For each triangle shown below, identify the method (Pythagorean theorem, law of sines, law of cosines) you would use to find each length x .

 <p>Triangle ABC is shown. Angle B is 82°, angle A is 75°, and side BC is 2.93. Side AC is labeled x.</p>	
 <p>Triangle ABC is shown. Angle B is 16°, side BC is 5.81, side AB is 5.95, and side AC is labeled x.</p>	

 <p>A right-angled triangle with a right angle symbol at the top vertex. The left leg has a length of 2.83, the right leg has a length of 4.24, and the hypotenuse at the bottom is labeled x. The angle at the bottom-right vertex is labeled 34.</p>	
 <p>A triangle with vertices labeled A, B, and C. Side AC has a length of 3.88, side BC has a length of 3.46, and side AB has a length of x. The angle at vertex A is 61, and the angle at vertex B is 79.</p>	
 <p>A triangle with vertices labeled A, B, and C. Side AB has a length of x, side BC has a length of 4.14, and side AC is the longest side. The angle at vertex B is 67, and the angle at vertex C is 28.</p>	

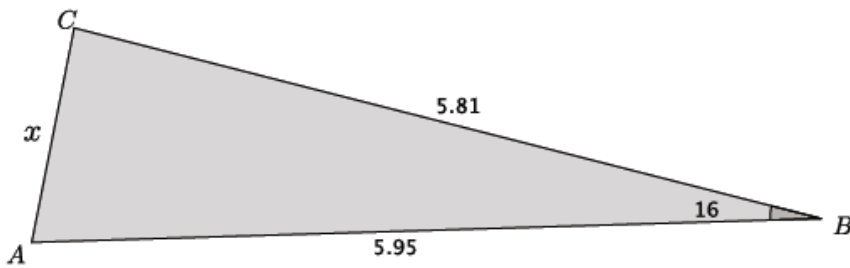
Example 1

Find the missing side length in $\triangle ABC$.



Example 2

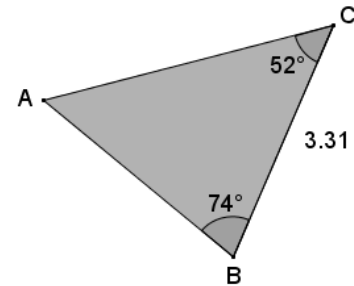
Find the missing side length in $\triangle ABC$.



Exercises 1–6

Use the Laws of sines and cosines to find all missing side lengths for each of the triangles in the Exercises below. Round your answers to the tenths place.

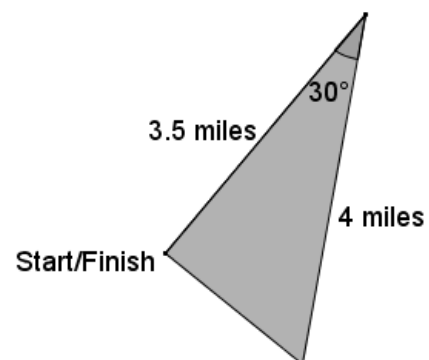
1. Use the triangle below to complete this exercise.
 - a. Identify the method (Pythagorean theorem, law of sines, law of cosines) you would use to find each of the missing lengths of the triangle. Explain why the other methods cannot be used.



- b. Find the lengths of \overline{AC} and \overline{AB} .

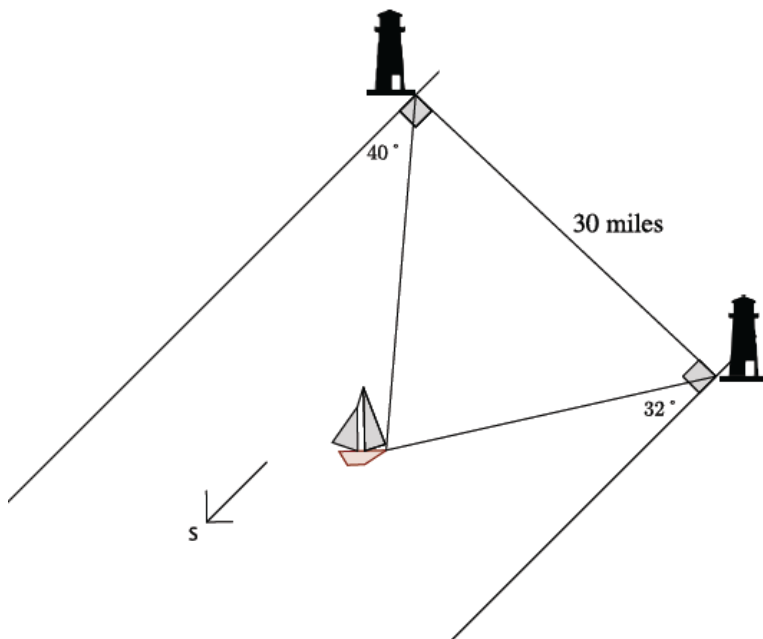
2. Your school is challenging classes to compete in a triathlon. The race begins with a swim along the shore, then continues with a bike ride for 4 miles. School officials want the race to end at the place it began so after the 4-mile bike ride racers must turn 30° and run 3.5 mi. directly back to the starting point.

- a. Identify the method (Pythagorean theorem, law of sines, law of cosines) you would use to find the total length of the race. Explain why the other methods cannot be used.

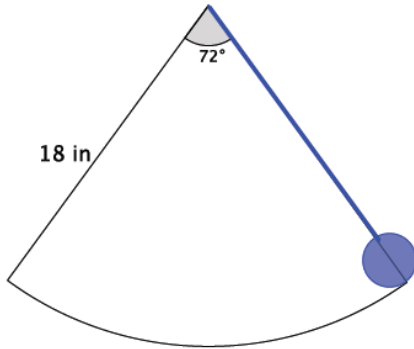


b. Determine the total length of the race. Round your answer to the tenths place.

3. Two lighthouses are 30 mi. apart on each side of shorelines running north and south, as shown. Each lighthouse keeper spots a boat in the distance. One lighthouse keeper notes the location of the boat as 40° east of south, and the other lighthouse keeper marks the boat as 32° west of south. What is the distance from the boat to each of the lighthouses at the time it was spotted? Round your answers to the nearest mile.



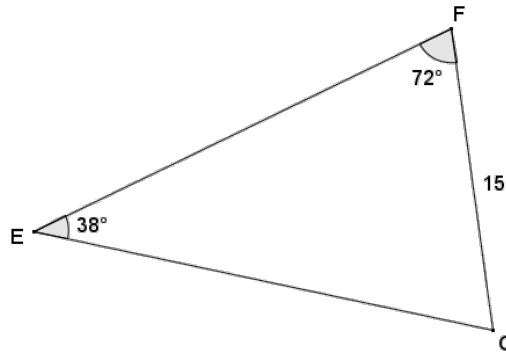
4. A pendulum 18 in. in length swings 72° from right to left. What is the difference between the highest and lowest point of the pendulum? Round your answer to the hundredths place, and explain how you found it.



5. What appears to be the minimum amount of information about a triangle that must be given in order to use the law of sines to find an unknown length?
6. What appears to be the minimum amount of information about a triangle that must be given in order to use the law of cosines to find an unknown length?

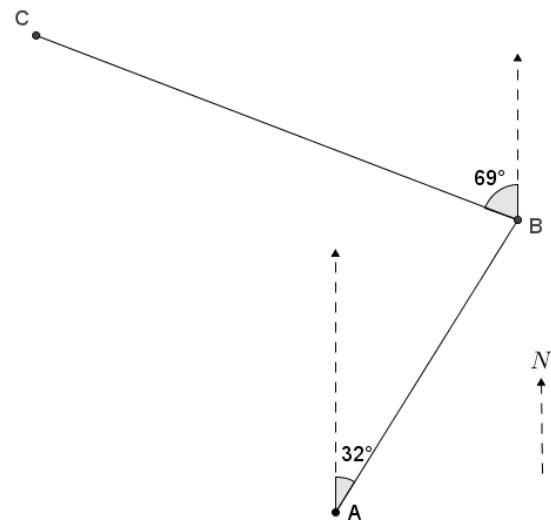
Problem Set

- Given triangle EFG , $FG = 15$, angle E has measure of 38° , and angle F has measure 72° , find the measures of the remaining sides and angle to the nearest tenth. Justify your method.

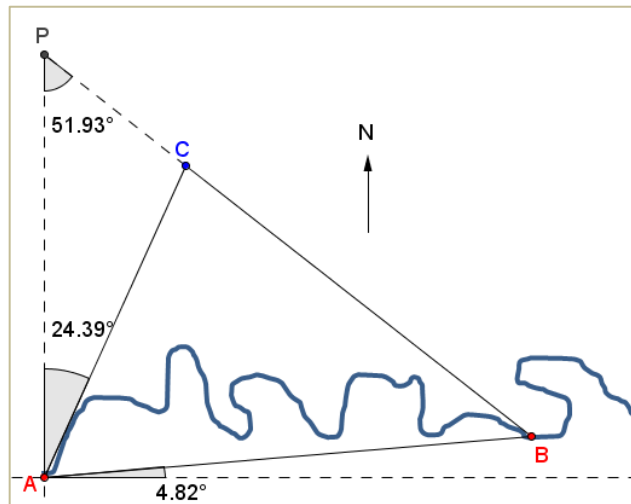


- Given triangle ABC , angle A has measure of 75° , $AC = 15.2$, and $AB = 24$, find BC to the nearest tenth. Justify your method.
- James flies his plane from point A at a bearing of 32° east of north, averaging speed of 143 miles per hour for 3 hours, to get to an airfield at point B . He next flies 69° west of north at an average speed of 129 miles per hour for 4.5 hours to a different airfield at point C .

- Find the distance from A to B .
- Find the distance from B to C ,
- Find the measure of angle ABC .
- Find the distance from C to A .
- What length of time can James expect the return trip from C to A to take?



4. Mark is deciding on the best way to get from point A to point B as shown on the map of Crooked Creek to go fishing. He sees that if he stays on the north side of the creek, he would have to walk around a triangular piece of private property (bounded by \overline{AC} and \overline{BC}). His other option is to cross the creek at A and take a straight path to B , which he knows to be a distance of 1.2 mi. The second option requires crossing the water, which is too deep for his boots and very cold. Find the difference in distances to help Mark decide which path is his better choice.



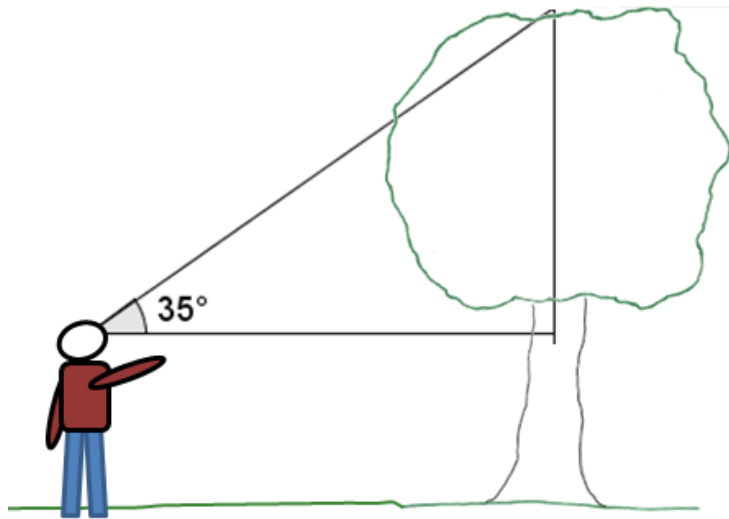
5. If you are given triangle XYZ , and the measures of two of its angles, and two of its sides, would it be appropriate to apply the law of sines or the law of cosines to find the remaining side? Explain.

Lesson 34: Unknown Angles

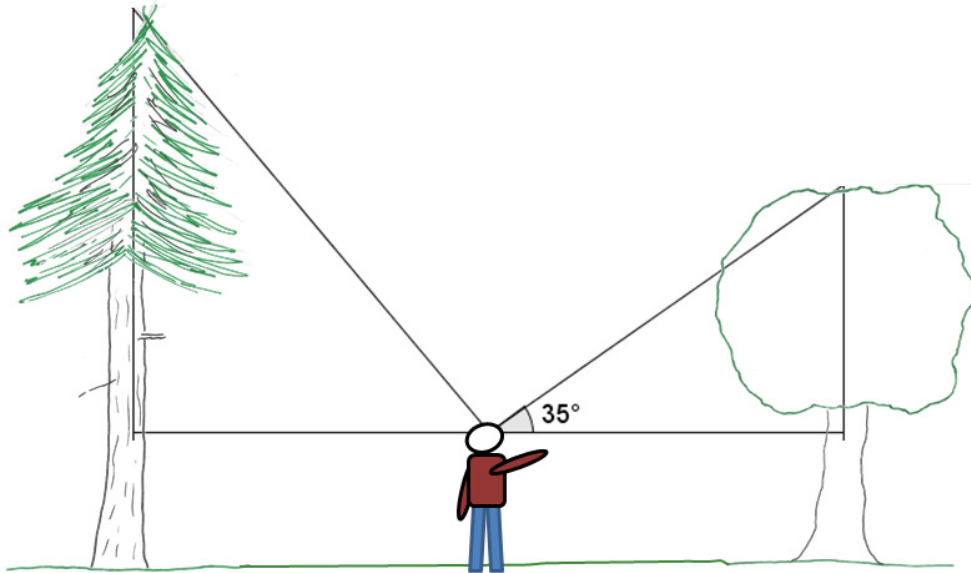
Classwork

Opening Exercise

- a. Dan was walking through a forest when he came upon a sizable tree. Dan estimated he was about 40 meters away from a tree when he measured the angle of elevation between the horizontal and the top of the tree to be 35 degrees. If Dan is about 2 meters tall, about how tall is the tree?



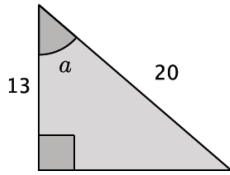
- b. Dan was pretty impressed with this tree ... until he turned around and saw a bigger one, also 40 meters away but in the other direction. “Wow,” he said. “I bet that tree is *at least* 50 meters tall!” Then he thought a moment. “Hmm ... if it is 50 meters tall, I wonder what angle of elevation I would measure from my eye level to the top of the tree?” What angle will Dan find if the tree is 50 meters tall? Explain your reasoning.



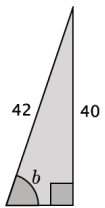
Exercises 1–5

1. Find the measure of angles a–d to the nearest degree.

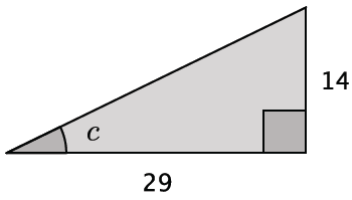
a.



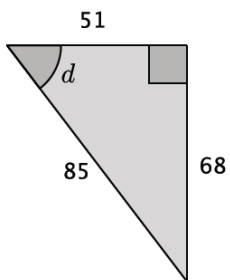
b.



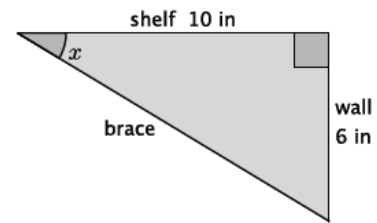
c.



d.



2. Shelves are being built in a classroom to hold textbooks and other supplies. The shelves will extend 10 in from the wall. Support braces will need to be installed to secure the shelves. The braces will be attached to the end of the shelf and secured 6 in below the shelf on the wall. What angle measure will the brace and the shelf make?



3. A 16 ft ladder leans against a wall. The foot of the ladder is 7 ft from the wall.
- Find the vertical distance from the ground to the point where the top of the ladder touches the wall.
 - Determine the measure of the angle formed by the ladder and the ground.

4. A group of friends have hiked to the top of the Mile High Mountain. When they look down, they can see their campsite, which they know is approximately 3 miles from the base of the mountain.
- Sketch a drawing of the situation.

- What is the angle of depression?

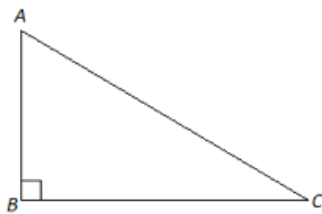
5. A roller coaster travels 80 ft of track from the loading zone before reaching its peak. The horizontal distance between the loading zone and the base of the peak is 50 ft.
- Model the situation using a right triangle.

- At what angle is the roller coaster rising according to the model?

Lesson Summary

In the same way that mathematicians have named certain ratios within right triangles, they have also developed terminology for identifying angles in a right triangle, given the ratio of the sides. Mathematicians will often use the prefix “arc” to define these because an angle is not just measured as an angle, but also as a length of an *arc* on the unit circle.

Given a right triangle $\triangle ABC$, the measure of angle C can be found in the following ways:



- $\arcsin\left(\frac{AB}{AC}\right) = m\angle C$
- $\arccos\left(\frac{BC}{AC}\right) = m\angle C$
- $\arctan\left(\frac{AB}{BC}\right) = m\angle C$

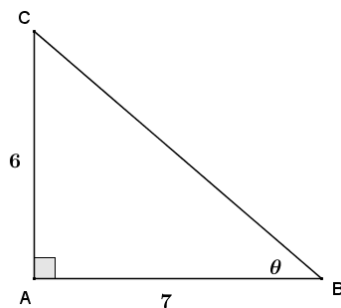
We can write similar statements to determine the measure of angle A .

We can use a calculator to help us determine the values of arcsin, arccos, and arctan. Most calculators show these buttons as “ \sin^{-1} ,” “ \cos^{-1} ,” and “ \tan^{-1} .” This subject will be addressed again in future courses.

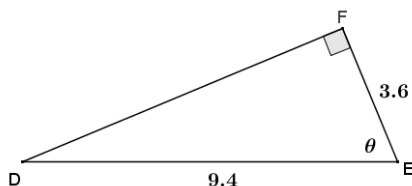
Problem Set

1. For each triangle shown, use the given information to find the indicated angle to the nearest degree.

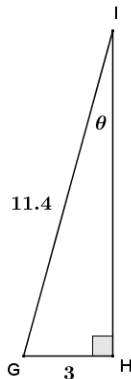
a.



b.



c.

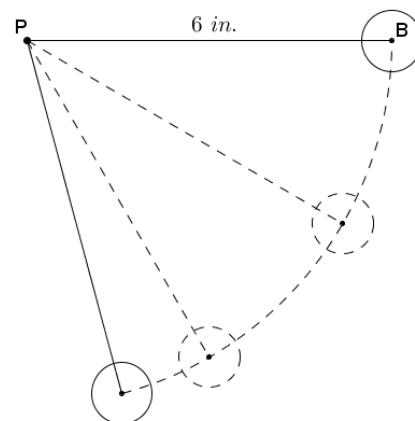


2. Solving a right triangle means using given information to find all the angles and side lengths of the triangle. Use \arcsin and \arccos , along with the given information, to solve right triangle ABC if leg $AC = 12$ and hypotenuse $AB = 15$.

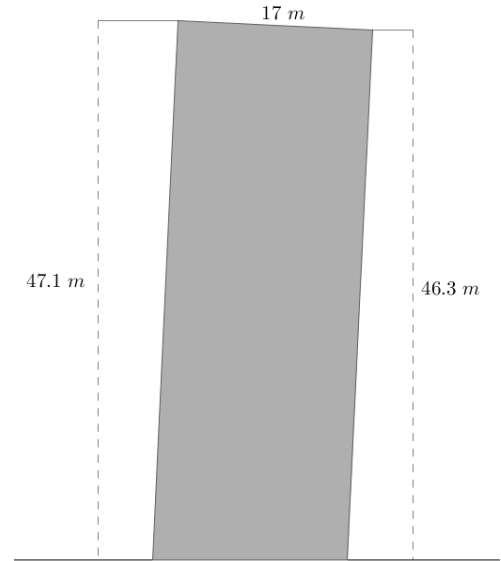
Once you have found the measure of one of the acute angles in the right triangle, can you find the measure of the other acute angle using a different method than those used in this lesson? Explain.

3. A pendulum consists of a spherical weight suspended at the end of a string whose other end is anchored at a pivot point P . The distance from P to the center of the pendulum's sphere, B , is 6 inches. The weight is held so that the string is taut and horizontal, as shown to the right, and then dropped.

- What type of path does the pendulum's weight take as it swings?
- Danni thinks that for every vertical drop of 1 inch that the pendulum's weight makes, the degree of rotation is 15° . Do you agree or disagree with Danni? As part of your explanation, calculate the degree of rotation for every vertical drop of 1 inch from 1 to 6 inches.



4. A stone tower was built on unstable ground and the soil beneath it settled under its weight causing the tower to lean. The cylindrical tower has a diameter of 17 meters. The height of the tower on the low side measured 46.3 meters and on the high side measured 47.1 meters. To the nearest tenth of a degree, find the angle that the tower has leaned from its original vertical position.



5. Doug is installing a surveillance camera inside a convenience store. He mounts the camera 8 ft above the ground and 15 ft horizontally from the store's entrance. The camera is being installed to monitor every customer that enters and exits the store. At what angle of depression should Doug set the camera to capture the faces of all customers?